Maps

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$$

also: iterated maps, difference equations, recursion relations

arise as

- discretization of continuous systems
  - eg - iterative methods for solving linear/nonlinear equations
  - numerical integration of differential equations
  - Poincaré sections of continuous-time dynamical systems
- models of natural phenomena with discrete dynamics
- simple mathematical models of chaotic behaviour

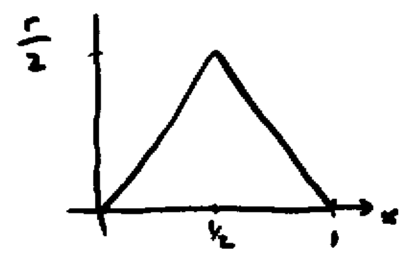
One-dimensional maps

$$x_{n+1} = f(x_n)$$

points x_n "hop" along their orbits - oscillations possible

Examples: • Tent map

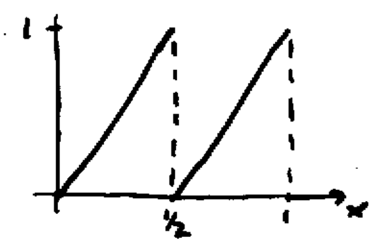
$$f(x) = \begin{cases} rx & 0 \le x \le \frac{1}{2} \\ r(1-x) & \frac{1}{2} < x \le 1 \end{cases}$$



If $0 \le r \le 2$, $f: [0,1] \to [0,1]$

• (Bernoulli) shift map

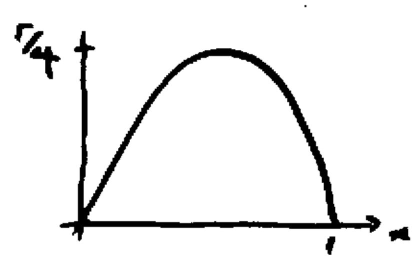
$$f(x) = 2x \bmod 1 = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2(x-\frac{1}{2}) & \frac{1}{2} \le x < 1 \end{cases}$$



$$f: [0,1) \to [0,1)$$

• Logistic map

$$f(x) = rx(1-x)$$



If $0 \le r \le 4$, $f: [0,1] \to [0,1]$

Fixed Points and Stability

$$x_{n+1} = f(x_n)$$

x^* is a fixed point : $\boxed{f(x^*) = x^*}$

Note:
$\dot{x} = f(x)$
f gives change in x
fixed point : $f(x) = 0$
$x_{n+1} = f(x_n)$
f gives new value of x
fixed point : $f(x) = x$

Stability:
Consider a small perturbation of x^*:
$$\eta_n = x_n - x^*$$
($\{x_n\}$ is a nearby orbit).

Then
$$x_{n+1} = f(x_n) \Rightarrow x^* + \eta_{n+1} = f(x^* + \eta_n)$$

$$\Rightarrow x^* + \eta_{n+1} = \underbrace{f(x^*)}_{= x^*} + f'(x^*)\eta_n + \tfrac{1}{2} f''(x^*)\eta_n^2 + \dots$$

$$\Rightarrow \quad \eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2)$$

Linearized map near x^* (okay if $|f''(x)|$ bounded near x^*)

$$\boxed{\eta_{n+1} = f'(x^*)\eta_n}$$

eigenvalue/multiplier $\lambda = f'(x^*)$

so $\eta_1 = \lambda \eta_0$, $\eta_2 = \lambda \eta_1 = \lambda^2 \eta_0$, ..., $\eta_n = \lambda^n \eta_0$

- If $|\lambda| = |f'(x^*)| < 1$: $\eta_n \to 0$ as $n \to \infty$
  ie $x_n \to x^*$:
  x^* is linearly stable

  (note: if $0 < \lambda < 1$, η_{n+1} has same sign as η_n
  $\Rightarrow$ iterates approach x^* monotonically

  if $-1 < \lambda < 0$ $\Rightarrow$ iterates oscillate about x^*)
- If $|\lambda| = |f'(x^*)| > 1$: $|\eta_n|$ grows with n
  x^* is unstable ($\lambda > 1$: monotonic growth, $\lambda < -1$: oscillatory instability)
- If $|\lambda| = |f'(x^*)| = 1$: marginal case
- If $\lambda = f'(x^*) = 0$: $\eta_{n+1} \sim \eta_n^2$ quadratic convergence
  x^* is superstable

eg $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$ near the root $x = 2$.

Possible iterations:

- $x_{n+1} = f(x_n) = x_n^2 - 2$ — $f'(x) = 2x$, $f'(2) = 4 > 1$ unstable, no convergence
- $x_{n+1} = f(x_n) = \sqrt{x_n + 2}$ — $f'(x) = \frac{1}{2\sqrt{x+2}}$, $f'(2) = \frac{1}{4}$ monotonic convergence
- $x_{n+1} = f(x_n) = 1 + 2/x_n$ — $f'(x) = -2/x^2$, $f'(2) = -1/2$ oscillatory convergence
- $x_{n+1} = f(x_n) = \frac{x_n^2 + 2}{2x_n - 1}$ — $f'(x) = \frac{2x^2 - 2x - 4}{(2x-1)^2}$, $f'(2) = 0$ quadratic convergence

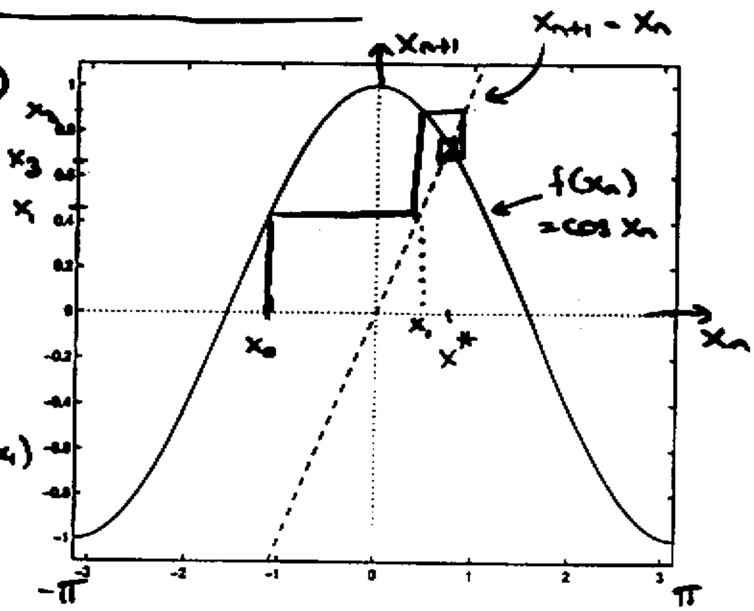
Newton's method for $g(x) = 0$:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = f(x_n)$$

$g'(x^*) \neq 0$: superstable.

Cobwebs for $x_{n+1} = f(x_n)$ given x_0.

- draw vertical line: intersects graph of f at $x_1 = f(x_0)$
- use x_1 as "input": draw horizontal line to the diagonal $x_{n+1} = x_n$, then move vertically to the curve of f again: $x_2 = f(x_1)$
- repeat: easy graphical approach to iteration



eg $x_{n+1} = \cos x_n$

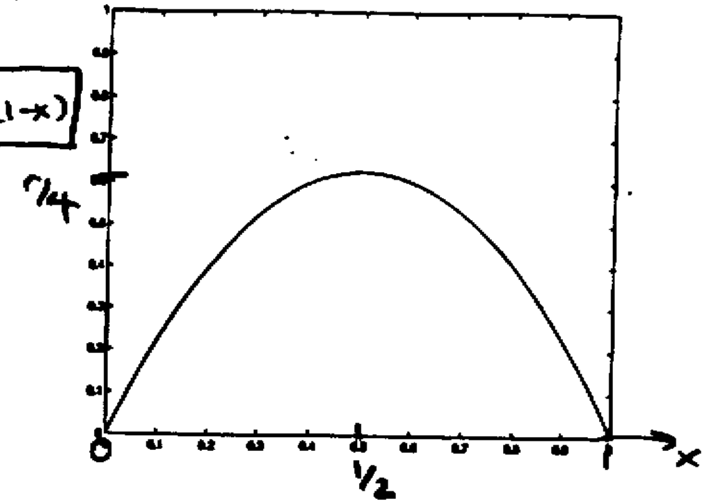
For any initial condition, $x_n \to x^*$. $x^* = 0.739085\ldots$ ← unique root of $x - \cos x = 0$.

