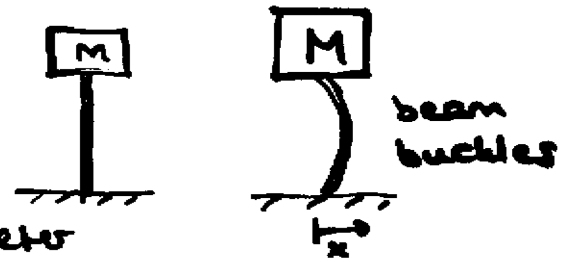
Bifurcations

A **bifurcation** - qualitative change in the character of solutions of a nonlinear system as a parameter is varied

In applications, models have parameters

eg buckling beam



weight m: control parameter
deflection: dynamical variable x

(spontaneous symmetry breaking)

Bifurcations of fixed points:

Flow on a line $\dot{x} = f(x) = f_r(x) = f(x,r)$

r: parameter(s)

Fixed point $x^*(r)$: $f(x^*(r), r) = 0$

Bifurcation point r_c :

A parameter value $r = r_c$ so that an arbitrarily small change in the parameter results in topologically distinct phase portraits and qualitatively different behaviour

- at r_c, system is structurally unstable

Structural stability: all "sufficiently close" systems have the same qualitative behaviour

- need hyperbolic fixed points, $f'(x^*, r) \neq 0$

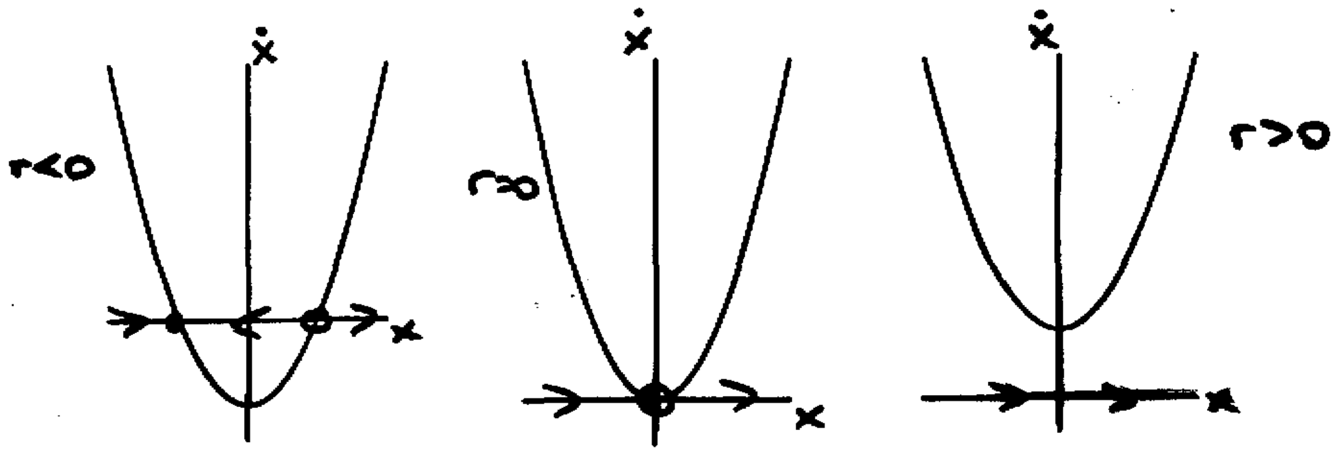
Saddle-Node Bifurcation

- creation and destruction of fixed points

Prototypical example:

$$\dot{x} = r + x^2$$

r: bifurcation parameter



$r<0$: Fixed points $x^* = \pm\sqrt{-r}$

Linear stability analysis: $f'(x^*) = 2x^*$,

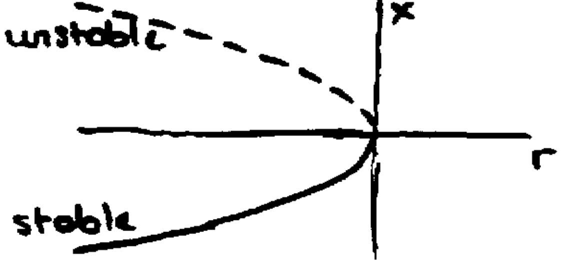
$x^* = -\sqrt{-r}$: $f'(x^*) < 0$ stable

$x^* = +\sqrt{-r}$: $f'(x^*) > 0$ unstable

Bifurcation at $r=0$ (of fixed point at $x=0$)

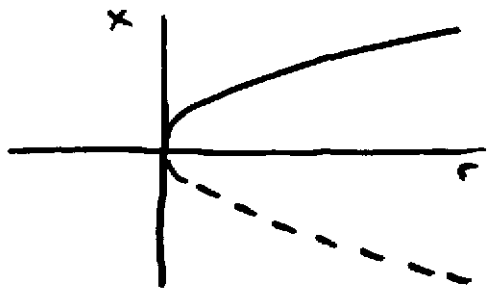
As r increases through 0, two fixed points approach each other, coalesce and vanish

Bifurcation diagram (fixed points as function of bifurcation parameter)



Similarly:

$\dot{x} = r - x^2$ fixed points $x = \pm\sqrt{r}$ for $r \geq 0$



- a pair of fixed points created as r increases through 0.

Bifurcation: split into two branches of fixed points

Names: $\left.\begin{array}{l}\text{Saddle-node} \\ \text{Turning point} \\ \text{Fold} \\ \text{"Blue sky"}\end{array}\right\}$ bifurcation

Note: the "prototypical examples"

$$\dot{x} = r + x^2$$

and

$$\dot{x} = r - x^2$$

are equivalent under the change of variables

$x \to -x, \quad r \to -r$

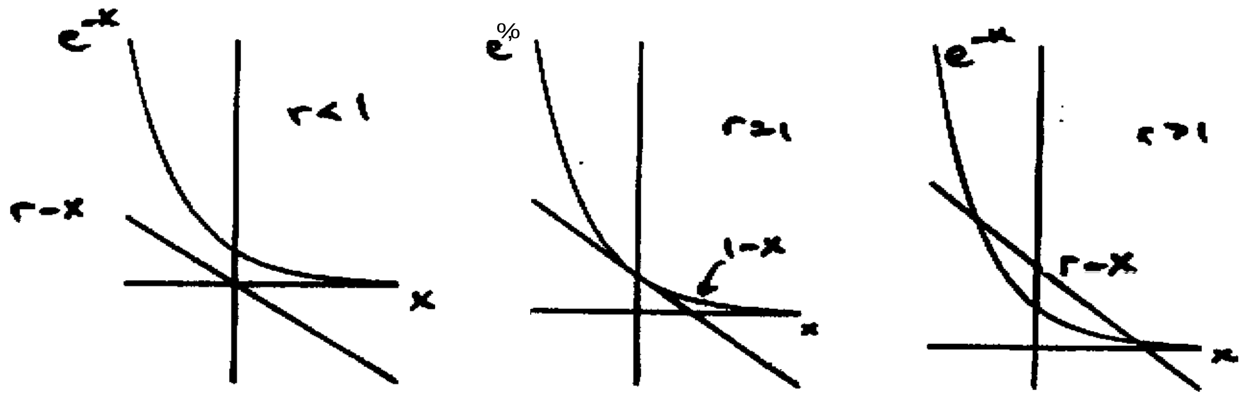
(sufficient to consider $\dot{x} = r - x^2$)

Example: $\dot{x} = r - x - e^{-x}$

- show this undergoes a saddle-node bifurcation; what is r_c?

Fixed points: $r - x^* - e^{-x^*} = 0$ transcendental equation for $x^*(r)$.

Graphical approach: plot $r-x$, e^{-x} simultaneously, look for intercepts.



Bifurcation point (x^*, r_c):

1. $f(x^*, r_c) = 0$ fixed point
2. Curves have same slope ie $-1 = -e^{-x}$

(fixed points coalesce: x^* is double zero)

ie $\left.\frac{\partial f}{\partial x}\right|_{x^*, r_c} = 0$.

From 2: $e^{-x^*} = 1 \Rightarrow x^* = 0$

1. $r_c - x^* = e^{-x^*} \Rightarrow r_c = 1$.

