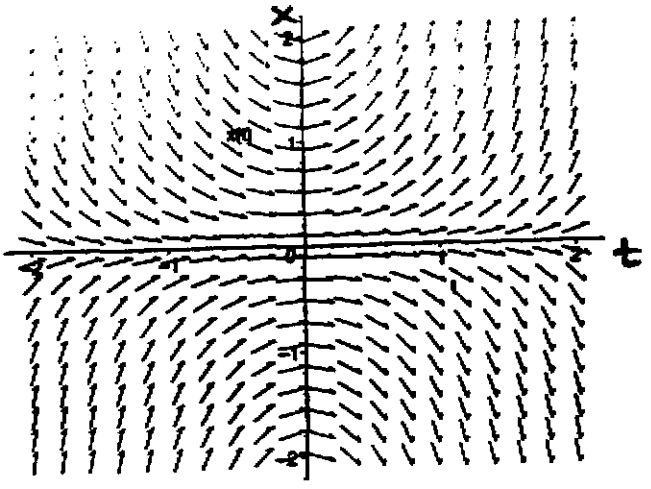
General 1st-order ODE $\dot{x} = f(x,t)$

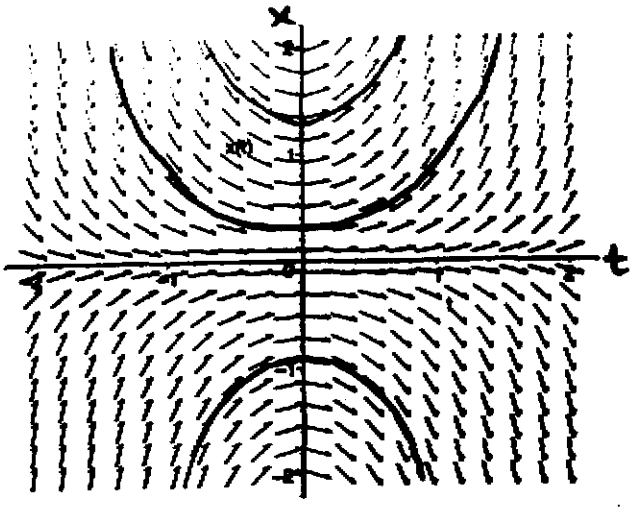
- solution $x(t)$ is a function whose graph in t-x plane (trajectory, solution curve) has slope $\dot{x} = f(x,t)$ at each (t,x) through which the curve passes.

eg $\dot{x} = 2xt$

Direction field (slope field)



Solution curves



Direction field tells us "which way to go" at (t,x)

Solution curves cannot cross (uniqueness)

Note: non-autonomous
- need values of x, t

$\Rightarrow$ 2-d dynamical system

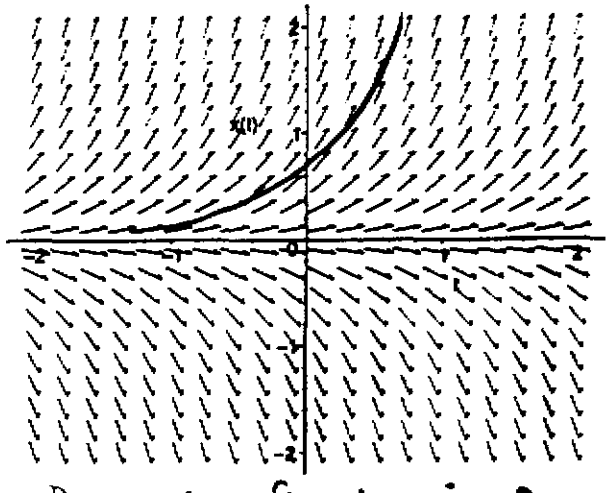
exact: $\frac{dx}{dt} = 2xt \Rightarrow \int \frac{dx}{x} = \int 2t\,dt = t^2 + c$

$\Rightarrow x(t) = x_0 e^{t^2}$)

Autonomous

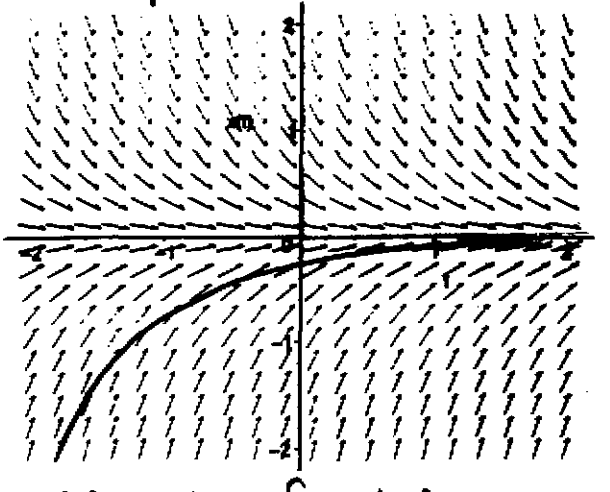
$\dot{x} = f(x)$ (first-order dynamical system)

- direction field independent of t



Direction field $\dot{x} = 2x$

$\Rightarrow x(t) = x_0 e^{2t}$



Direction field $\dot{x} = -2x$

$\Rightarrow x(t) = x_0 e^{-2t}$

Flows on the Line

$n = 1$

one-dimensional dynamical systems

Geometric thinking vs analytical solutions

eg $\dot{x} = \sin x$ Note $-1 \le \dot{x} \le 1$

Separable: $\int \frac{dx}{\sin x} = \int dt \Rightarrow t = \int \operatorname{cosec} x \, dx + c$

$x \neq n\pi$

$= -\ln|\operatorname{cosec} x + \cot x| + c$

Initial condition $x(0) = x_0$

$$\Rightarrow t = \ln\left|\frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right|$$