Multiplier: $|f'(x^*)| = |-\sin x^*| < 1$: stable

convergence through damped oscillations

Logistic Map (Robert May, 1976)

$x_{n+1} = f(x_n)$, $\boxed{f(x) = rx(1-x)}$



- Graph of f: parabola
- $\boxed{f'(x) = r(1-2x)}$

Maximum of f at $x = \frac{1}{2}$:
$f(\frac{1}{2}) = \frac{r}{4}$.

We restrict our attention to $\underline{0 \le r \le 4}$: then $f: [0,1] \to [0,1]$
(f maps unit interval into itself)

[If $r > 4$, $x_n \to -\infty$ for most initial conditions x_0]

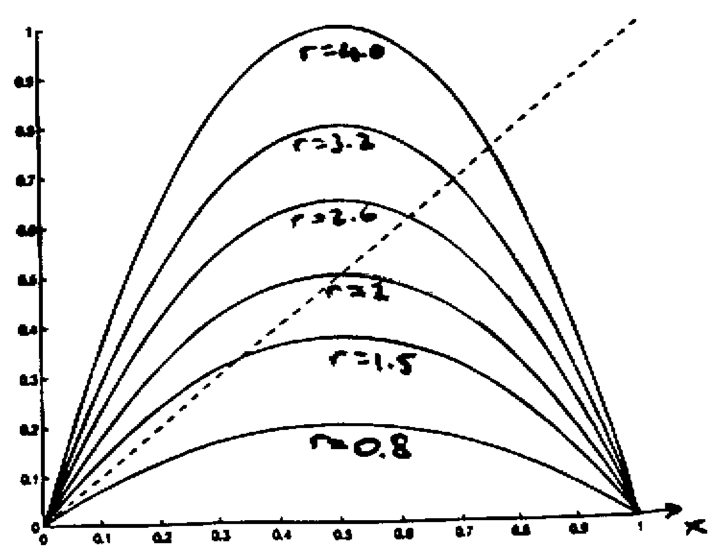
- <u>Fixed points</u> $x = f(x) = rx(1-x)$

$\Rightarrow (r-1)x - rx^2 = 0$
$\Rightarrow x^* = 0$ or $x^* = \frac{r-1}{r} = 1 - \frac{1}{r}$ ← in $[0,1]$ for $r \ge 1$.

- <u>Stability</u>

$x^* = 0$: $f'(0) = r$ so $x^* = 0$ is $\begin{cases} \text{stable}, & 0 \le r < 1 \\ \text{unstable}, & r > 1 \end{cases}$

(at $r = 1$: $f(x) = x - x^2 < x$
for $0 < x \le 1$
$\Rightarrow x^* = 0$ is stable)



$x^* = 0$ becomes unstable through a <u>transcritical bifurcation</u> at $r = 1$.

<u>Stability</u> of $x^* = 1 - 1/r$: compute multiplier $\lambda = f'(x^*)$:

$$f'(x^*) \quad f'(1-1/r) = r(1 - 2(1-1/r)) = 2 - r$$

$x^* = 1 - 1/r$ is $\begin{cases} \text{stable if } |2-r|<1 \text{ ie } 1<r<3 \\ \underline{\text{unstable}} \text{ if } r>3. \end{cases}$ ← note: superstable if $r=2$

For $r>3$, <u>both</u> fixed points are linearly unstable
(cannot occur for flows)

$0 < f'(x^*) < 1$

$1<r<2$
stable,
monotonic approach

$f'(x^*) = 0$

$r=2$
superstable

$-1 < f'(x^*) < 0$

$2<r<3$
stable,
oscillatory

$f'(x^*) < -1$

$r>3$
2-cycle

What happens for $r>3$?

$r = 3.2$ Iterates osc<u>illate</u>

between two values
$x_1 < x^* = 1 - 1/r$, $x_2 > x^*$: period-2 cycle
or <u>2-cycle</u>

<u>Period-doubling bifurcation</u> at $r = 3$. ← $f'(x^*) = -1$

Larger r : eg $r = 3.5$: iterates oscillate between f<u>our</u> values
4-cycle – another period-doubling

Further period-doublings to period 8, 16, 32, 64, ...

Let r_n be the value of r at which a 2^n-cycle first appears:

Numerically:

		Birth of cycle of period
r_1	$= 3$	2
r_2	$= 1+\sqrt{6} = 3.449\ldots$	4
r_3	$= 3.54409\ldots$	8
r_4	$= 3.5644\ldots$	16
r_5	$= 3.568759\ldots$	32
$\vdots$		
r_∞	$= 3.569946\ldots$	∞

Successive bifurcations occur faster and faster:

$r_n \to r_\infty$ as $n \to \infty$.

Convergence is geometric:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$ Feigenbaum constant.

$r > r_\infty$: Chaos

Aperiodic long-term dynamics:

Sequence $\{x_n\}$ never settles down to a fixed point or periodic orbit

Sensitive dependence on initial conditions:

two trajectories starting close together rapidly diverge from each other

($\Rightarrow$ long-term prediction impossible: small uncertainties are amplified exponentially fast)

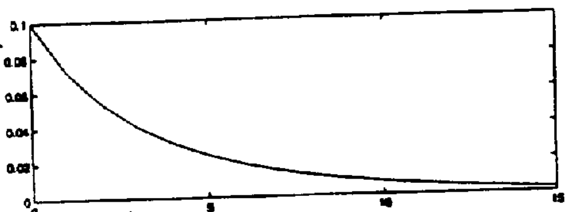
Irregularity due to nonlinearity, not noise.

But: periodic windows eg period 3 for $r = 3.84$

Chaos: Aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence to initial conditions.

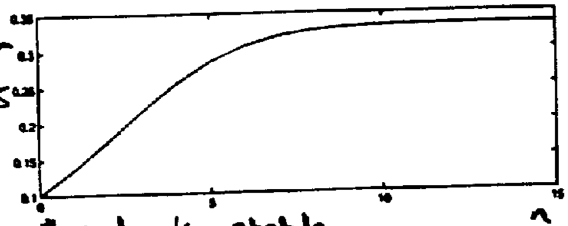
Iterates for the logistic map $x_{n+1} = r x_n (1 - x_n)$

$r = 0.8$



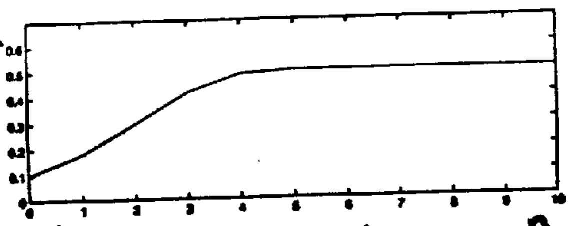
$x^* = 0$ stable

$r = 1.5$



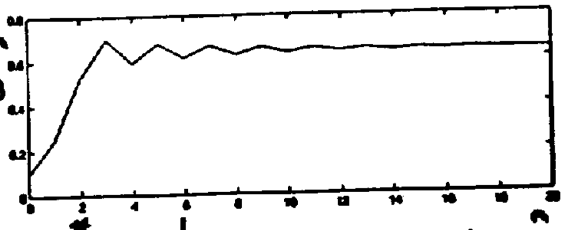
$x^* = 1 - \frac{1}{r} = \frac{1}{3}$ stable : monotonic