Bifurcation $(x^*, r_c) = (0,1)$

Expand near x^*, r_c:

$$f(x,r) = r - x - e^{-x} = \cancel{1} + (r-1) - \cancel{x}$$
$$- \left(\cancel{1} - \cancel{x} + \tfrac{1}{2}x^2 - \tfrac{1}{3!}x^3 + \dots\right)$$

$$= (r-1) - \tfrac{1}{2}x^2 + O(x^3)$$

$\frac{1}{3!}x^3 - \frac{1}{4x^4} + \dots$

$$\dot{x} = f(x,r) = r - x - e^{-x}$$
$$= (r-1) - \tfrac{1}{2}x^2 + O(x^3)$$

same form as $\dot{X} = R - X^2$

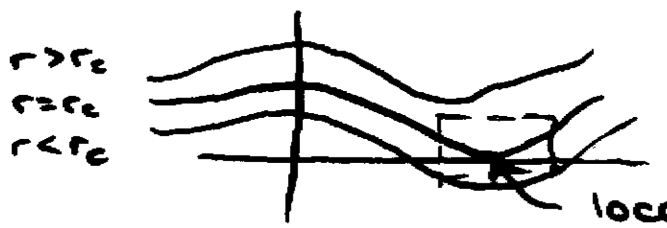
In fact: $X = \tfrac{1}{2}x$, $R = \tfrac{1}{2}(r-1)$
$\Rightarrow x = 2X$, $r = 2R+1$

$$\Rightarrow 2\dot{X} = 2R - \tfrac{1}{2}(2X)^2 + \dots$$
$$\Rightarrow \underbrace{\boxed{\dot{X} = R - X^2}}_{\text{normal form for saddle-node bifurcation}} + \dots$$

("generic")

Normal form: representative of all saddle-node bifurcations

(close to bifurcation point, dynamics look like $\dot{x} = r \pm x^2$)



Algebraically: Taylor expand near bifurcation point x^*, r_c

$$\dot{x} = f(x,r) = f(x^*, r_c) + (x - x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)}$$
$$+ \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \dots$$

At bifurcation point: $f(x^*, r_c) = 0$ (x^* is a fixed point)

$\frac{\partial f}{\partial x}(x^*, r_c) = 0$ (tangency condition: x^* is not simple zero, $f(x)$ is tangent to 0 at x^*.

Bifurcation of fixed point at x^*, r_c: $\quad f(x^*, r_c) = 0$

$$\frac{\partial f}{\partial x}(x^*, r_c) = 0$$

$$\Rightarrow \quad \dot{x} = (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \frac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \ldots$$

Also, generically (in absence of special symmetries ...)

expect
$$\frac{\partial f}{\partial r}(x^*, r_c) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(x^*, r_c) \neq 0$$

locally parabolic

Let $a = \frac{\partial f}{\partial r}(x^*, r_c)$, $b = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x^*, r_c)$:

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \ldots$$

Rescale:

$$\left.\begin{aligned} X &= -b(x - x^*) \\ R &= -ab(r - r_c) \end{aligned}\right\} \Rightarrow \boxed{\dot{X} = R - X^2} + \underbrace{\ldots}_{O(X^3)}$$

normal form for saddle-node bif^n.

Normal form theory

- can systematically eliminate higher-order terms by near-identity coordinate changes

eg. $\dot{x} = r - x^2 + ax^3 + O(x^4)$, $\quad |x| \ll 1$, $r \neq 0$

- define $x = X + bX^4$ $\quad$ - choose b to eliminate cubic term - get $\dot{X} = r - X^2 + O(X^4)$

$$\Rightarrow X = x - bX^4 = x - b(x - bX^4)^4 = x - bx^4 + O(x^7)$$

$$\begin{aligned} \dot{X} &= (1 - 4bx^3 + \ldots)\dot{x} = (1 - 4bx^3 + \ldots)(r - x^2 + ax^3 + \ldots) \\ &= (1 - 4b(X + bX^4)^3 + \ldots)(r - (X + bX^4)^2 + a(X + bX^4)^3 + \ldots) \\ &= r - X^2 + (\underbrace{a - 4br}_{0})X^3 + O(X^4) \end{aligned}$$

Choose $\boxed{b = a/4r}$

Summary: conditions for saddle-node bifurcation of $\dot{x} = f(x,r)$:

At (x^*, r_c)

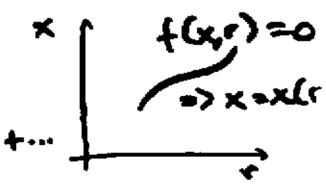
$$f = 0, \quad \frac{\partial f}{\partial x} = 0; \quad \frac{\partial f}{\partial r} \neq 0, \quad \frac{\partial^2 f}{\partial x^2} \neq 0.$$

An alternative perspective: (continuous) dependence of zeros on r

Equilibria of $\dot{x} = f(x,r)$ $\Leftrightarrow$ zeros of f : $f(x,r) = 0$.

Suppose x_0 is a fixed point for a given r_0.
Then the equation $f(x^*, r) = 0$ defines a unique function $x^*(r)$ of r implicitly in some neighborhood containing r_0 (a continuous, single-valued function, satisfying $f(x^*(r), r) = 0$) provided $\frac{\partial f}{\partial x} \neq 0$.

$\Rightarrow$ bifurcations of fixed points require $\underline{f = 0}$ and $\underline{\frac{\partial f}{\partial x} = 0}$.



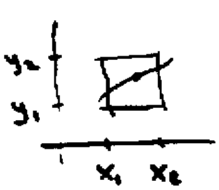
Idea:

$$\underset{\text{want } = 0}{f(x^*, r)} = \underbrace{f(x_0, r_0)}_{=0} + (x^* - x_0) \frac{\partial f}{\partial x}\bigg|_{(x_0, r_0)} + (r - r_0) \frac{\partial f}{\partial r}\bigg|_{(x_0, r_0)} + \cdots$$

- can solve for $x^* = x^*(r)$ provided $\frac{\partial f}{\partial x} \neq 0$.

<u>Implicit Function Theorem</u>:

Assume $g(x_0, y_0) = 0$, $\frac{\partial g}{\partial y}(x_0, y_0) \neq 0$, and that $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ are continuous. Then there exists a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ containing (x_0, y_0) s.t. for each x, $x_1 \le x \le x_2$, the equation $g(x,y) = 0$ determines a unique $y = h(x)$ with $y_1 \le y \le y_2$ satisfying $y_0 = h(x_0)$ and $g(x, h(x)) = 0$ for $x \in [x_1, x_2]$, where $h(x)$ is continuous and has continuous derivative.



Transcritical Bifurcation

Often a fixed point exists for all values of the parameter r

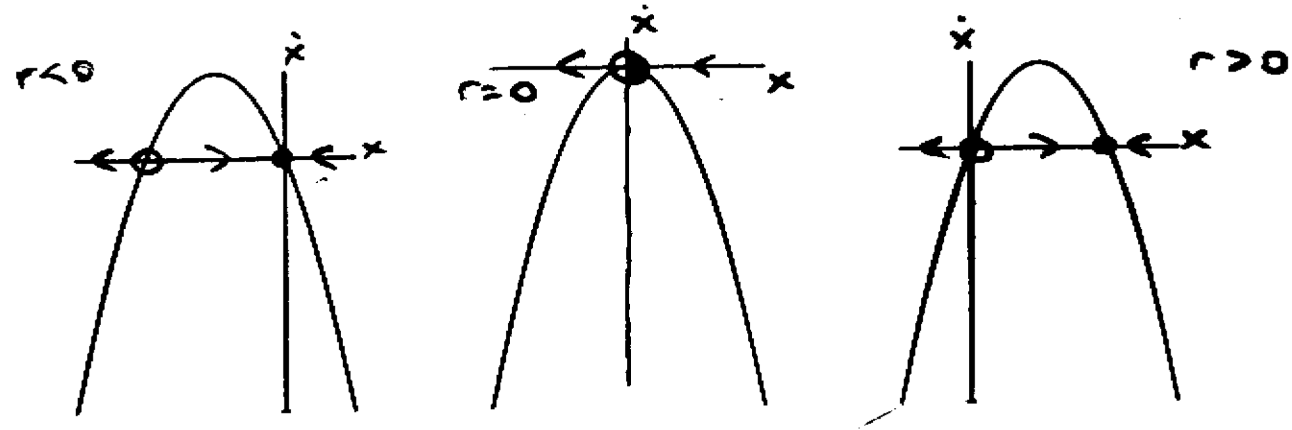
$\dot{x} = f(x,r)$: fixed point $x^*(r)$, all r

$f(x^*,r) = 0 \quad \left(\Rightarrow \frac{\partial f}{\partial r}(x^*,r) = 0 \right)$

eg logistic model $\dot{N} = rN(1 - N/K)$: $N=0$ is always a fixed point

– but change of stability

Normal form: $\boxed{\dot{x} = rx - x^2}$ fixed points $x^* = 0, r$

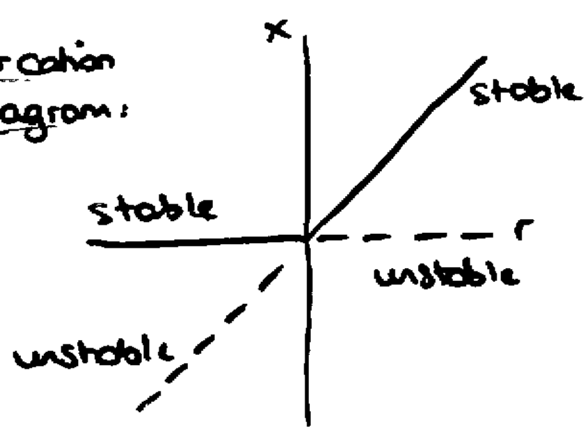


$x^* = 0$ stable
$x^* = r$ unstable

$x^* = 0$ half-stable

$x^* = 0$ unstable
$x^* = r$ stable

Bifurcation diagram:



Exchange of stabilities

Normal form: $\boxed{\dot{x} = rx - x^2} + \ldots$

Local analysis near (x^*, r_c):

$$\dot{x} = f(x,r) = \underbrace{f(x^*,r_c)}_{0} + (x-x^*)\left.\frac{\partial f}{\partial x}\right|_{0} + (r-r_c)\left.\frac{\partial f}{\partial r}\right|_{0}$$

$$+ \underbrace{\tfrac{1}{2}(x-x^*)^2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{\neq 0} + \underbrace{(x-x^*)(r-r_c)}\left.\frac{\partial^2 f}{\partial x \partial r}\right|_{\neq 0} + \tfrac{1}{2}(r-r_c)^2\left.\frac{\partial^2 f}{\partial r^2}\right|_{0} + \ldots$$

eg $\dot{x} = r\ln x + x - 1$ $\quad x^* = 1$ is a fixed point for all r

Analysis near $x=1$:

$x = 1+u$ $\quad (|u| \ll 1)$

$$\Rightarrow \dot{u} = \dot{x} = r\ln x + x - 1 = r\ln(1+u) + (1+u) - 1$$
$$= r\left[u - \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 - \tfrac{1}{4}u^4 + \dots\right] + u$$
$$= (r+1)u - \tfrac{1}{2}ru^2 + O(u^3)$$

Transcritical bifurcation at $r_c = -1$.

Put into normal form - need $\dot{X} = RX - X^2 + \dots$ no R dependence

- rescale u to remove coefficient of u^2:

$u = av$
$$\Rightarrow \dot{v} = \frac{1}{a}\dot{u} = \frac{1}{a}\left[(r+1)(av) - \tfrac{1}{2}r(av)^2 + O(v^3)\right]$$
$$= (r+1)v - \tfrac{1}{2}arv^2 + O(v^3)$$

- set $\frac{1}{2}ar = 1 \Rightarrow a = 2/r$

Then $\dot{v} = (r+1)v - v^2 + O(v^3)$

Now let $R = r+1$, $X = v$ $\quad \left(\Rightarrow X = \frac{1}{a}u = \frac{r}{2}(x-1)\right)$

$$\Rightarrow \boxed{\dot{X} = RX - X^2} + O(X^3)$$

normal form

For $R \neq 0$, we can successively eliminate higher-order terms by near identity changes of variables $y = X + bX^k$

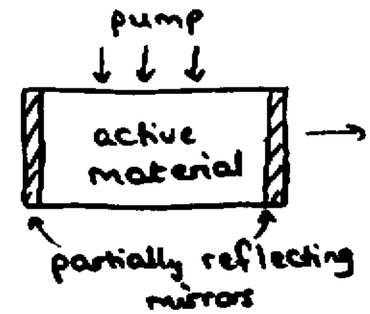
Summary: Transcritical bifurcation of $\dot{x} = f(x,r)$:

At (x^*, r_c)

$$f = 0,\ \frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial r} = 0,\ \frac{\partial^2 f}{\partial x^2} \neq 0,\ \frac{\partial^2 f}{\partial x \partial r} \neq 0$$

Laser Threshold

Solid-state laser



Atoms emit photons via
- spontaneous emission (incoherent)
- stimulated emission (coherent)

Beyond a certain pumping threshold, atoms oscillate in phase – laser. (self-organizing)

Simplified model: $n(t)$ – no. of coherent photons in laser field
$N(t)$ – no. of excited atoms

$$\dot{n} = \text{gain} - \text{loss} = \underset{\text{gain coefficient}}{G}\, nN - \underset{\text{loss coefficient}}{k}\, n$$

G: gain coefficient

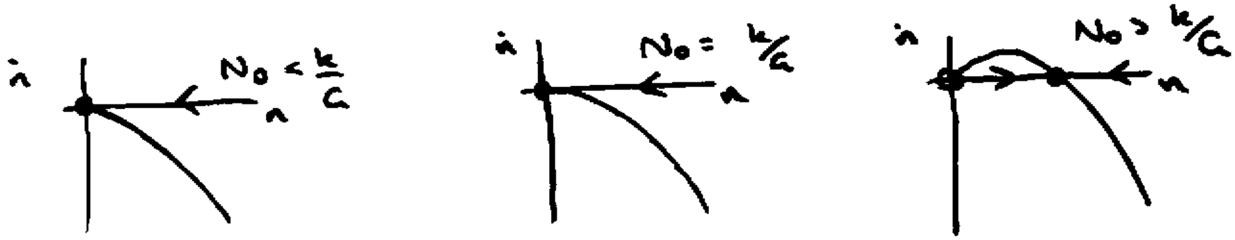
k: loss coefficient: rate of escape of photons through mirror

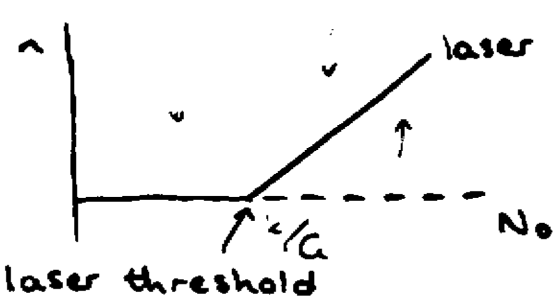
$$N(t) = N_0 - \alpha n(t)$$

N_0: pump

α: rate of stimulated emission: atoms drop to ground state

$$\Rightarrow \dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - G\alpha n^2$$

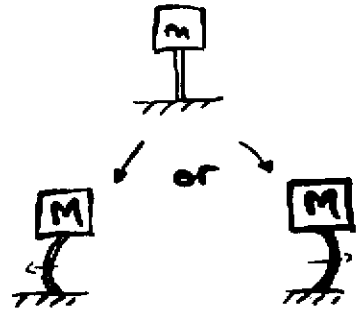




$n^* = 0$ undergoes a transcritical bifurcation at $N_0 = \frac{G}{k}$

Pitchfork Bifurcation

Bifurcation in the presence of symmetry (eg left-right)



$f(-x,r) = -f(x,r)$ (odd)

$\Rightarrow$ $\dot{x} = f(x,r)$ is invariant under $x \to -x$.

Consequences: • $x^* = 0$ is always a fixed point $f(0,r)=0$, all r

$$\Rightarrow \left.\frac{\partial^k f}{\partial r^k}\right|_{(x^*,r_c)} = 0, \quad k \geq 1$$

$$\bullet \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,r_c)} = \left.\frac{\partial^4 f}{\partial x^4}\right|_{(x^*,r_c)} = \dots = 0$$

(and $f(x^*,r_c) = \frac{\partial f}{\partial x}(x^*,r_c) = 0$: bifurcation at (x^*,r_c), $x^* = 0$)

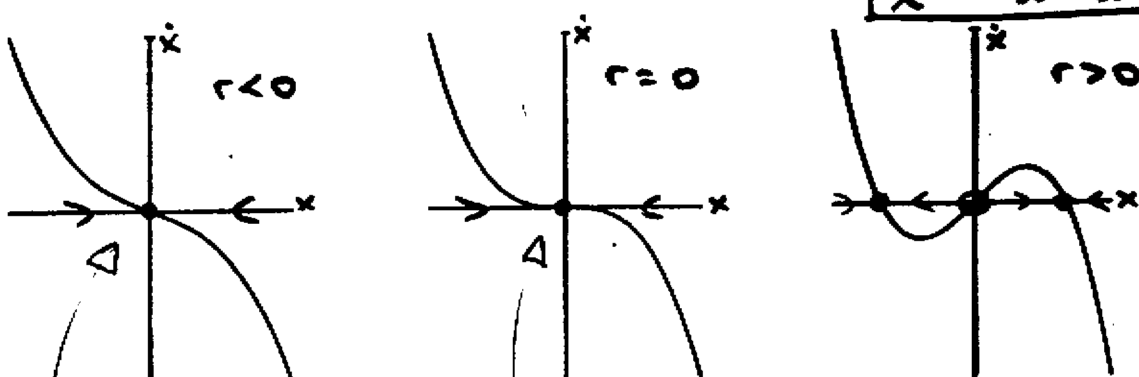
$$\Rightarrow \dot{x} = f(x,r) = (x-x^*)(r-r_c)\left.\frac{\partial^2 f}{\partial x \partial r}\right|_{(x^*,r_c)} + \frac{1}{3!}(x-x^*)^3 \left.\frac{\partial^3 f}{\partial x^3}\right|_{(x^*,r_c)} + \dots$$