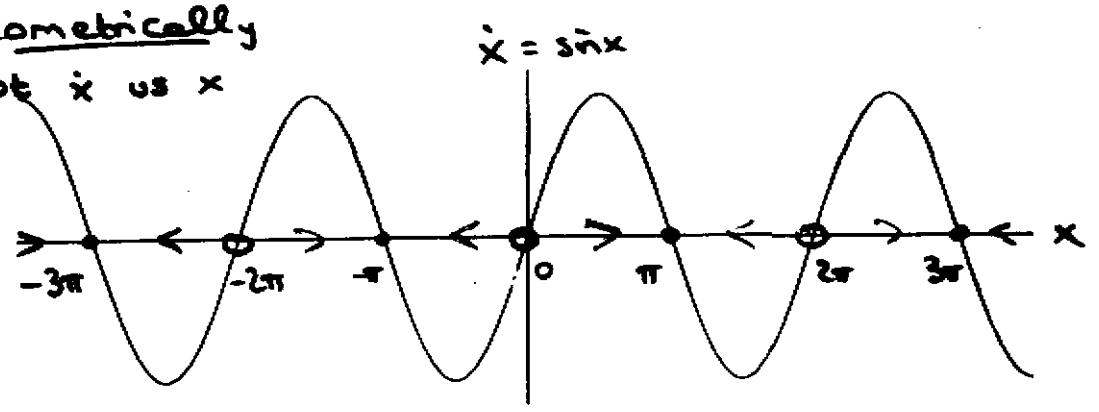
– difficult to solve for $x(t)$, to interpret.

Questions

eg 1. Given $x_0 = \pi/4$, what is qualitative behaviour, $x(t)$ as $t \to \infty$?

2. Describe the behaviour qualitatively for any x_0.

Geometrically plot $\dot{x}$ vs x



$\dot{x} = f(x) = \sin x$

vector field or flow

When $\dot{x} = 0$, no flow – fixed point

$\dot{x} > 0$ – flow to right $\rightarrow$

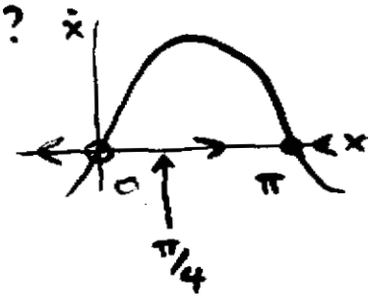
$\dot{x} < 0$ – flow to left $\leftarrow$

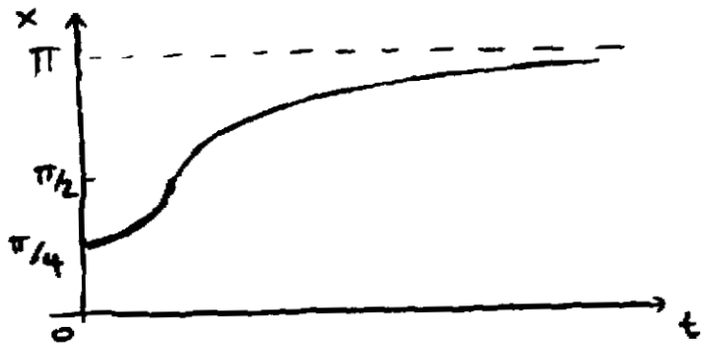
o unstable fixed pt (repeller, source)

• stable fixed pt (attractor, sink)

Answers to questions.

1. Given $x_0 = \pi/4$, qualitative behaviour?





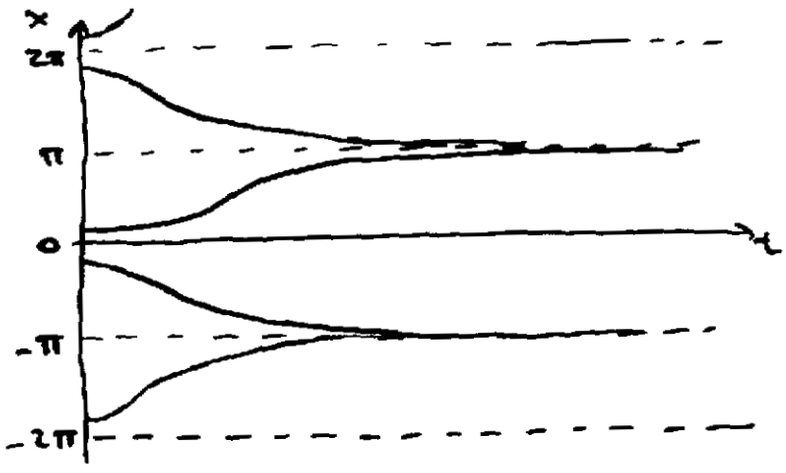
- picture gives qualitative behaviour, but not, say $x(2)$, or time when $x = 0.9\pi$, ...

2. Behaviour for general x_0?

- if $\dot{x} > 0$ initially, solution increases (goes to right) and asymptotically approaches the nearest stable fixed point. if $x_0 \in (2k\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$

- if $\dot{x} < 0$ initially, solution decreases (moves to left) and approaches an attractor

- if $\dot{x} = 0$ $(x_0 = k\pi,\ k \in \mathbb{Z})$
  x remains constant



Autonomous systems

$$\dot{x} = f(x)$$

- if $x(t)$ is a solution, so is $x(t - t_0)$ for any constant t_0 - behaviour depends only on x, not explicitly on t

(so without loss of generality, assume $t_0 = 0$)
- so consider $\dot{x}$ as function of x

<u>Interpretation</u>:

"phase point" in a fluid, local velocity $f(x)$
starting at x_0 at $t=0$

Watch how phase point is carried by the <u>flow</u>
- it moves according to $x(t)$: $\frac{dx}{dt} = f(x(t))$

Trajectory $x(t)$

Set of all qualitatively different trajectories:
Phase portrait.

Emphasize dependence of flow on initial condition
- write $\varphi_t(x_0)$

for the trajectory $x(t)$ through x_0. $(x(0) = x_0)$

$\dot{x} = f(x)$

$f(x) > 0 \Rightarrow x$ is increasing (flow to right)

$f(x) < 0 \Rightarrow x$ is decreasing (flow to left)

$f(x) = 0 \Rightarrow x$ is constant

Fixed points and Stability

$f(x^*) = 0$: x^* is a critical point of the flow / vector field.