$r = 2.0$



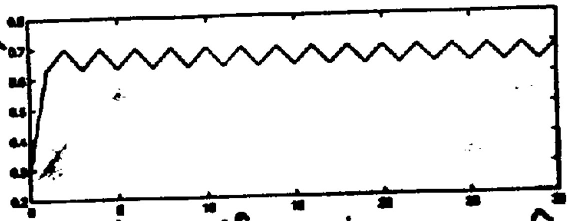
$x^* = \frac{1}{2}$ superstable

$r = 2.8$



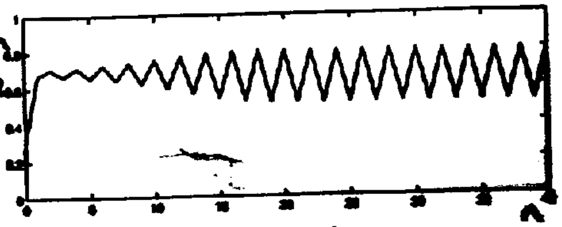
$x^* = 1 - \frac{1}{2.8}$ stable : oscillatory

$r = 3.10$



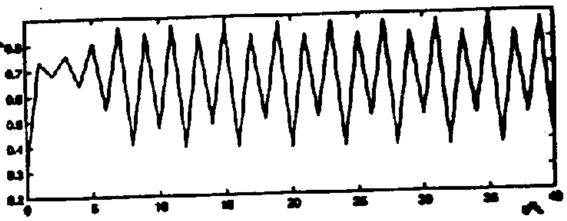
$x^* = \frac{2}{3}$ · flip bifurcation

$r = 3.2$



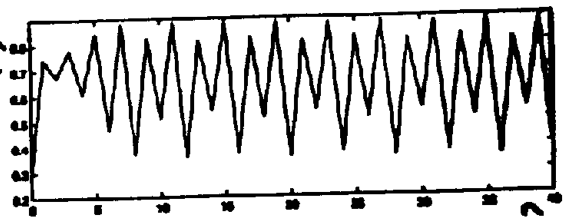
stable 2-cycle

$r = 3.5$



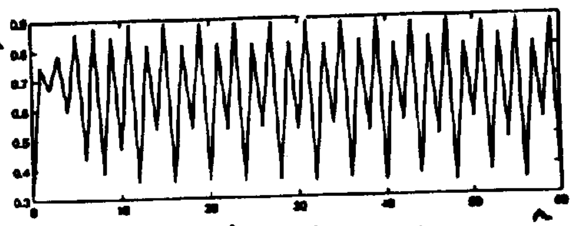
stable 4-cycle

$r = 3.55$



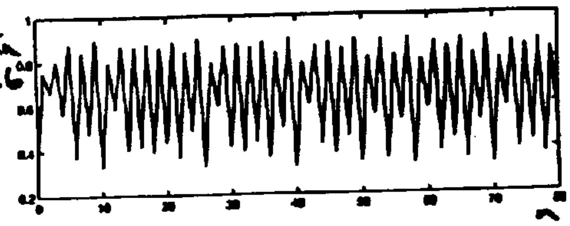
stable 8-cycle

$r = 3.565$



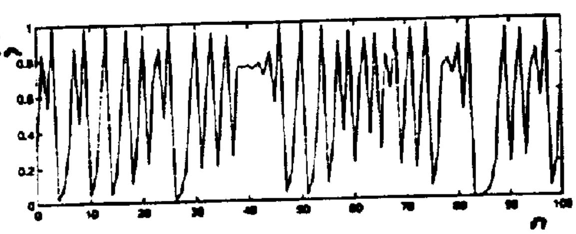
stable 16-cycle

$r = 3.6$



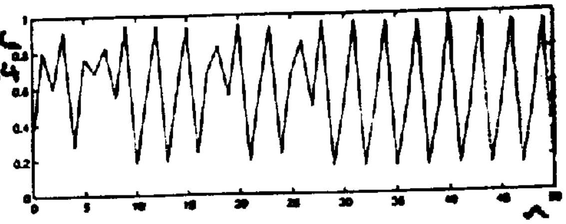
$r > r_\infty$ · chaos

$r = 4.0$