Two types: supercritical, subcritical (sign of $\frac{\partial^3 f}{\partial x^3}$)

Supercritical Pitchfork Bifurcation

Normal form

$$\boxed{\dot{x} = rx - x^3}$$

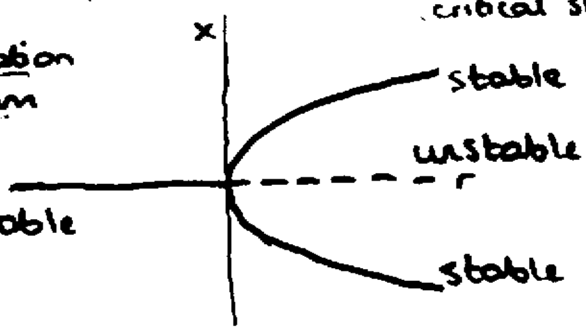


$x^* = 0$ linearly stable exponential decay

$x^* = 0$ algebraic decay $x(t) \sim t^{-1/2}$ (critical slowing down)

$x^* = 0$ unstable
$x^* = \pm\sqrt{r}$ stable

Bifurcation diagram

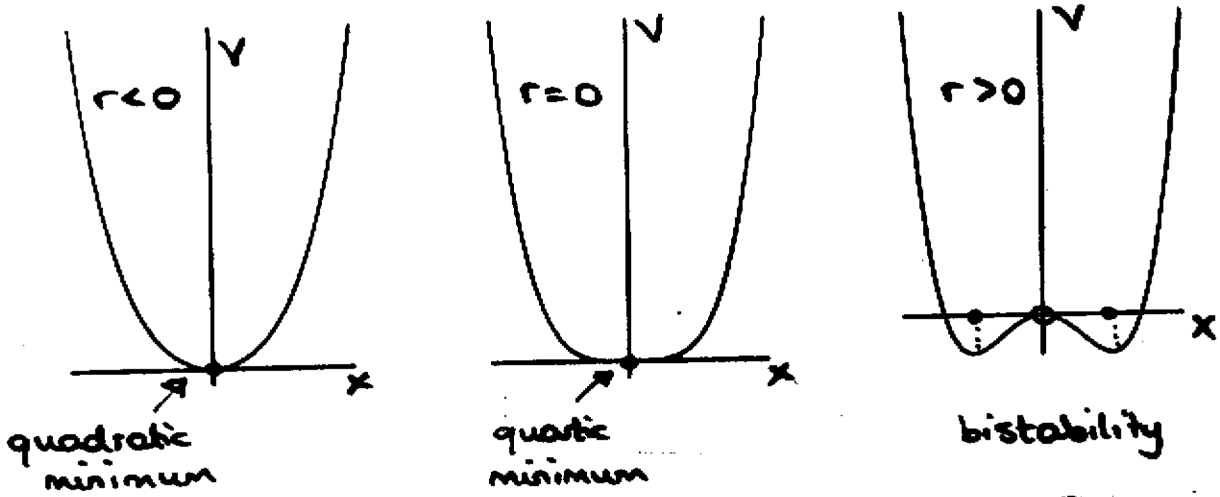


"supercritical" pitchfork bifurcation

($r>0$: $x^* = 0$ unstable
stable nontrivial fixed points created)

Potential: $\dot{x} = rx - x^3 = -\dfrac{\partial V}{\partial x}$

$$\Rightarrow V(x) = -\tfrac{1}{2}rx^2 + \tfrac{1}{4}x^4$$



eg $\dot{x} = -x + \beta \tanh x$

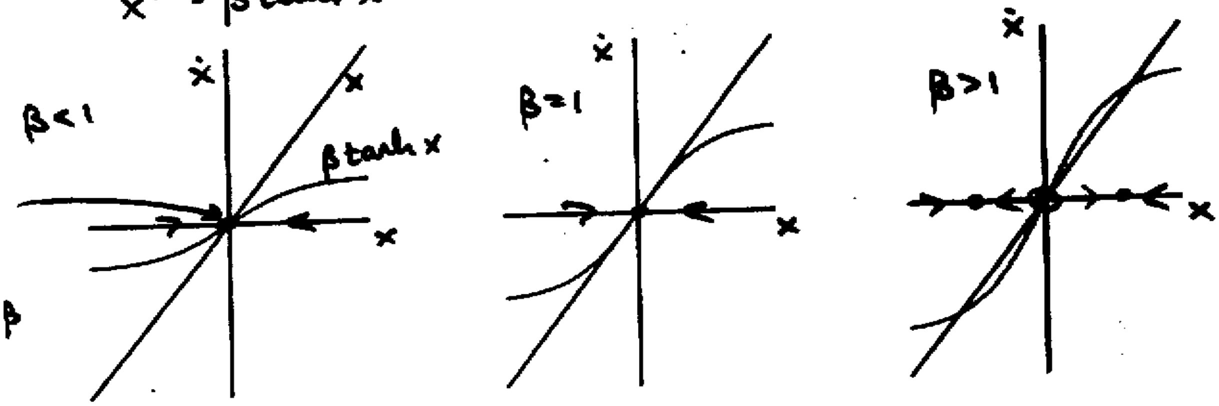
$x^* = 0$ - fixed point for all β

symmetry $x \to -x$ - expect pitchfork bifurcation

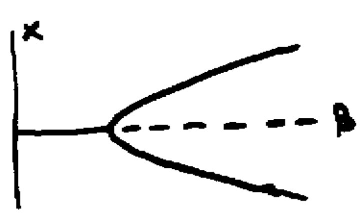
Fixed points:

$x^* = \beta \tanh x^*$

slope of $\beta \tanh x$ at $x = 0$: $\beta\, \mathrm{sech}^2 0 = \beta$



Bifurcation diagram: supercritical pitchfork



$x^* = \beta \tanh x^*$: difficult to solve for $x^*(\beta)$ (Newton)

- plot $\beta = \dfrac{x^*}{\tanh x^*}$ (parametric)

("forward" bifurcation
"soft / safe"
related to second-order / continuous phase transitions)

eg ferromagnetism

Phase transitions

-- may be modelled by equations such as

$$\dot{x} = -x + \beta \tanh x$$

eg Ferromagnetism

x/β related to magnetization

$1/\beta$ related to temperature

Simple model: spins on a 1-d lattice (Ising model)

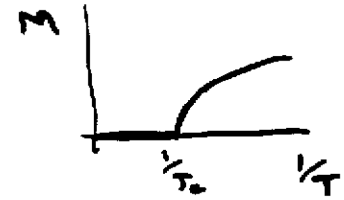
↑ ↓ ↑ ↑ ↑ ↓ ↓ ↑ ↓ ↓ "up-down" symmetry

At high temperature ($\beta<1$) disorder, random alignment of spins - zero average magnetization

At low temperature ($\beta>1$) spontaneous order, alignment of spins - nonzero magnetization

- phase transition at critical temperature T_c.
- symmetry-breaking: ordered state has lower symmetry (either up or down)

Continuous phase transition: magnetization increases continuously from zero for $T< T_c$ (no jump)



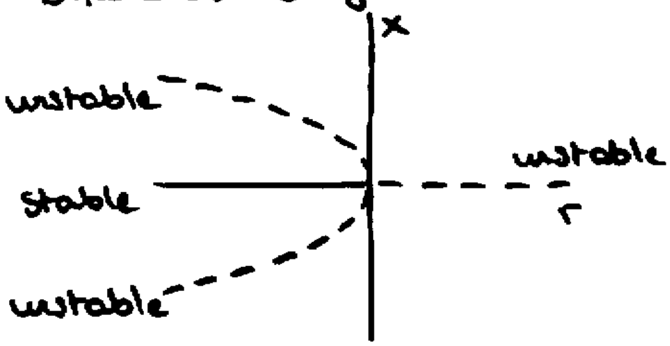
Supercritical pitchfork bifurcation: $\dot{x} = rx - x^3$

- cubic term is stabilizing
- nontrivial fixed points for $r > r_c$.