Then

$x(t_0) = x^* \Rightarrow x(t) = x^*$ for all t (assuming uniqueness - later)

- a constant solution (fixed point, equilibrium point, steady state, ...)

- The fixed point x^* is <u>stable</u> sometimes: asymptotically stable if all sufficiently small disturbances decay in time (all nearby trajectories approach the fixed point)

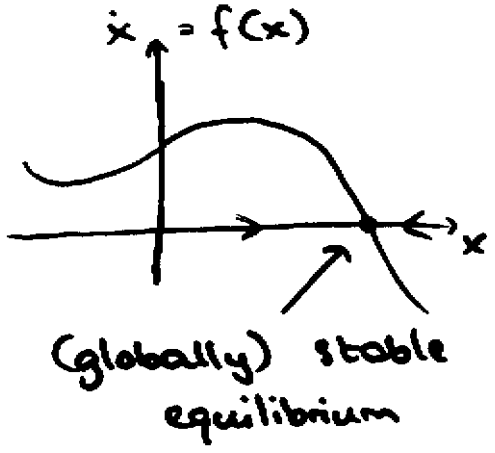
$$\left[\exists\, \delta > 0 \text{ s.t. } \forall x_0 \text{ with } |x_0 - x^*| < \delta, \quad \lim_{t\to\infty} |\underbrace{x(t)}_{\varphi_t(x_0)} - x^*| = 0 \right]$$

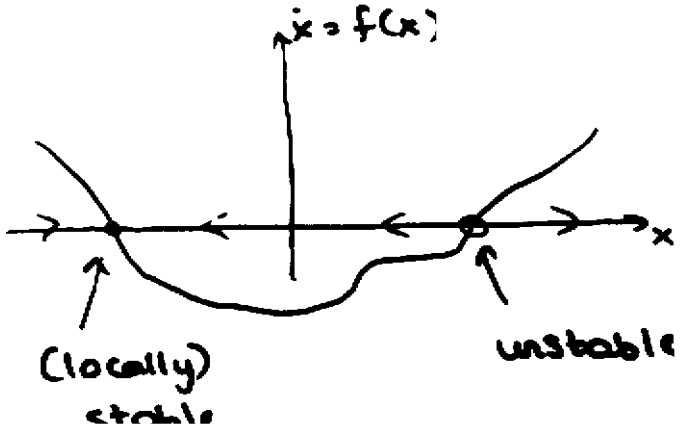
"there exists" "such that" "for all"

- The fixed point x^* is <u>unstable</u> if (some) arbitrarily small perturbations grow in time (nearby trajectories move away)

- The fixed point x^* is <u>neutrally stable</u> sometimes: stable if all trajectories that start sufficiently nearby stay close to the fixed point

$$\left[\forall \varepsilon > 0 \ \exists\, \delta > 0 \text{ s.t. } |x_0 - x^*| < \delta \Rightarrow |x(t) - x^*| < \varepsilon \text{ for } t > 0 \right]$$





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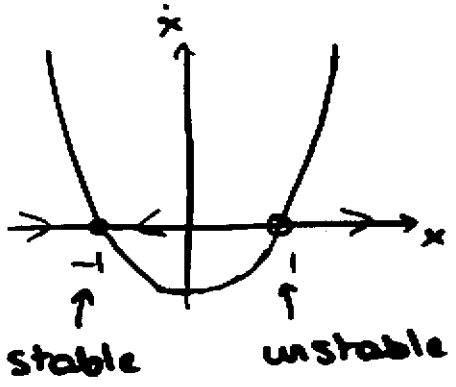
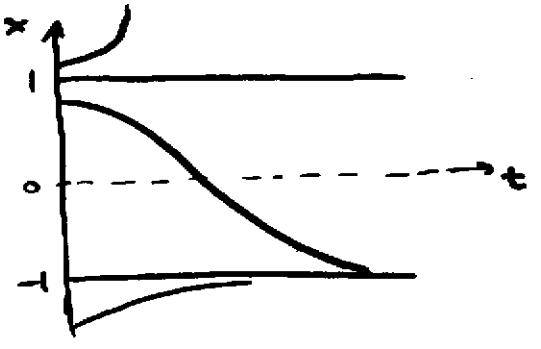
eg $\dot{x} = x^2 - 1$

fixed points: $x^2 - 1 = 0$
$\Rightarrow x = \pm 1$



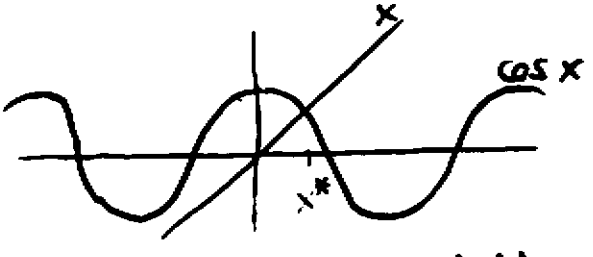
Phase portrait

Solution curves



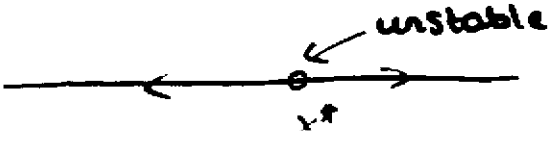
eg $\dot{x} = x - \cos x$

fixed point: $x^* = \cos x^*$ (≈ 0.74)

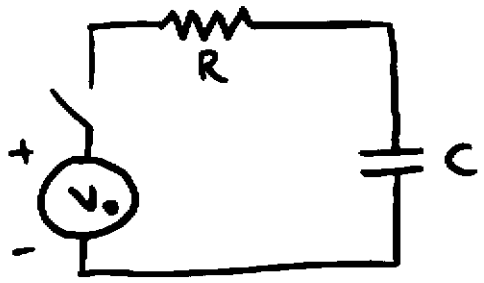


$x > x^* \Rightarrow \dot{x} > 0$

$x < x^* \Rightarrow \dot{x} < 0$



eg RC circuit



$Q(t)$ - charge on capacitor

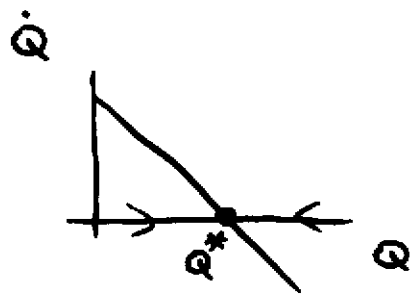
What is $Q(t)$ as $t \to \infty$?