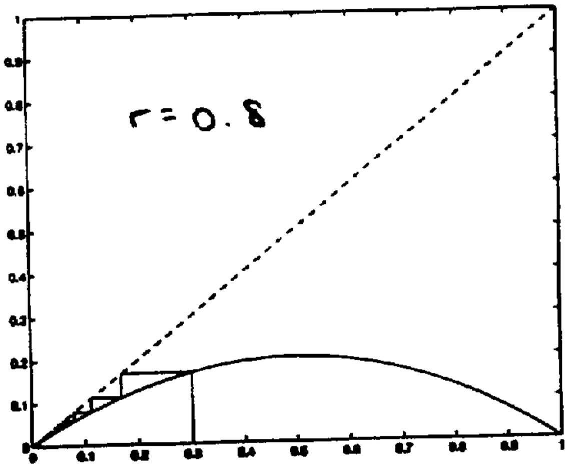
Chaos

$r = 3.84$

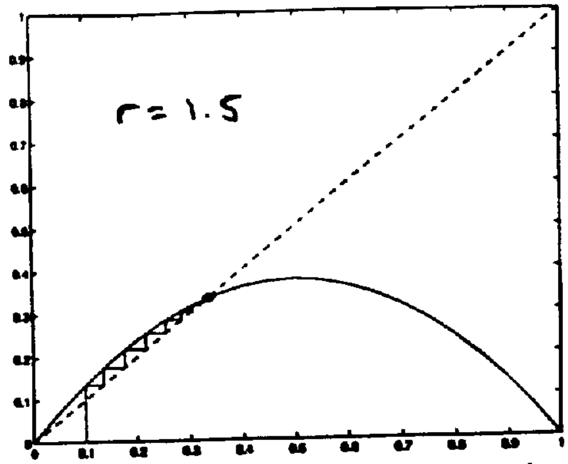


stable 3-cycle

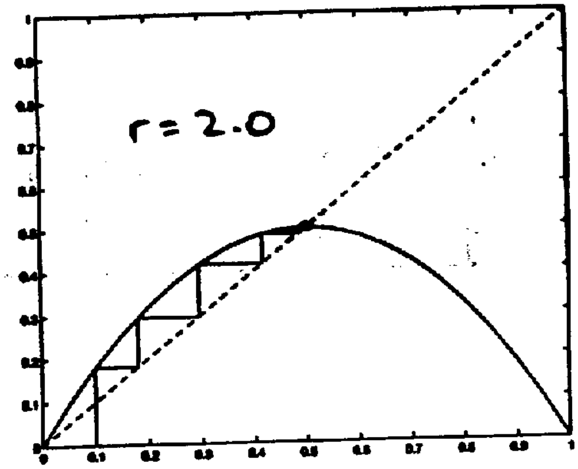
Cobweb diagrams for the logistic map



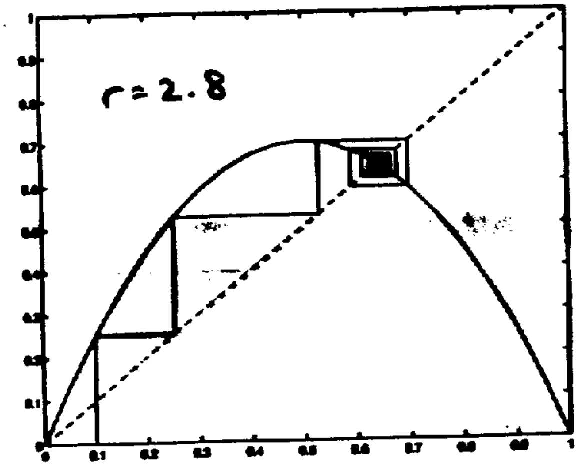
$x^* = 0$ stable



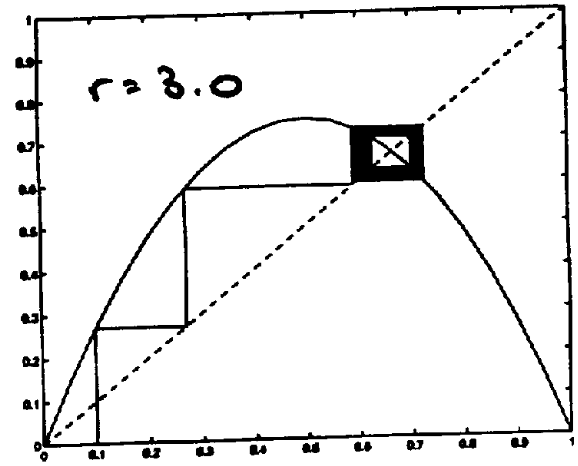
$x^* = 1/3$ stable fixed point



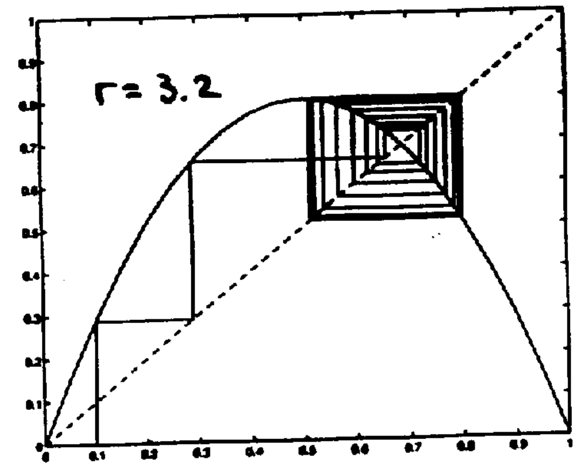
$x^* = \frac{1}{2}$ superstable fixed point



x^* stable fixed point

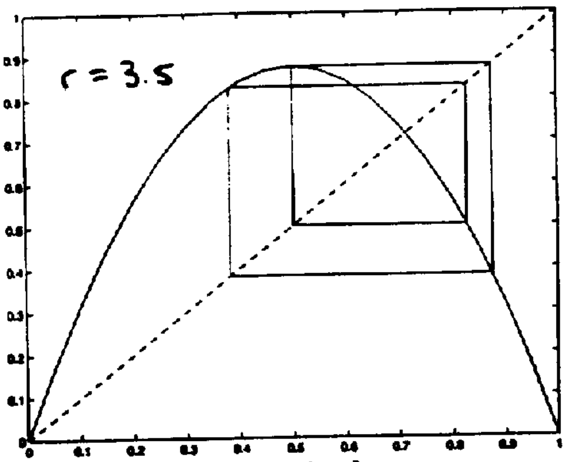


flip bifurcation

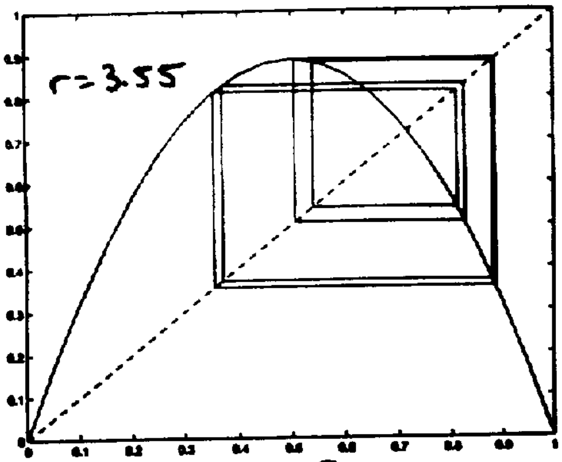


period-2 cycle

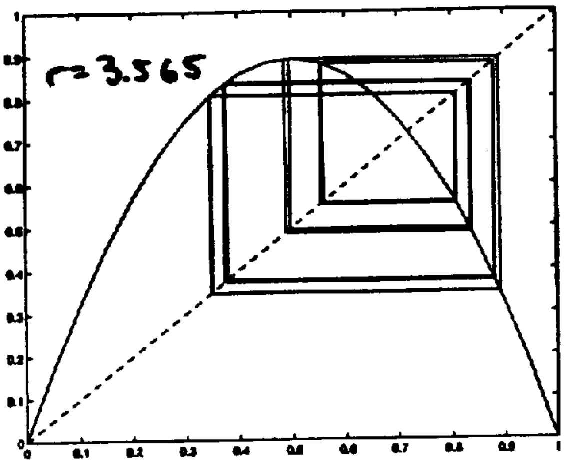
Cobweb diagrams for the logistic map (no transients)