Subcritical Pitchfork Bifurcation

$$\dot{x} = rx + x^3$$ - cubic term is destabilizing

Bifurcation diagram



"subcritical" : $r>0$: $x^*=0$ unstable

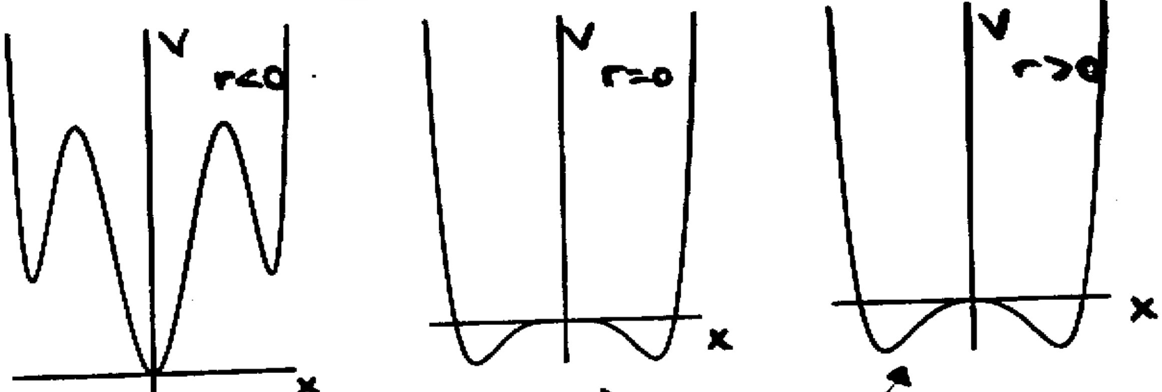
nontrivial fixed points exist below bifurcation, for $r<0$

nonzero fixed points: $x^* = \pm\sqrt{-r}\,,\ r<0$

Note: $\dot{x} \sim x^3$ for large x: blow-up in finite time
-need a stabilizing higher-order term. $-x^5$

Canonical example: $\boxed{\dot{x} = rx + x^3 - x^5}$ [respecting symmetry $x \to -x$

Potential

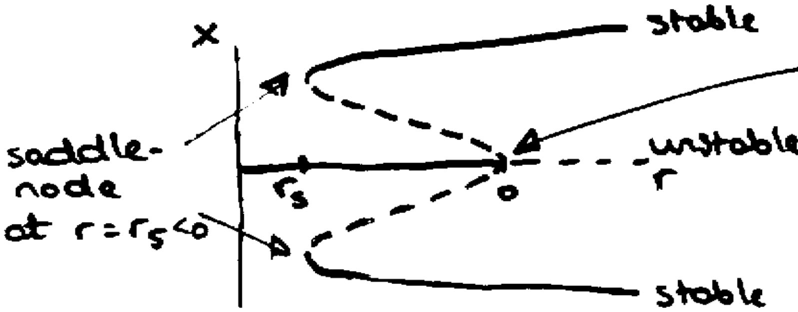


3 stable, 2 unstable

2 stable, 1 unstable at $x^*=0$

$$V(x) = -\frac{1}{2}rx^2 - \frac{1}{4}x^4 + \frac{1}{6}x^6$$

"backward/inverted" bifurcation
"hard/dangerous"
first-order/discontinuous phase transitions



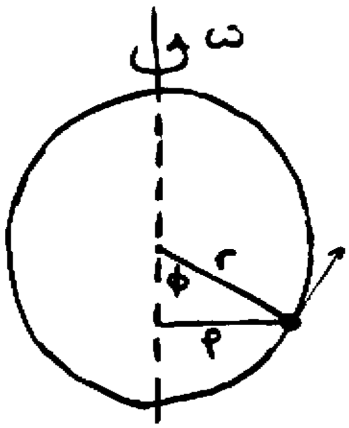
Subcritical pitchfork

stable large-amplitude branches for $r > r_s$

Note: • Bistability $r_s < r < 0$ - two qualitatively different stable states: origin locally, not globally stable

• Jumps and Hysteresis as r is varied
-lack of reversibility

Overdamped Bead on a Rotating Hoop



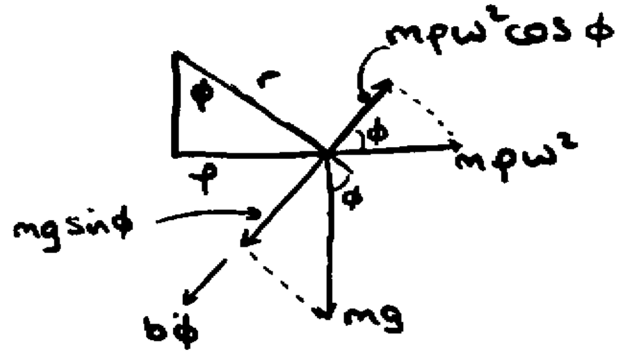
rigid hoop rotating about vertical

tangential velocity	$r\dot{\phi}$	↗
forces: gravitational	mg	↓
centrifugal	$m\rho\omega^2$	→
frictional	$b\dot{\phi}$	↙

left-right symmetry

Balance of tangential forces:

$\rho = r\sin\phi$



Newton's Law:

$$\boxed{mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi}$$

invariant under $\phi \to -\phi$

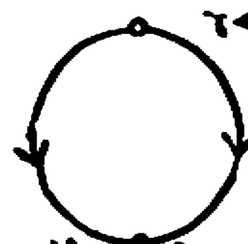
Second-order ODE. can we neglect $mr\ddot{\phi}$ term?

Assume (for now) $|mr\ddot{\phi}| \ll |b\dot{\phi}|$

$$\Rightarrow \quad b\dot{\phi} = mr\omega^2\sin\phi\cos\phi - mg\sin\phi$$

$$= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

slow rotation $\gamma < 1$



centrifugal force weak

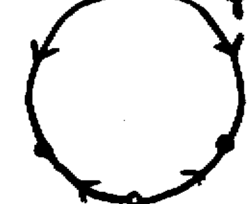
Fixed points: $\sin\phi = 0 \Rightarrow \phi^* = 0$ (bottom)
$\phi^* = \pi$ (top)

If $\gamma \equiv \dfrac{r\omega^2}{g} > 1$, two additional fixed points

$$\cos\phi^* = 1/\gamma = g/r\omega^2$$

(note: $\gamma \to \infty \Rightarrow \phi^* \to \pm\pi/2$)

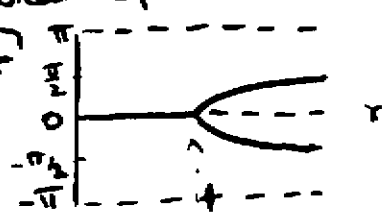
fast rotation $\gamma > 1$



centrifugal force balances gravity

Condition for additional equilibria:

$\omega > \omega_{crit} = \sqrt{g/r}$



supercritical pitchfork bifurcation at $\gamma = 1$

symmetry-broken solutions

Dimensional Analysis and Scaling

When can we neglect $mr\ddot{\phi}$? (what is "small")?

$m \to 0$ is not appropriate - ignores gravitational and centrifugal forces.

Scaling and Nondimensionalization :

- estimate typical / characteristic scales of the dependent variables
- estimate characteristic time (or length) scales; relative to these scales, derivatives are $O(1)$
- reduce the number of relevant parameters to dimensionless groups (pure numbers) which determine dynamics
- there may be more than one choice of scales, different nondimensionalizations, appropriate in different regions

In dimensionless system - units are unimportant
- "large" and "small" have a definite meaning

Nondimensionalize -(express equation in dimensionless form)

ϕ : in radians (already a pure number)

Characteristic time scale T - new dimensionless time $\tau = t/T$

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d}{d\tau} \quad \left(= \frac{d}{d(T\tau)} \right)$$

$$\Rightarrow \dot{\phi} = \frac{d\phi}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}, \qquad \ddot{\phi} = \frac{d^2\phi}{dt^2} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$$

Substitute: $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

$$\Rightarrow \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

Balance of forces - divide by force mg

$$\Rightarrow \underbrace{\left(\frac{r}{gT^2}\right)}_{\text{dimensionless groups}}\frac{d^2\phi}{d\tau^2} = -\underbrace{\left(\frac{b}{mgT}\right)}_{\text{dimensionless groups}}\frac{d\phi}{d\tau} - \sin\phi + \underbrace{\left(\frac{r\omega^2}{g}\right)}_{\gamma}\cos\phi\sin\phi$$

dimensionless groups

Choice of natural time scale T:

- so terms on r.h.s. are comparable ie $\frac{b}{mgT} = O(1)$

(since $\sin\phi = O(1)$, and by assumption $\phi_\tau, \phi_{\tau\tau} = O(1)$)

- choose $T = b/mg$

We can neglect $\frac{d^2\phi}{d\tau^2}$ term on l.h.s. (to get a first-order system

if $\frac{r}{gT^2} = \frac{r}{g}\left(\frac{mg}{b}\right)^2 = \frac{rm^2g}{b^2} \ll 1$

Define $\varepsilon = \frac{r}{gT^2} = \frac{rm^2g}{b^2}$ (dimensionless)

Dimensionless formulation

$$\boxed{\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\cos\phi\sin\phi}$$

(5 parameters $\rightarrow$ 2 dimensionless group:
m, g, r, b, ω $\quad$ ε, γ

Overdamped limit

$\varepsilon \ll 1 \Leftrightarrow b^2 \gg rm^2g$

("strong damping" / "small mass" made precise)

From dimensional analysis, we expect:

In overdamped limit $\underline{\varepsilon \to 0}$, the dynamics of the bead should be well-approximated by the first-order system

$$\frac{d\phi}{d\tau} = f(\phi) = \sin\phi(\gamma\cos\phi - 1)$$

Paradox: Can $\frac{d\phi}{d\tau} = f(\phi)$ uniformly describe (up to $O(\varepsilon)$ corrections) the motion of the bead? No...