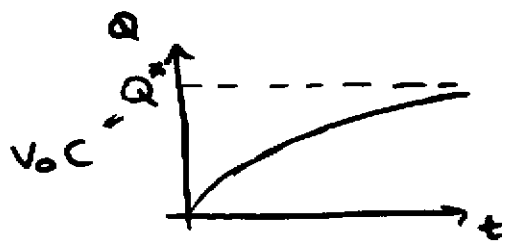
Voltage balance: $V_0 = IR + Q/C$

$I = \dot{Q}$

$$\Rightarrow \dot{Q} = \underbrace{\frac{V_0}{R} - \frac{Q}{RC}}_{f(Q)}$$

fixed point: $f(Q) = 0$ when $Q^* = V_0 C$





so $Q(t) \to Q^* = V_0 C$ as $t \to \infty$

(for any Q_0)

Population Growth

$N(t)$ - population at time t

$$\dot{N} = \beta N - \delta N = \underbrace{(\beta - \delta)}_{r} N$$

birth rate (births per unit population per unit time)

death rate

growth rate

$$\dot{N} = rN \Rightarrow N(t) = N_0 e^{rt}$$

exponential growth, characteristic time $\tau = 1/r$

$\dot{N} = rN$, $r > 0$ constant - predicts unbounded growth

Better:

$r = \dot{N}/N$ decreases as N increases,

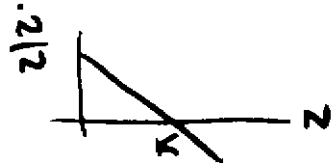
relative growth rate

due to $\begin{cases} \text{overcrowding} \\ \text{competition} \\ \text{limited resources} \end{cases}$

Carrying capacity K : $\dot{N} < 0$ if $N > K$

Simplest model : $\dot{N}/N$ decreases linearly with N :

$$\Rightarrow \frac{\dot{N}}{N} = r\left(1 - \frac{N}{K}\right)$$



$$\Rightarrow \dot{N} = rN\left(1 - \frac{N}{K}\right)$$

logistic equation

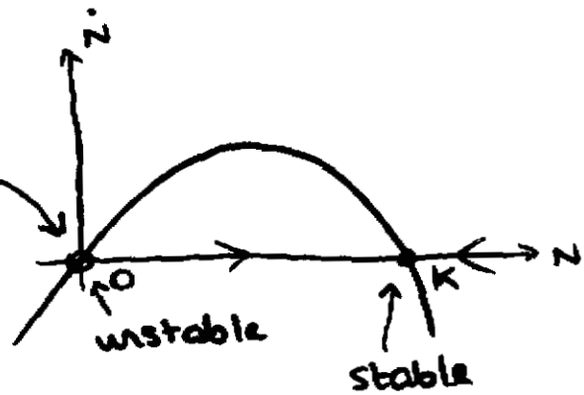
(Verhulst 1838)

$(= aN - bN^2)$

Logistic equation $\dot{N} = rN\left(1 - \frac{N}{K}\right)$

Fixed points: $\dot{N} = 0 \Rightarrow N^* = 0$ or $N^* = K$

slope r : exponential growth for small N



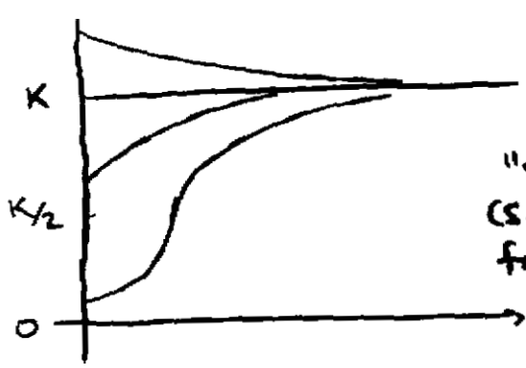
$N_0 > 0 \Rightarrow N(t) \to K$ as $t \to \infty$

population approaches carrying capacity

$N_0 = 0 \Rightarrow N(t) = 0$ for all t

$\dot{N}$ increases (accelerated growth) for $N < K/2$

$\dot{N}$ decreases for $N > K/2$



"sigmoid" (S-shaped) for $N_0 < K/2$

Logistic model - a "metaphor"

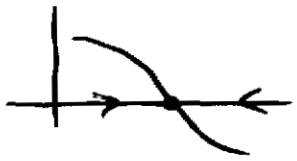
- okay for bacteria, yeast, .. (simple life cycle)

- not so good for fruit flies, flour beetles, ... (complex life cycles)

Linear Stability Analysis

- gives quantitative information about rate of growth / decay at fixed point, analytical stability information

Stable

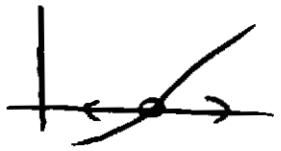


$f(x) > 0$ for $x < x^*$

$f(x) < 0$ for $x > x^*$

$\Rightarrow \quad f'(x^*) < 0$

Unstable



$f(x) < 0$ for $x < x^*$

$f(x) > 0$ for $x > x^*$

$\Rightarrow \quad f'(x^*) > 0$

---

x^* : fixed point $\qquad f(x^*) = 0$

Growth/decay of small perturbations ?

$$\eta(t) = x(t) - x^* \qquad \text{Perturbation}$$

$$\Rightarrow \quad \dot{\eta} = \frac{d}{dt}(x(t) - x^*) = \dot{x} = f(x) = f(x^* + \eta)$$