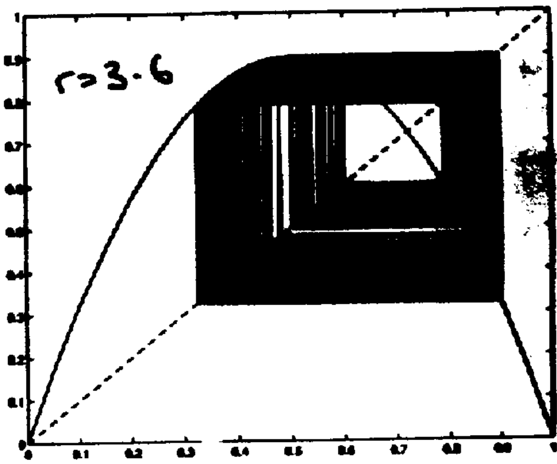
period 4



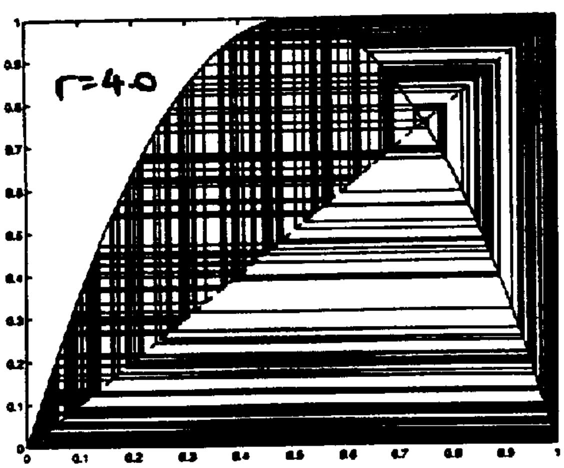
period 8



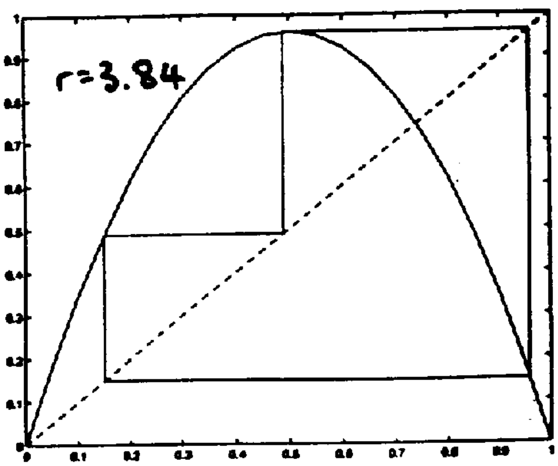
period 16



chaos



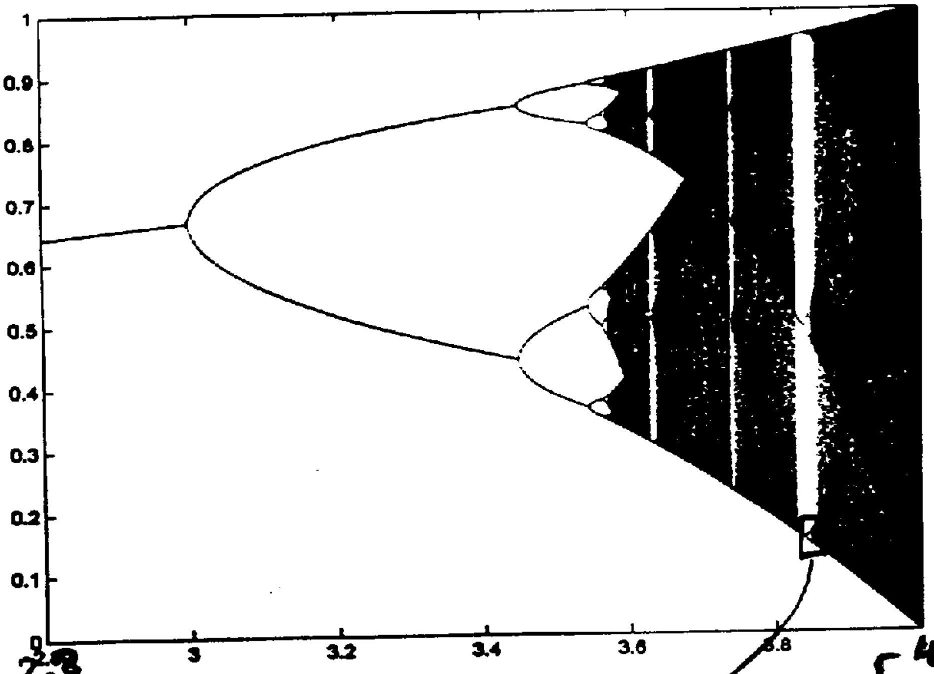
chaos



period 3

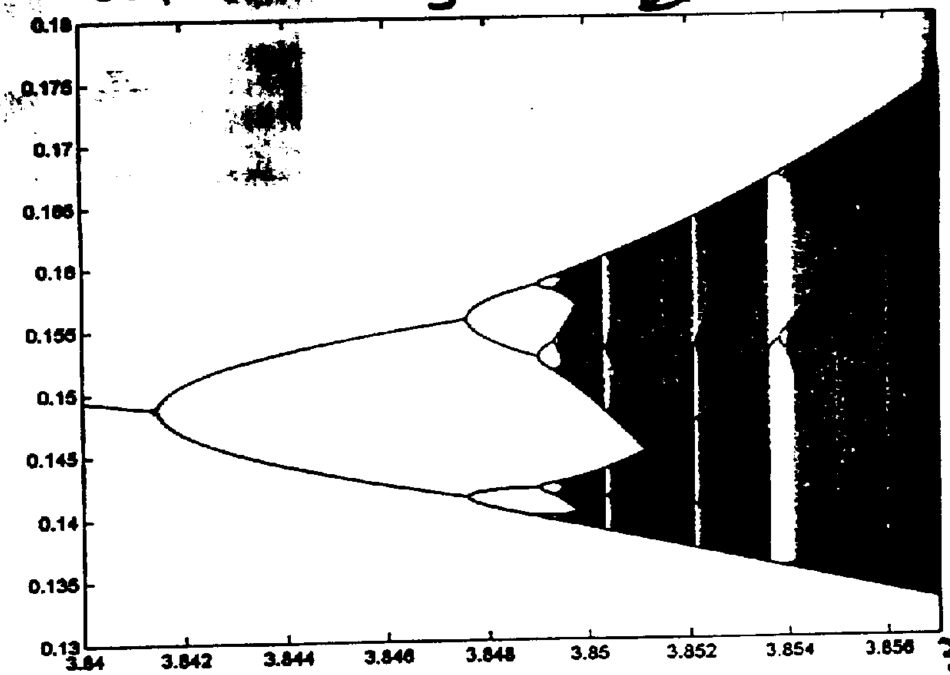
Orbit diagram for the logistic map

$x_{n+1} = r x_n (1 - x_n)$



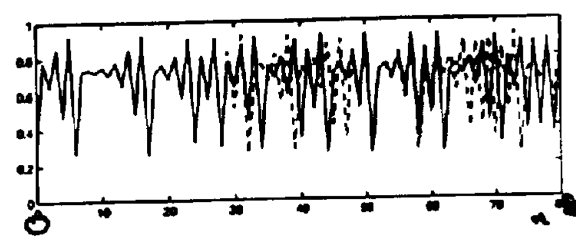
(Plot of long-time behaviour of iterates as a function of r)

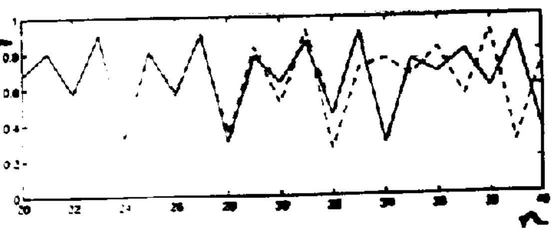
self-similarity



Chaos : sensitive dependence on initial conditions

$r = 3.7$





(iterates x_n, $n = 20 - 40$)

Solid line — : $x_0 = 0.3$

Dashed line --- : $x_0 = 0.300001$

Period-2 orbit for $r > 3$:

Fixed point at $x^* = 1 - 1/r$ undergoes a flip bifurcation

at $r = 3$: $f'(x^*) = -1$

$\Rightarrow$ a period-2 orbit is born:

there are $x_1 = p$, $x_2 = q$ so that $f(p) = q$, $f(q) = p$

$$\Rightarrow p = f(f(p)) = (f \circ f)(p) = f^2(p)$$ composition, not power

ie p (and q) is a fixed point of the second iterate map

$g(x) \equiv f^2(x) = f(f(x))$

$f(x) = rx(1-x)$ quadratic $\Rightarrow f^2(x) = rf(x)(1-f(x))$ quartic

Computation of 2-cycle: Find p, q: solve $x = f^2(x)$ (fixed points of f^2)

$$x = f(f(x)) = rf(x)(1-f(x)) = r^2x(1-x)(1-rx(1-x))$$

Quartic polynomial equation!

Note: fixed points $x^* = 0$ and $x^* = 1 - 1/r$ of f are also fixed points of f^2 ie (trivially) period-2 orbits (also 2-cycles)

$$\left[f^2(x^*) = f(\underbrace{f(x^*)}_{= x^* \text{ fixed point}}) = f(x^*) = x^*\right]$$

so we already know two roots:

factor x and $[x - (1 - 1/r)]$ by long (polynomial) division

$$x - f^2(x) = x - r^2x\left(1 - (1+r)x + 2rx^2 - rx^3\right)$$
$$= x\left(rx - (r-1)\right)\underbrace{\left(r^2x^2 - r(r+1)x + (r+1)\right)}_{\text{quadratic}} = 0$$

p, q are roots

$$\Rightarrow p, q = \frac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$$

real for $r > 3$; $-1 < r < 3$: roots complex, 2-cycle doesn't exist

$\Rightarrow$ 2-cycle exists for $r > 3$.