Second-order ODE $\phi'' = \tilde{f}(\phi, \phi')$ - specify 2 initial conditions $\phi(0), \phi'(0)$

First-order ODE $\phi' = f(\phi)$ - specify 1 initial condition $\phi(0)$
- compute $\phi'(0) = f(\phi(0))$

In general, solution of $\phi' = f(\phi)$ cannot satisfy both conditions

Phase Plane Analysis

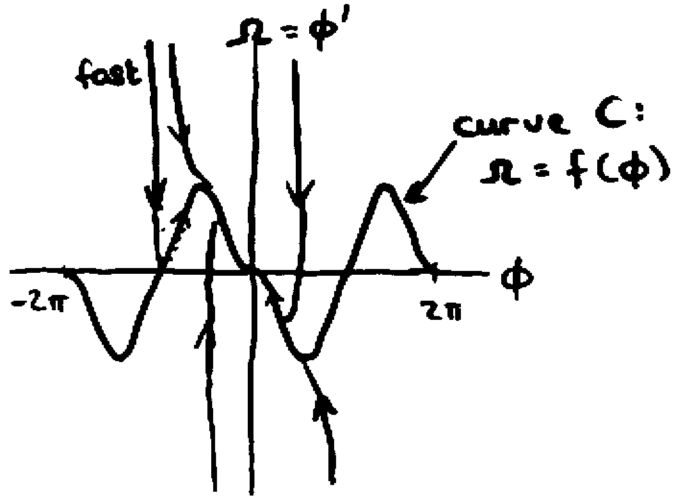
- interpret 2^{nd} order ODE as a vector field in the plane:

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} + f(\phi) \qquad , \quad \varepsilon \ll 1$$

Introduce $\Omega \equiv \phi' = \frac{d\phi}{d\tau}$ $\quad \left(\Rightarrow \varepsilon\Omega' = \varepsilon\phi'' = -\underbrace{\phi'}_{\Omega} + f(\phi)\right)$

then
$$\begin{cases} \phi' = \Omega \\ \Omega' = \frac{1}{\varepsilon}\left(f(\phi) - \Omega\right) \end{cases}$$

If $f(\phi) - \Omega$ is $O(1)$, then Ω' is $O(1/\varepsilon) \gg 1$ ie Ω' is very large, so Ω changes rapidly, the phase point zaps towards $f(\phi) - \Omega = O(\varepsilon)$, then it evolves along $\Omega \approx f(\phi)$, ie. according to $\phi' = f(\phi)$.



Typical trajectory:

- rapid initial transient on timescale ε (dimensional: $T_{fast} = \varepsilon T = \frac{mr}{b}$)
  - cannot neglect $\varepsilon \frac{d^2\phi}{d\tau^2}$
- slow drift along curve where $\phi' = f(\phi)$ on timescale 1 (dimensional: $T_{slow} = T = \frac{b}{mg}$)

Dynamics on fast timescale: $\sigma = \tau/\varepsilon$

$$\Rightarrow \begin{cases} \frac{d\phi}{d\sigma} = \varepsilon\Omega \\ \frac{d\Omega}{d\sigma} = f(\phi) - \Omega \end{cases} \Rightarrow \begin{array}{l} \phi = \text{constant}, \\ \Omega \text{ decays exp. to } f(\phi) \end{array}$$

Note: $\varepsilon \to 0$ is a singular limit

(highest order derivative is lost)

Mathematical treatment: singular perturbation theory

eg boundary layers in fluid mechanics for "small viscosity" (dimensionless: high Reynolds number)

Imperfect Bifurcations

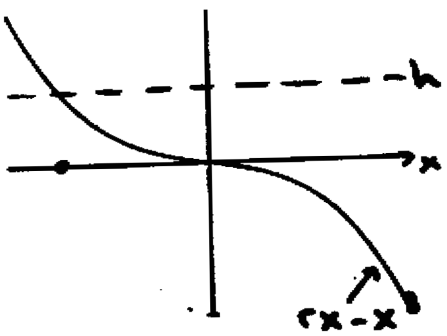
Symmetry $x \to -x$: pitchfork
What if symmetry is approximate?

$$\dot{x} = h + rx - x^3 \qquad \text{(2 parameters)}$$

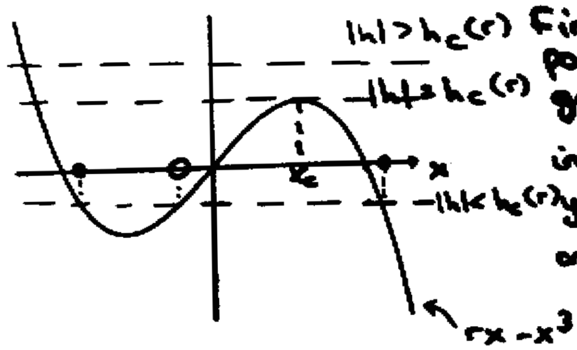
h ↑ imperfection parameter

($h=0$: supercritical pitchfork bifurcation at $r=0$.)

Fix r, vary h:



$r \leq 0$
1 fixed point



Find fixed points graphically: intersection of $y = rx - x^3$ and $y = -h$.

$r > 0$
1, 2 or 3 fixed points

$r>0$: Critical value $h_c(r)$ (saddle-node bifurcation)
- where $h = \text{const.}$ is tangent to $rx - x^3$:

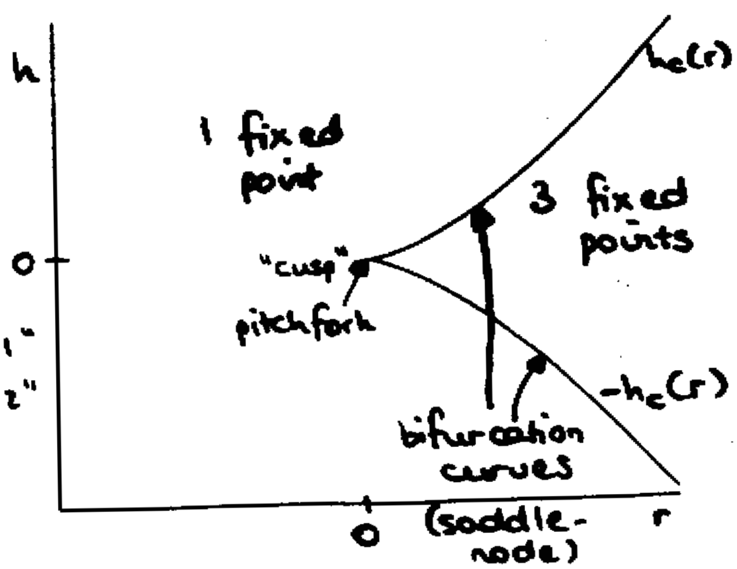
$$\frac{\partial}{\partial x}(rx - x^3) = r - 3x^2 = 0 \quad \Rightarrow \quad x = \pm x_c = \sqrt{r/3}$$

value at x_c: $h_c = rx_c - x_c^3 = r\sqrt{\frac{r}{3}} - \left(\sqrt{r/3}\right)^3 = \frac{2r}{3}\sqrt{\frac{r}{3}}$

Saddle-node at $h = \pm h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}$: $27h^2 = 4r^3$

Stability Diagram

(plot of types of behaviour, no. of fixed points in parameter space)



$h \neq 0$: saddle-node "codimension 1"

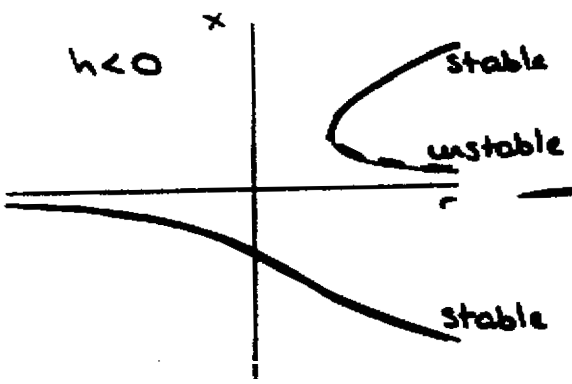
$h=0$, $r=0$: pitchfork "codimension 2"