Taylor expand near x^*

$$= \cancel{f(x^*)}^{=0} + f'(x^*)\eta + \tfrac{1}{2} f''(x^*)\eta^2 + \dots$$

$$\dot{\eta} = f'(x^*)\eta + \mathcal{O}(\eta^2) \leftarrow \tfrac{1}{2} f''(x^*)\eta^2 + \dots$$

$f'(x^*) \neq 0$: as $\eta \to 0$, $\mathcal{O}(\eta^2)$ terms negligible relative to $f'(x^*)\eta$ - neglect higher-order terms

Linearization near x^*: $\boxed{\dot{\eta} \simeq f'(x^*)\eta}$

["Big-Oh" notation: $\mathcal{O}(\eta^2)$ - quadratic and higher-order terms in η as $\eta \to 0$. We say $g(\eta)$ is $\mathcal{O}(h(\eta))$ as $\eta \to 0$ if $\exists\, K < \infty$ s.t. $\lim_{\eta \to 0} |g(\eta)/h(\eta)| < K$]

Linearization:

about the fixed point x^* $\quad \dot{\eta} \approx f'(x^*)\,\eta \quad \Rightarrow \eta(t) \approx \eta_0 e^{f'(x^*)t}$

$f'(x^*) > 0$: exponential growth unstable

$f'(x^*) < 0$: decay stable

- the sign of $f'(x^*)$ determines stability.

- the magnitude determines time scales ("how stable/unstable")

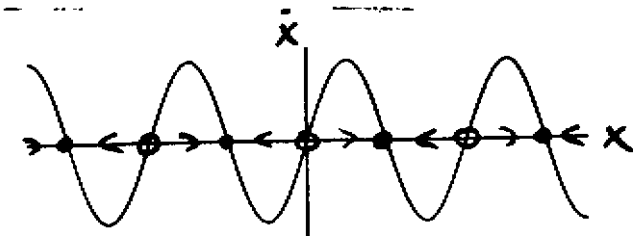
Characteristic time scale for evolution near x^*:

$$\tau = 1/|f'(x^*)|$$

Definition: A fixed point is hyperbolic if $f'(x^*) \neq 0$.

(x^* is a simple zero)

eg $\dot{x} = \sin x$



Fixed points: $x^* = k\pi, \quad k \in \mathbb{Z}$

Stability: $f'(x^*) = \cos x\big|_{x=k\pi} = (-1)^k = \begin{cases} +1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases}$

$x^* = k\pi$: k even - unstable
k odd - stable

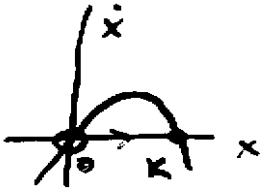
eg Logistic equation

$$\dot{N} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{f(N)}$$

fixed points.
$N^* = 0, K$

$$f'(N) = r\left(1-\frac{2N}{K}\right)$$

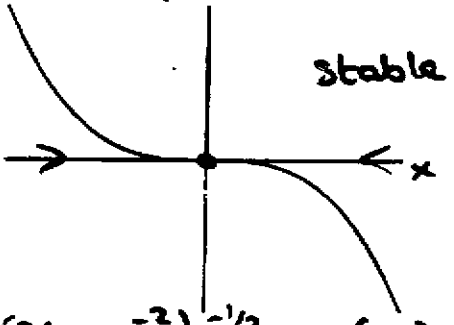
$\Rightarrow$ $f'(0) = r > 0$ unstable
$f'(K) = -r < 0$ stable



$f'(x^*) = 0$: nothing can be said in general

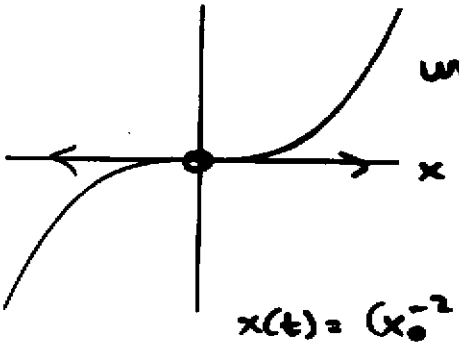
Important examples

$\dot{x} = -x^3$



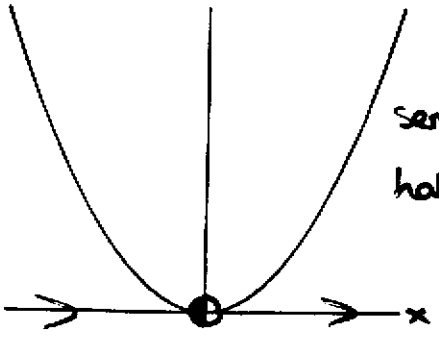
$x(t) = (2t + x_0^{-2})^{-1/2} \cdot \mathrm{sgn}(x_0)$

$\dot{x} = x^3$



$x(t) = (x_0^{-2} - 2t)^{-1/2} (\mathrm{sgn}\, x_0)$

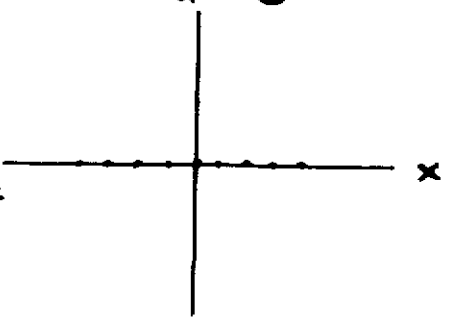
$\dot{x} = x^2$



semistable/
half-stable

$x(t) = \dfrac{1}{x_0^{-1} - t}$

$\dot{x} = 0$



$x(t) = \text{const}$

Existence and Uniqueness

eg $\dot{x} = x^{1/3}$, $x(0) = 0$ — $f'(x) = \frac{1}{3} x^{-2/3}$

$x(t) \equiv 0$ is a solution.