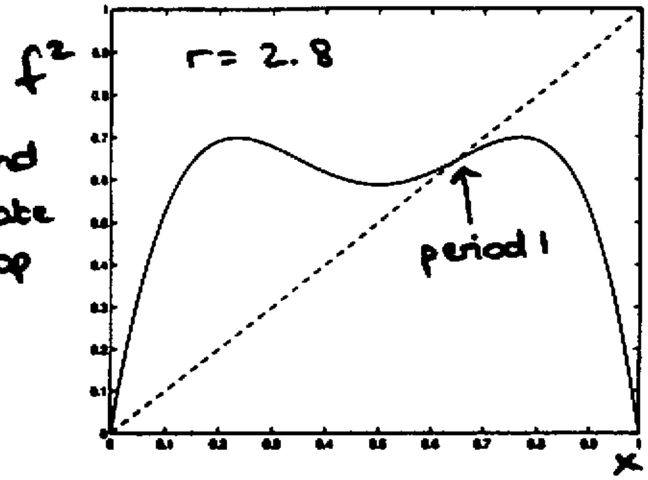
At $r = 3$: $p, q = \frac{r+1}{2r} = \frac{2}{3} = 1 - \frac{1}{3} = x^*$: 2-cycle bifurcates continuously from x^*

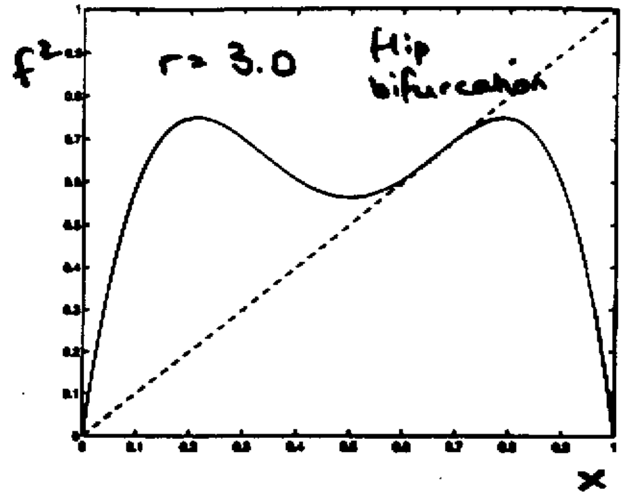
Period-doubling via a supercritical flip bifurcation

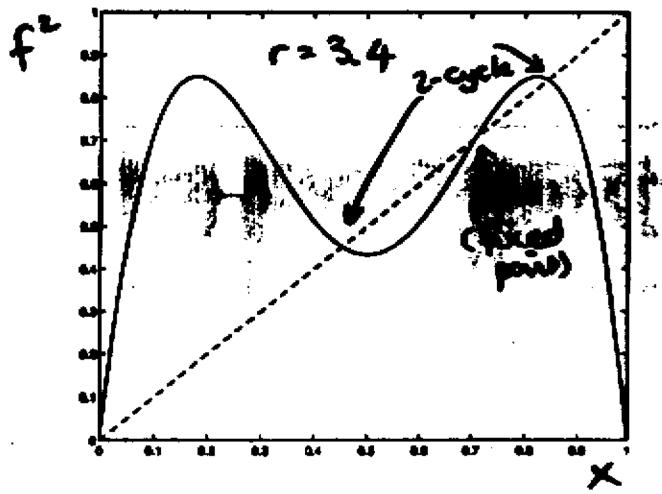
Flip (period-doubling) bifurcation:

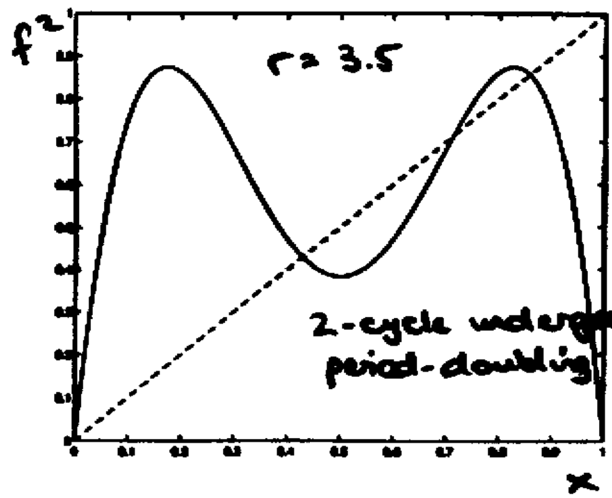
supercritical pitchfork for f^2: creation of new fixed points

second iterate map



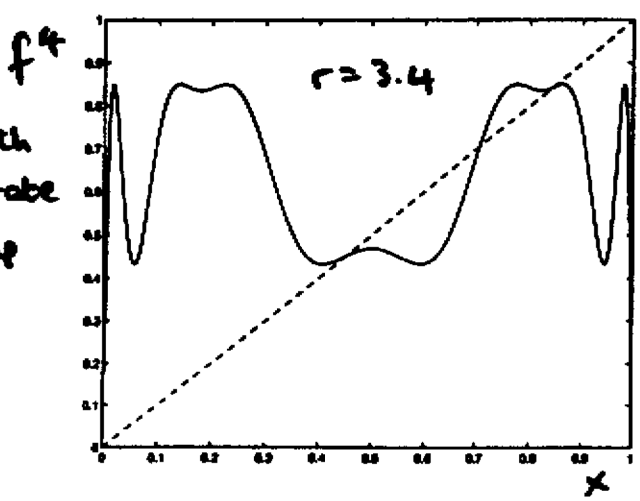


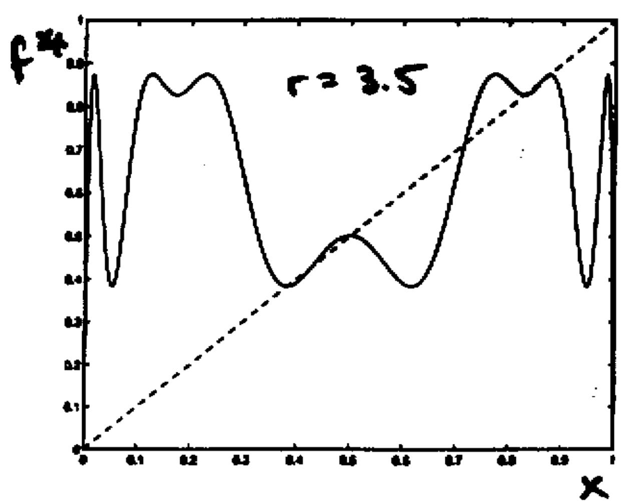




$r > 1 + \sqrt{6} = 3.449\ldots$: 2-cycle becomes unstable in flip bifurcation

fourth iterate map





Stability of 2-cycle

- reduce question of the stability of a cycle to stability of a fixed point!

$p = f(q),\ q = f(p) \Rightarrow p, q$ are fixed points of f^2.
second iterate map

The 2-cycle is stable $\Leftrightarrow$ p,q are stable fixed points of f^2

Compute multiplier (for p):

$$\lambda = \frac{d}{dx}\left[f^2(x)\right]\Big|_{x=p} = \frac{d}{dx}\left[f(f(x))\right]\Big|_{x=p} = f'(f(p))f'(p) = f'(q)f'(p)$$

Note: multiplier is same at $x=p$ and $x=q$:
p, q branches bifurcate simultaneously.

Now $f'(x) = r(1-2x)$, $\quad p,q = \dfrac{(r+1) \pm \sqrt{(r+1)(r-3)}}{2r}$ $\left[\text{roots of } r^2x^2 - r(r+1)x + r+1 = 0\right.$

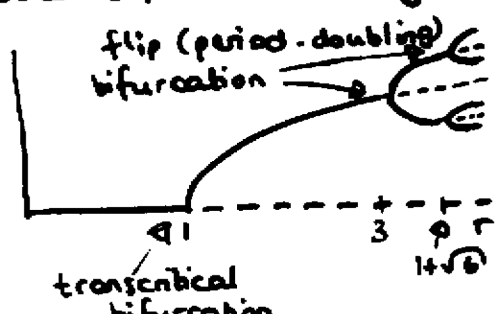
$$\Rightarrow \lambda = f'(p)f'(q) = r(1-2p)r(1-2q)$$
$$= r^2(1-2(p+q)+4pq)$$
$$= r^2\left(1 - 2\frac{r(r+1)}{r^2} + 4\frac{r+1}{r^2}\right) = r^2 - 2r(r+1) + 4(r+1)$$
$$= -r^2 + 2r + 4$$

The 2-cycle is linearly stable $\Leftrightarrow$ $|\lambda| = |-r^2+2r+4| < 1$

ie for $\underset{r_1}{3} < r < \underset{r_2}{1+\sqrt{6}} = 3.449\ldots$

$\underbrace{r = 1+\sqrt{6}}_{\lambda = -1}$: fixed points of f^2 undergo period-doubling bifurcation : creation of 4-cycle (for f)