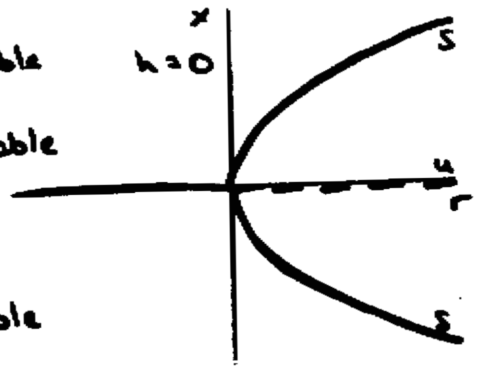
(need to tune 2 parameters)

$(r,h) = (0,0)$: cusp point
(curves meet tangentially)

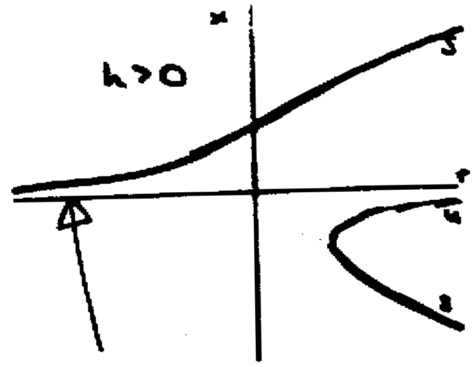
Bifurcation diagrams

Fix h: x^* as function of r $\qquad \left(r = \frac{x^3 - h}{x}\right)$



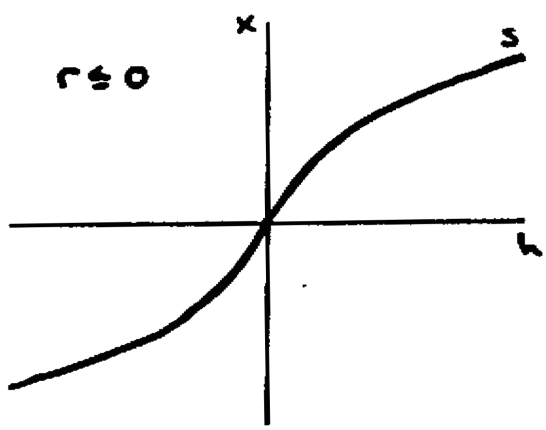


supercritical pitchfork bifurcation

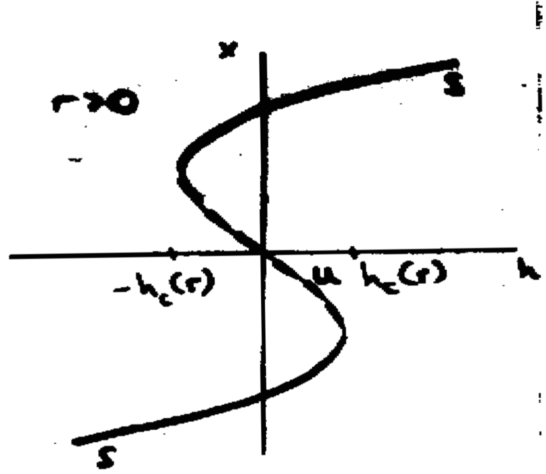


no sharp transition as r increases from $r<0$ – stay on stable branch

Fix r: x^* as function of h $\qquad (h = -rx + x^3)$

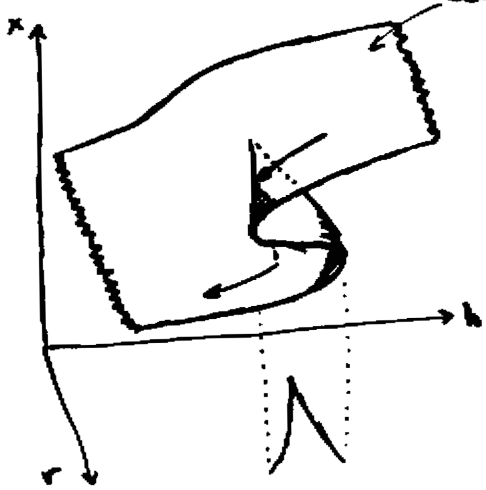


1 stable fixed point for each h



3 fixed points when $|h| < h_c(r)$

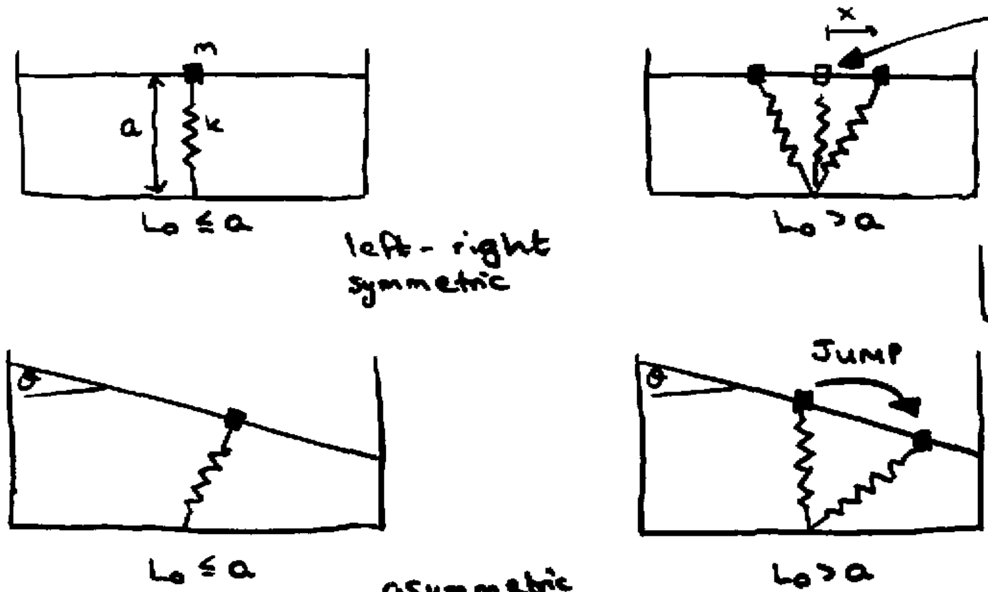
catastrophe surface



cusp catastrophe

Catastrophe: discontinuous change in the state of a system due to smooth change of parameters eg **Bead on a tilted wire**

L_0 : natural (relaxed) length of spring
stiffness k



left-right symmetric

midpoint unstable
(spring compressed)
- pitchfork bifurcation

asymmetric

increase θ sufficiently: upper equilibrium disappears

a: bifurcation parameter

θ: imperfection parameter

Insect Outbreak (Ludwig et al, 1978)

Spruce budworm attacks leaves of balsam fir

Budworm: fast timescale · characteristic doubling time ~ months

Fir trees: slow timescale
- replace foliage ~7-10 yrs
- life span ~ 100-150 yrs

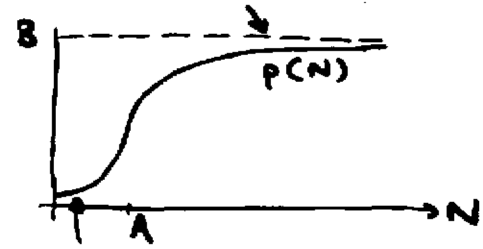
- view forest variables as "constant" parameters, not dynamic variables

Model:

$N(t)$ - budworm population

$$\frac{dN}{dt} = \underbrace{RN\left(1-\frac{N}{K}\right)}_{\substack{\text{logistic:}\\ \text{growth rate } R\\ \text{carrying capacity } K}} - \underbrace{p(N)}_{\substack{\text{predation}\\ \text{(by birds)}}}$$

saturation: birds eat at maximal rate



birds seek food elsewhere

$$\frac{dN}{dt} = RN\left(1-\frac{N}{K}\right) - p(N)$$

$p(N)$ - sigmoid function - models a "switch", threshold A

- use $\quad p(N) = B \cdot \dfrac{N^2}{A^2+N^2}$

$$\Rightarrow \quad \dot N = RN\left(1-\frac{N}{K}\right) - B\frac{N^2}{A^2+N^2}$$: 4 parameters $A, B, R, K > 0$

Nondimensionalize!