Separate: $\int \frac{dx}{x^{1/3}} = \int dt \Rightarrow \frac{3}{2} x^{2/3} = t + c$

$x(0) = 0 \Rightarrow c = 0$

so $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ is a solution;

also $x(t) = -\left(\frac{2}{3}t\right)^{3/2}$

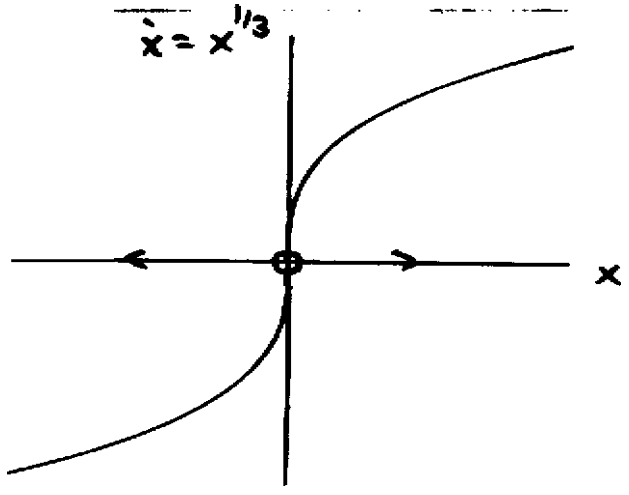
Infinitely many solutions! $x(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ \pm\left(\frac{2}{3}(t - t_0)\right)^{3/2} & t \ge t_0 \end{cases}$

is a C^1 solution for any $t_0 \ge 0$.

Difficulty?

The fixed point $x^* = 0$ is very unstable:

$f'(0)$ is infinite



Existence - Uniqueness Theorem:

Initial-value problem (IVP) $\dot{x} = f(x)$, $x(0) = x_0$

? If $f(x)$ is continuous in an open interval R of $\mathbb{R}$, containing x_0, then the IVP has a solution in some interval $I = (-\tau, \tau)$, for some $\tau = \tau(x_0) > 0$.

! If in addition $f'(x)$ is continuous in R, the solution is unique.

Flow "backwards in time" : solution $x(t)$ for $t < 0$:

We can interpret this as the forward flow induced by $-f$
(same direction field, all arrows reversed) :

$$\frac{dx}{dt} = f(x) \quad : \quad \text{set } t' = -t \qquad t' \geq 0 \Leftrightarrow t \leq 0$$

$$\Rightarrow \frac{dx}{dt'} = -\frac{dx}{dt} = -f(x)$$

eg a trajectory moving away from a fixed point in forward time, approaches it as $t \to -\infty$

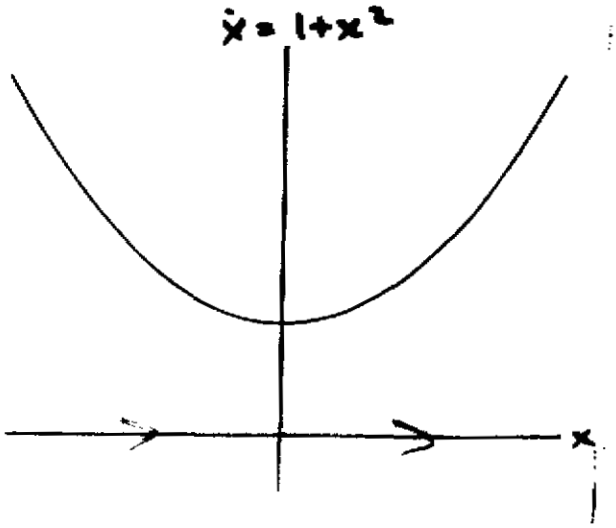
Note: The theorem guarantees local existence and uniqueness, not global
- time interval $(-\tau, \tau)$ depends on x_0.
(global existence : $\tau = \infty$)

eg $\dot{x} = 1 + x^2$, $x(0) = 0$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\Rightarrow \arctan x = t + c^{=0}$$

$$\Rightarrow x(t) = \tan t$$



Solution exists for $-\pi/2 < t < \pi/2$

Finite-time singularity / blow-up of solutions at $t = \pi/2$
due to extremely fast growth
(a nonlinear effect)

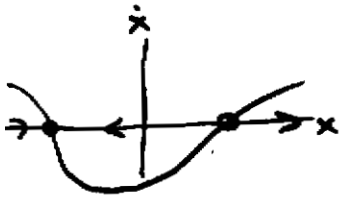
Impossibility of Oscillations

Flow on a line: $\dot{x} = f(\underline{x})$

- the direction of the flow never reverses

- so trajectories must increase or decrease monotonically, or stay constant

Possibilities for flow on a line:

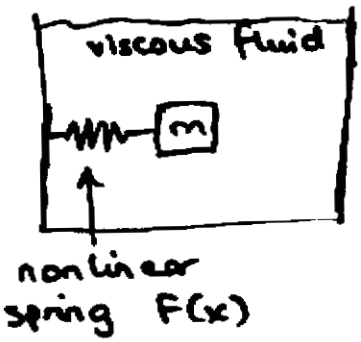


- trajectory approaches a fixed point
- trajectory diverges to $\pm\infty$

- no periodic / oscillatory solutions to $\dot{x} = f(x)$
  - a topological result

Note: different for flow on a circle – later.