Partial bifurcation diagram



Fixed points of $f^4(x) = f(f(f(f(x))))$
16^{th} order polynomial equation
(4 roots known: $x^* = 0, 1-1/r$; p, q)

- analytical methods become unwieldy

Further period-doublings at $r = r_3, r_4, r_5, \ldots$ to cycles of period $8, 16, 32, \ldots$
- bifurcation points accumulate : Chaos at $r = r_\infty$

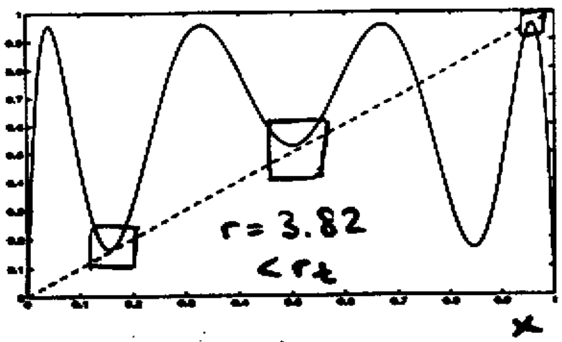
Periodic windows and intermittency for $r > r_\infty$:

Stable period-3 window: $3.8284\ldots < r < 3.8415\ldots$

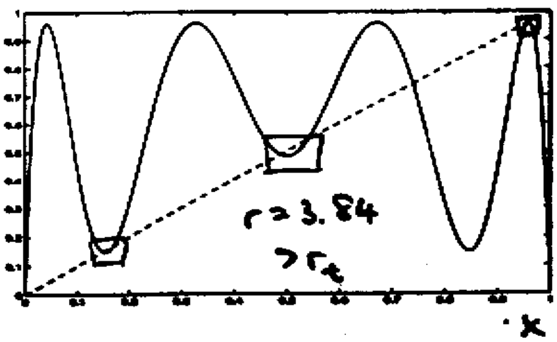
$\uparrow r_t$

- how does a stable 3-cycle appear from chaos?

f^3



f^3

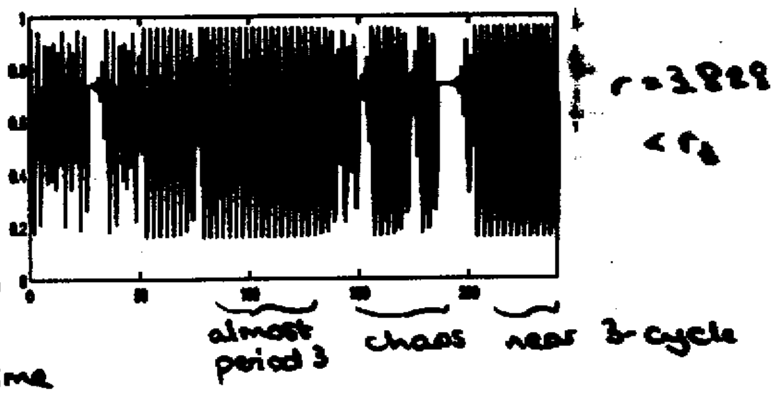


As we decrease r, intersections vanish through a

<u>tangent bifurcation</u> at $r = r_t = 1+\sqrt{8}$

fold, saddle-node

For $r < r_t$: Intermittency



- iterates remain near 3-cycle for a long time, then move away

Orbit spends much time in channel: <u>ghost</u> of saddle-node bifurcation

- intermittent bursts of chaos (orbit escapes) before iterates return near the 3-cycle

<u>Experimental observation</u>: (eg lasers)

- nearly periodic motion interspersed by irregular bursts, statistically distributed (in a deterministic system)

- Intermittency Route to Chaos

Period-doubling in the window: 3-cycle is start of a period-doubling sequence, to $3\cdot 2^k$-cycles

<u>Note</u>: "Period three implies chaos" (Li & Yorke 1975, Šarkovskii 1964)

If f has an orbit of period 3, it has an orbit of each period $m \in \mathbb{N}$ and aperiodic orbits (which don't approach any periodic orbit)

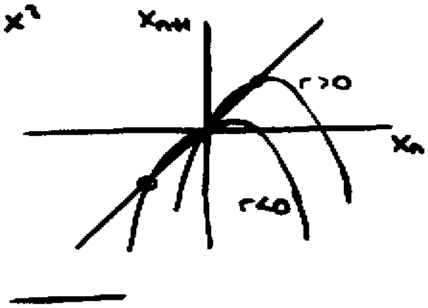
Bifurcations of maps

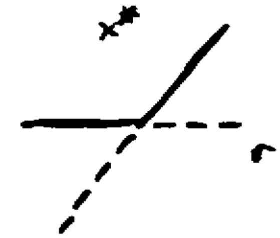
Transcritical bifurcation

$$x_{n+1} = f(x_n) = (1+r)x_n - x_n^2$$
(normal form)

fixed points: $x = f(x) = (1+r)x - x^2$

$\Rightarrow x^* = 0, r$





stability: $f'(0) = 1+r$

$f'(r) = 1-r$

Transcritical bifurcation at $r=0$

Fold bifurcation (tangent, saddle-node)

$$x_{n+1} = f(x_n) = -r + x_n + x_n^2$$

fixed points: $x = -r + x + x^2 \Rightarrow x^2 = r$

$\Rightarrow x^* = \pm\sqrt{r}, \ r \geq 0$

stability: $f'(\sqrt{r}) = 1+2\sqrt{r} > 1$ unstable

$f'(-\sqrt{r}) = 1-2\sqrt{r}$ stable $0<r<1$

Flip bifurcation

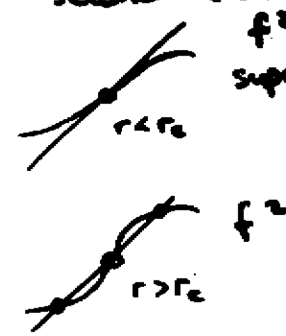
- need $f(x^*) = x^*, \ f'(x^*) = -1$

eg $x_{n+1} = f(x_n) = -(1+r)x_n \pm x_n^3$

fixed point at $x^* = 0$: $f'(0) = -(1+r)$

So $x^* = 0$ is stable for $r<0$, unstable for $r>0$

second iterate map f^2



Check: • for "+" $[f(x) = -(1+r)x + x^3]$

f^2 has two stable fixed points for $r>0$: stable 2-cycle for f

$\Rightarrow$ supercritical flip bifurcation (period-doubling)

- corresponds to supercritical pitchfork bifurcation for f^2

• for "−" $[f(x) = -(1+r)x - x^3]$

f^2 has two unstable fixed points for $r<0$: unstable 2-cycle

$\Rightarrow$ subcritical flip bifurcation

subcritical pitchfork bifurcation for f^2

Liapunov Exponent

<u>Chaos</u>: Sensitive dependence on initial conditions

i.e. Neighbouring orbits diverge exponentially fast (on average)

Quantify sensitive dependence:

Consider initial condition x_0 (iterates $x_1 = f(x_0), x_2 = f^2(x_0), \ldots$),

nearby point $x_0 + \delta_0$: <u>small</u> initial separation δ_0

δ_n : separation after n iterates.

If $\boxed{|\delta_n| \approx |\delta_0| e^{n\lambda}}$, λ: Liapunov exponent

λ: measure of exponential rate of convergence/divergence.