K, A have units of N : different choices of nondimensionalizatio

- choose to put parameters into logistic part, not into predation part :

$x = \dfrac{N}{A}$ (and divide by B)

$N = Ax$:

$$\Rightarrow \quad \frac{A}{B}\frac{dx}{dt} = \frac{RA}{B}x\left(1-\frac{A}{K}x\right) - \frac{x^2}{1+x^2}$$

Dimensionless parameters $r = \dfrac{RA}{B}$, $k = \dfrac{K}{A}$; time scale $\dfrac{A}{B}$: $\tau = \dfrac{B}{A}t$: $\dfrac{dx}{d\tau} = \dfrac{A}{B}\dfrac{dx}{dt}$

$\Rightarrow$ Dimensionless system:

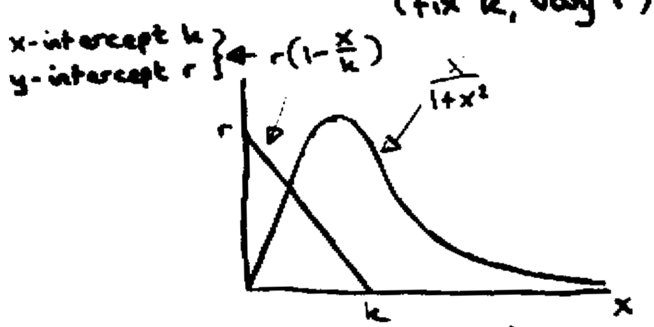
$$\boxed{\frac{dx}{d\tau} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}} = f(x,r,k)$$

2 dimensionless parameters: $r, k > 0$

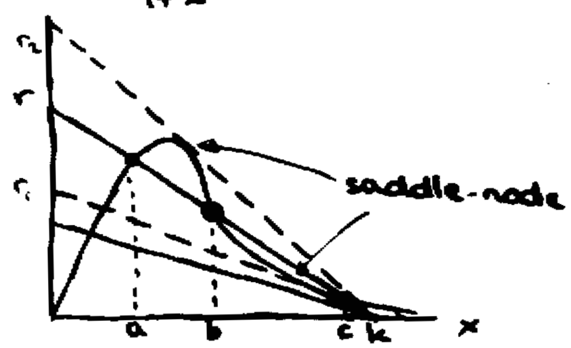
Fixed points:

- $x^* = 0$, always unstable $\left.\dfrac{\partial f}{\partial x}\right|_{x=0} = r > 0$ (small x: low predation exponential growth

- $x \neq 0$: Graphical analysis : (fix k, vary r) $\quad r\left(1-\dfrac{x}{k}\right) = \dfrac{x}{1+x^2}$



small k: one (nontrivial) steady stat



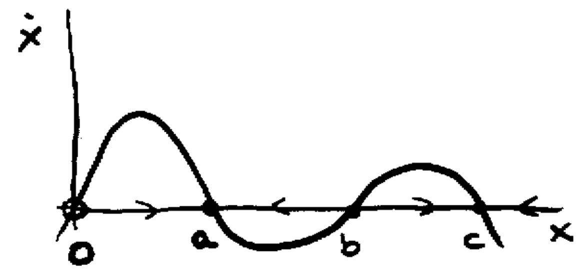
large k: 1,2 or 3 solutions

For sufficiently large k: saddle-node bifurcations

at $\begin{cases} r = r_1(k) & \text{(between fixed points } b \text{ and } c) \\ r = r_2(k) & \text{(between fixed points } a \text{ and } b) \end{cases}$

Stability: - alternates between stable and unstable

$$\begin{array}{ccccccc} 0 < & a & < & b & < & c \\ u & s & & u & & s \end{array}$$



stable equilibria: $x^* = a$: refuge

$x^* = c$: outbreak

Pest control: wish to keep budworm population at a, away from c

Initial conditions: $\left. \begin{array}{l} x(0) < b \Rightarrow x(t) \to a \\ x(0) > b \Rightarrow x(t) \to c \end{array} \right\} x^* = b$: threshold

If parameters change so r increases beyond $r_2(k)$:

outbreak triggered by saddle-node bifurcation of a and b

(due to hysteresis : if r decreases, population stays at c)

Saddle-node bifurcation curves:

require $\underbrace{r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}}_{\text{nontrivial fixed point}}$ ① and $\frac{\partial}{\partial x}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{\partial}{\partial x}\left[\frac{x}{1+x^2}\right]$

tangency (i.e. $\frac{\partial f}{\partial x} = 0, x \neq 0$)

$$\Rightarrow \quad -\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2} \quad ②$$

Substitute ② in ①:

$$r = \frac{r}{k}x + \frac{x}{1+x^2} = \frac{x^2-1}{(1+x^2)^2}x + \frac{x}{1+x^2} \quad \Rightarrow \quad r = \frac{2x^3}{(1+x^2)^2} \quad ③$$

Substitute ③ in ②:

$$k = -r \cdot \frac{(1+x^2)^2}{1-x^2} = \frac{2x^3}{(1+x^2)^2} \frac{(1+x^2)^2}{x^2-1} \quad \Rightarrow \quad k = \frac{2x^3}{x^2-1} \quad ④$$

③, ④: parametric representation $(r(x), k(x))$ of bifurcation curves

Note: $k > 0 \Rightarrow x > 1$. $(N > A)$

$$\frac{dr}{dx} = \ldots = \frac{-2x^2(x^2-3)}{(1+x^2)^3}, \quad \frac{dk}{dx} = \frac{2x^2(x^2-3)}{(x^2-1)^2} \Rightarrow \text{max. of } r, \text{ min. of } k$$

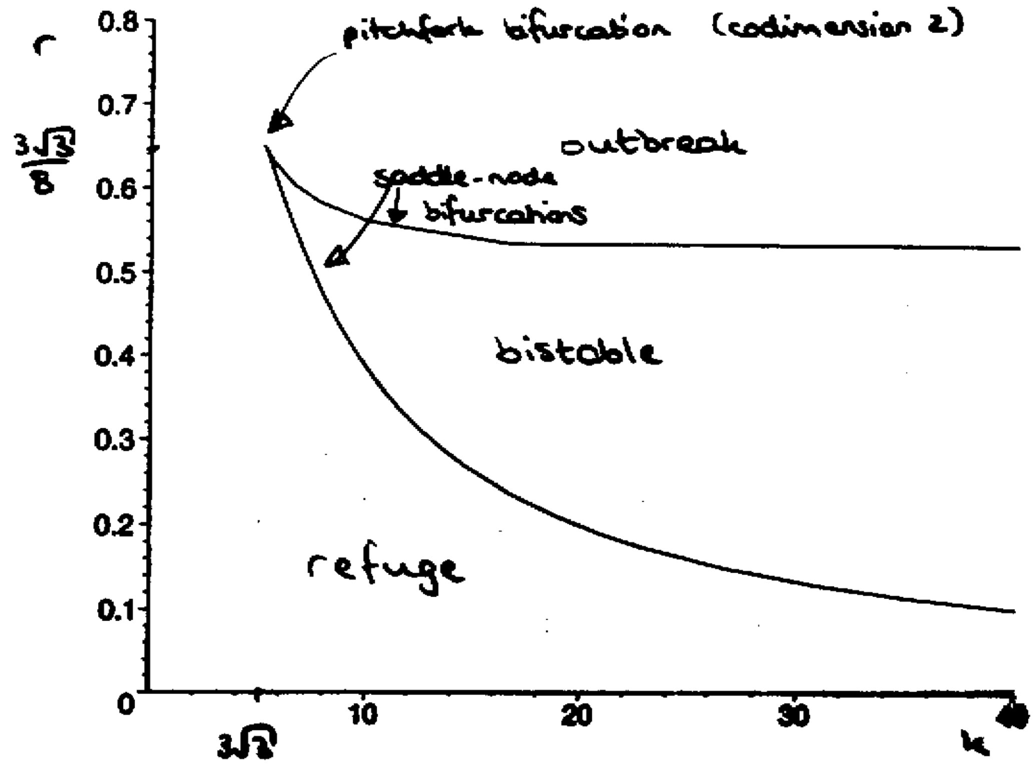
at $x = \sqrt{3}$:

$r_{max} = 3\sqrt{3}/8$, $k_{max} = 3\sqrt{3}$

Bifurcation curves: (parametric form)

$$\left.\begin{aligned} r(s) &= \frac{2s^3}{(1+s^2)^2} \\ k(s) &= \frac{2s^3}{s^2-1} \end{aligned}\right\} s>1$$

Stability Diagram:



Biologically plausible parameter values?

Approximations:
neglected · tree dynamics
· spatial effects

S : average size of trees (total surface area of foliage in a stand)

Expect carrying capacity K, half-saturation parameter A to be proportional to foliage area

$K = K'S, \quad A = A'S$

$\Rightarrow$ dimensionless parameters $\quad r = \frac{RA}{B} = \frac{RA'}{B}S, \quad k = \frac{K}{A} = \frac{K'}{A'}$

As the forest grows, expect: r increases, k fixed

Young forest: $k \approx 300$, $r < 1/2$ - bistable

More mature forest: k increases - parameter drift to $r \approx 1$
- outbreak!

- fir trees die, birch trees take over, forest takes 50-100 yrs to recover