Mechanical analogue: overdamped system



$$m\ddot{x} + b\dot{x} = F(x)$$

$m\ddot{x} \ll b\dot{x}$ viscous damping strong – neglect inertia

$$\Rightarrow \dot{x} \propto \frac{1}{b}F(x) = f(x)$$

overdamped, no oscillations

exponential decay to equilibrium

(Note: neglect $m\ddot{x}$ only after rapid initial transient)

Potentials

$\dot{x} = f(x)$

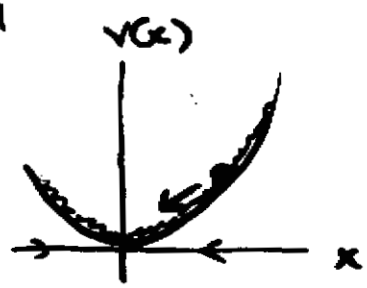
visualize flow using potential energy

Define potential $V(x)$ by $f(x) = -\dfrac{dV}{dx}$

$$\Rightarrow \dot{x} = -\frac{dV}{dx}$$

Motion of particle: (heavily damped - in "goo")

down the slope / potential well



Change of $V = V(x(t))$ along trajectory of particle:

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}\left(-\frac{dV}{dx}\right) = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

V decreases with the flow (monotonically)

no oscillations

Critical points of V, $\frac{dV}{dx} = 0$: fixed points

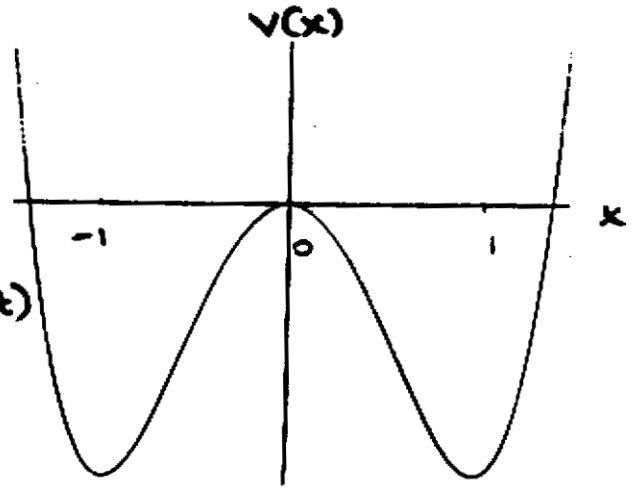
Local $\left\{\begin{matrix}\text{minima}\\ \text{maxima}\end{matrix}\right\}$ of V : $\left\{\begin{matrix}\text{stable}\\ \text{unstable}\end{matrix}\right\}$ equilibria

eg $\dot{x} = x - x^3$ critical points $\pm 1, 0$

$$\frac{dV}{dx} = -(x - x^3)$$

$$\Rightarrow V(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + \underbrace{C}_{=0}$$

(potential defined up to arbitrary constant)



Double-well potential:

bistable

(spontaneous symmetry breaking...)

Numerical integration of $\dot{x} = f(x,t)$

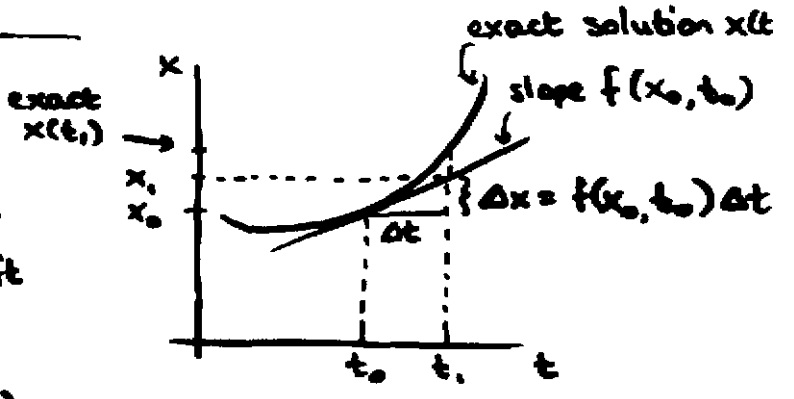
Problem: Given $\dot{x} = f(x,t)$, $x(t_0) = x_0$

$t_n = t_0 + n\,\Delta t$, $n = 0, \ldots, N$

- find approximations x_n to $x(t_n)$
- estimate $x(t_N)$ given $x(t_0)$

Single-step methods: given x_n, estimate x_{n+1}

Euler's method:



Tangent line approximation: estimate derivative at left end of interval, to get

$$x_1 = x_0 + \Delta t\, f(x_0, t_0)$$

Equivalent: approximation of derivative

$$\dot{x} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} = f(x(t),t) \Rightarrow x(t+\Delta t) \approx x(t) + \Delta t\, f(x(t),t))$$

$$\boxed{x_{n+1} = x_n + \Delta t\, f(x_n, t_n)}$$

$n = 1, 2, 3, \ldots$

forward (explicit) Euler

Taylor series:

$$x(t+\Delta t) = x(t) + \Delta t\, \dot{x}(t) + \tfrac{1}{2}(\Delta t)^2\, \ddot{x}(t) + \ldots$$
$$= x(t) + \Delta t\, f(x(t),t) + \mathcal{O}(\Delta t^2)$$

Local truncation error $O(\Delta t^2)$ (per time step)

Number of steps from $t_0 = a$ to $t_N = b$ is $O(1/\Delta t)$

$\Rightarrow$ Global (cumulative) truncation error is $\underline{O(\Delta t)}$

Euler's method is first-order.

Problems: • need small stepsize (first-order accurate)

with forward Euler] • <u>unstable</u> if Δt is too large

Error: define $e_n = \underset{\text{exact}}{x(t_n)} - \underset{\text{approximation}}{x_n}$

$$e_{n+1} = x(t_{n+1}) - x_{n+1} = x(t_n + \Delta t) - x_{n+1}$$

$$= x(t_n) + \Delta t\, \dot{x}(t_n) + O(\Delta t^2) - [x_n + \Delta t\, f(x_n, t_n)]$$

$$= x(t_n) - x_n + \Delta t\,[f(x_n + e_n, t_n) - f(x_n, t_n)] + O(\Delta t^2)$$

$$= e_n + \Delta t\left[e_n \frac{\partial f}{\partial x}(x_n, t_n) + O(e_n^2)\right] + O(\Delta t^2)$$

$$= \left[1 + \Delta t \frac{\partial f}{\partial x}\right] e_n + \ldots$$

Example: $\dot{x} = -ax$, $a > 0$

$\Rightarrow x(t) = x_0 e^{-at}$: $x(t) \to 0$ as $t \to \infty$

Stable fixed point at $x = a$, decay time $\tau = 1/a$

but

$$e_{n+1} \approx (1 - a\Delta t)\, e_n \quad , \quad |e_{n+1}| \approx |1 - a\Delta t|\,|e_n|$$

$|e_n| \to 0$ if $|1 - a\Delta t| < 1$ ie $\Delta t < 2/a$ stable

$|e_n| \to \infty$ if $|1 - a\Delta t| > 1$ ie $\Delta t > 2/a$ <u>unstable</u> iteration

Stepsize restriction!