Positive λ: sign<u>ature</u> of chaos.

$$\frac{\delta_n}{\delta_0} = \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \underset{\delta_0 \to 0}{\approx} (f^n(x_0))'$$

chain rule $= f'(f^{n-1}(x_0))\, f'(f^{n-2}(x_0)) \ldots f'(f(x_0))\, f'(x_0)$

$$= f'(x_{n-1})\, f'(x_{n-2}) \cdots f'(x_1) f'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

$$\Rightarrow \lambda \approx \frac{1}{n} \ln \left|\frac{\delta_n}{\delta_0}\right| \approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i)\right|$$

If the limit as $n \to \infty$ exists, we use this to <u>define</u> the

Lyapunov exponent: $$\boxed{\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|}$$

a property of an <u>orbit</u> (depends on x_0)

- If f has a <u>stable p-cycle</u> containing x_0, then <u>$\lambda < 0$</u>:

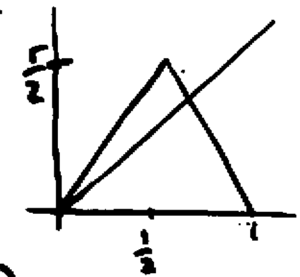
p-cycle is stable $\Rightarrow$ x_0 is a stable fixed point of f^p $\Rightarrow |(f^p)'(x_0)| < 1$

For a p-cycle, $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| = \frac{1}{p} \sum_{i=0}^{p-1} \ln |f'(x_i)|$

$$= \frac{1}{p} \ln \prod_{i=0}^{p-1} |f'(x_i)| = \frac{1}{p} \ln |(f^p)'(x_0)| < 0.$$

<u>Super</u>stable cycle $\Rightarrow (f^p)'(x_0) = 0 \Rightarrow \lambda = -\infty$.

eg tent map: $x_{n+1} = f(x_n) = \begin{cases} r x_n & 0 \le x_n \le \frac{1}{2} \\ r(1-x_n) & \frac{1}{2} \le x_n \le 1 \end{cases}$



Note: $|f'(x)| = r$ for all $x \in [0,1]$

(ie for $r > 1$, both fixed points are unstable)

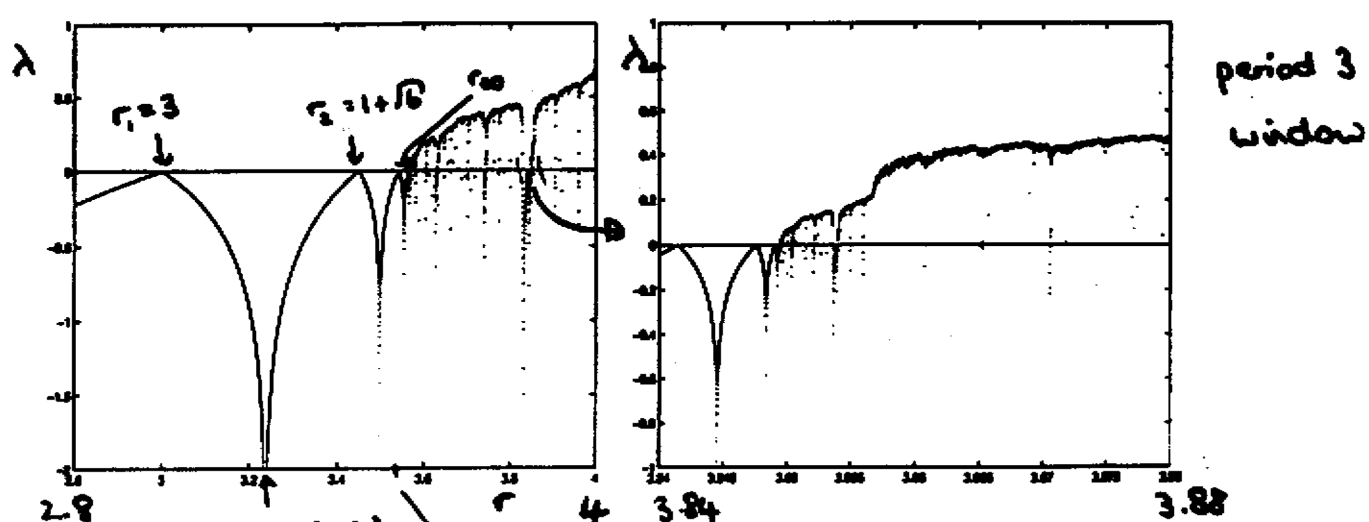
Liapunov exponent: $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln r = \ln r$$

$\Rightarrow \lambda = \ln r$ (independent of x_0)

$\Rightarrow$ tent map has chaotic solutions for all $r > 1$

Lyapunov exponent for the logistic map

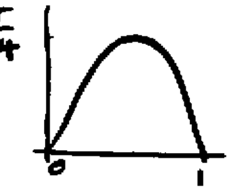


superstable 2-cycle at $r = 1+\sqrt{5}$

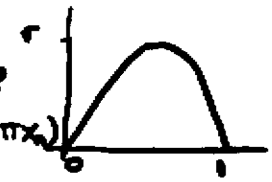
onset of chaos at $r_\infty \approx 3.57$: Liapunov exponent λ becomes positive

Universality

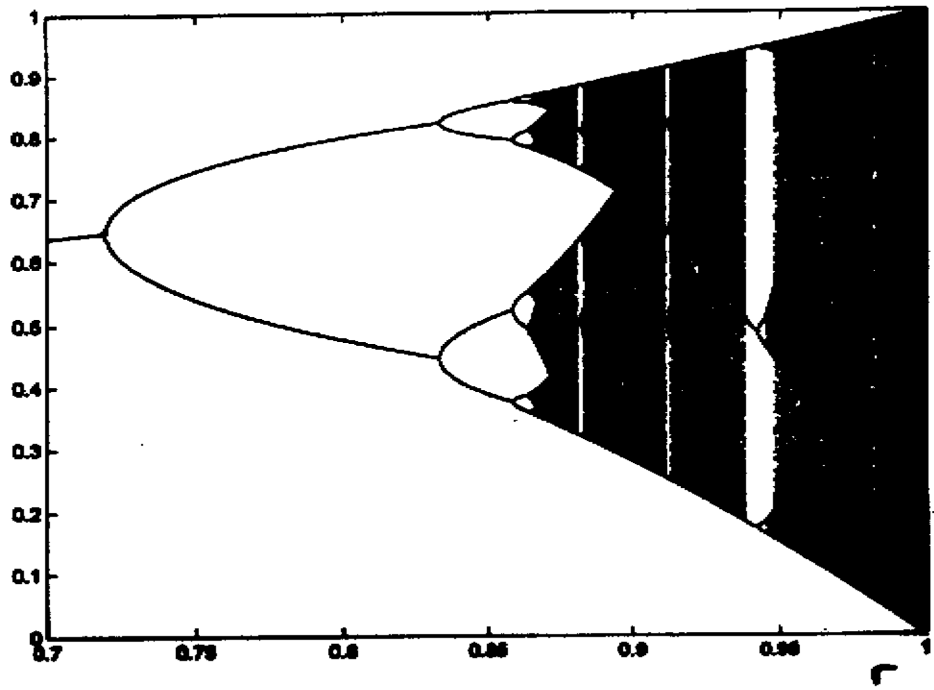
Compare
Logistic map
$x_{n+1} = r x_n (1-x_n)$



Sine map
$x_{n+1} = r \sin(\pi x_n)$



Orbit diagram for sine map $x_{n+1} = r \sin(\pi x_n)$



Qualitative dynamics (period-doubling route to chaos, periodic windows, ...) identical for logistic map and sine map.

Universality: different unimodal maps have qualitatively similar dynamics and routes to chaos

single peak, smooth, convex

for a popular discussion, see Gleick "Chaos: Making a New Science"

also observed in continuous systems, experiments!

eg • periodic attractors (stable cycles) appear in the same sequence (U-sequence) Metropolis et al 1973

(for unimodal maps $x_{n+1} = r f(x_n)$, $f(0) = f(1) = 0$, there is a universal sequence of attractors: up to period 6:

1, 2, 2×2, 6, 5, 3, 2×3, 5, 6, 4, 6, 5, 6)

• universal convergence rate of period-doubling bifurcation points r_n: $\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$ Feigenbaum, 1975

• theory: based on renormalization (Feigenbaum)

• experiment: period-doubling routes to chaos observed in ODEs ($n \geq 3$), chemical systems, convection (hydrodynamics), electronic ...