(similarly, if x^* is a stable fixed point, $f'(x^*) < 0$, $\tau = \frac{1}{|f'(x^*)|}$

need $\Delta t \lesssim 2/|f'(x^*)|$ for stability of forward Euler)

Note: larger $a \Rightarrow x = 0$ is more stable (rapid decay)

but need a smaller step size ...

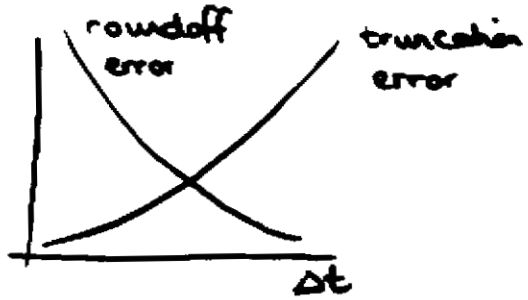
(in higher dimensions, <u>stiffness</u> in problems with widely varying time constants - difficulty for explicit methods)

Smaller step size ?

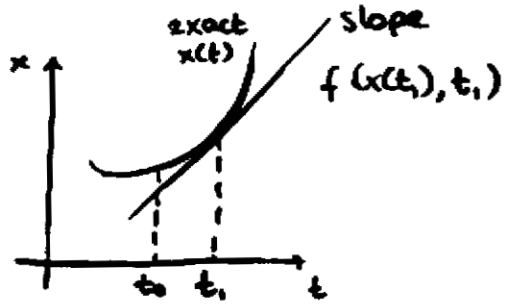
Problems - • computation time

• roundoff error due to finite precision arithmetic

(error per step $\begin{cases} \sim 10^{-7}-10^{-8} & \text{single precision} \\ \sim 10^{-16} & \text{double precision} \end{cases}$)



Backward Euler method



- approximate derivative at right end of interval

$$\frac{x(t+\Delta t)-x(t)}{\Delta t} \approx \dot{x}(t+\Delta t) = f(x(t+\Delta t), t+\Delta t)$$

$$\Rightarrow \quad \boxed{x_{n+1} = x_n + \Delta t\, f(x_{n+1}, t_{n+1})}$$

<u>backward</u> <u>(implicit) Euler</u>

- still first-order
- have to solve implicit (nonlinear) equation for x_{n+1}
- but stable!

as before

$$e_{n+1} = e_n + \Delta t \frac{\partial f}{\partial x} e_{n+1} + \cdots \quad \Rightarrow e_{n+1} \approx \frac{e_n}{1-\Delta t \frac{\partial f}{\partial x}}$$

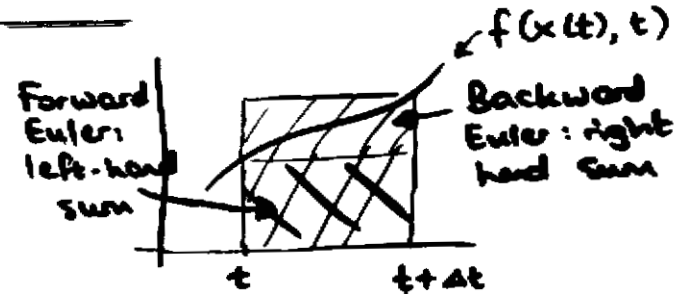
eg $\dot{x} = -ax$, $a>0$: $\quad e_{n+1} = \frac{1}{1+a\Delta t} e_n$

$\Rightarrow e_n \to 0$: stable for all Δt.

Analogy: numerical integration

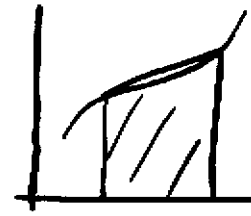
$$\dot{x} = f(x,t)$$

$$\Rightarrow x(t+\Delta t) = x(t) + \int_t^{t+\Delta t} f(x(s),s)\,ds$$



Improved methods:

- Better approach: estimate
  <u>average</u> derivative



Improved Euler ~ trapezoid rule

Trial (predictor) step (forward Euler)

$$\tilde{x}_{n+1} = x_n + \Delta t\, f(x_n, t_n)$$

Real (corrector) step

$$x_{n+1} = x_n + \tfrac{1}{2}\Delta t\left[f(x_n,t_n) + f(\tilde{x}_{n+1}, t_{n+1})\right]$$

<u>Improved Euler</u> (second order: truncation error $O(\Delta t^2)$)

- Estimate derivative at midpoint

2nd order RK ~ midpoint rule

$$\begin{cases} \tilde{x}_{n+1/2} = x_n + \frac{\Delta t}{2} f(x_n, t_n) \\ x_{n+1} = x_n + \Delta t\, f(\tilde{x}_{n+1/2}, t_n + \frac{1}{2}\Delta t) \end{cases}$$

<u>Second-order Runge-Kutta</u> error $O(\Delta t^2)$

Methods of arbitrarily high order exist
- complicated, many function evaluations.

A good/popular compromise between accuracy and speed/simplicity

<u>Fourth-order Runge-Kutta</u> (~ Simpson's rule)

error $O(\Delta t^4)$

$$\begin{cases} k_1 = f(x_n, t_n) \\ k_2 = f(x_n + \frac{\Delta t}{2} k_1, t_n + \frac{\Delta t}{2}) \\ k_3 = f(x_n + \frac{\Delta t}{2} k_2, t_n + \frac{\Delta t}{2}) \\ k_4 = f(x_n + \Delta t\, k_3, t_n + \Delta t) \\ x_{n+1} = x_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

Further improvements: adaptive methods - stepsize control