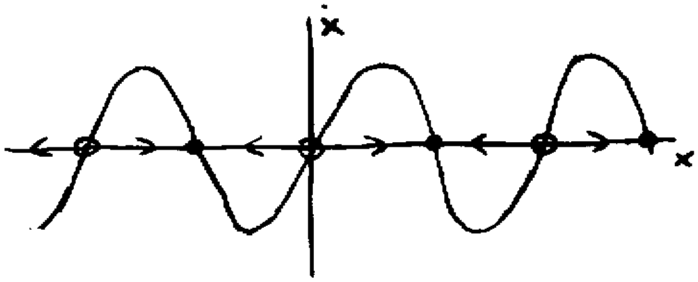
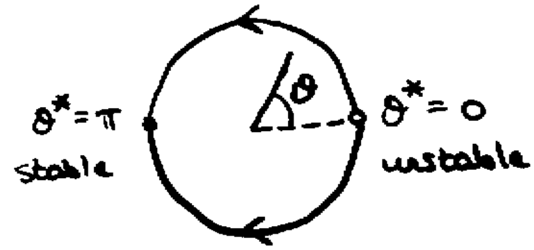
Flows on the Circle

eg $\dot{x} = \sin x$ periodic in x

$\dot{\theta} = \sin\theta$





$\dot{\theta} = f(\theta)$: vector field on circle S^1

if $f(\theta)$ is 2π-periodic, $f(\theta + 2\pi) = f(\theta)$, all θ

(unique velocity $\dot{\theta}$ for each point θ on the circle)

Oscillations possible!

(unlike flows on line: on circle, can return to starting place by flowing in one direction – periodic solutions)

θ : angle, phase of oscillation

(no amplitude variable : 1-d dynamics)

Uniform Oscillator : phase changes uniformly

$$\dot{\theta} = \omega$$

$$\Rightarrow \theta(t) = \omega t + \theta_0$$

Periodic solution : Period $T = \frac{2\pi}{\omega}$ (time for $\theta(t)$ to change by 2π)

eg two joggers, periods $T_1 = 2\pi/\omega_1$, $T_2 = 2\pi/\omega_2$ $\quad T_1 < T_2$

$\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$

Phase difference $\phi = \theta_1 - \theta_2$: $\dot{\phi} = \omega_1 - \omega_2 > 0$

Jogger 1 overtakes jogger 2 in $T = \frac{2\pi}{\omega_1 - \omega_2} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1}$

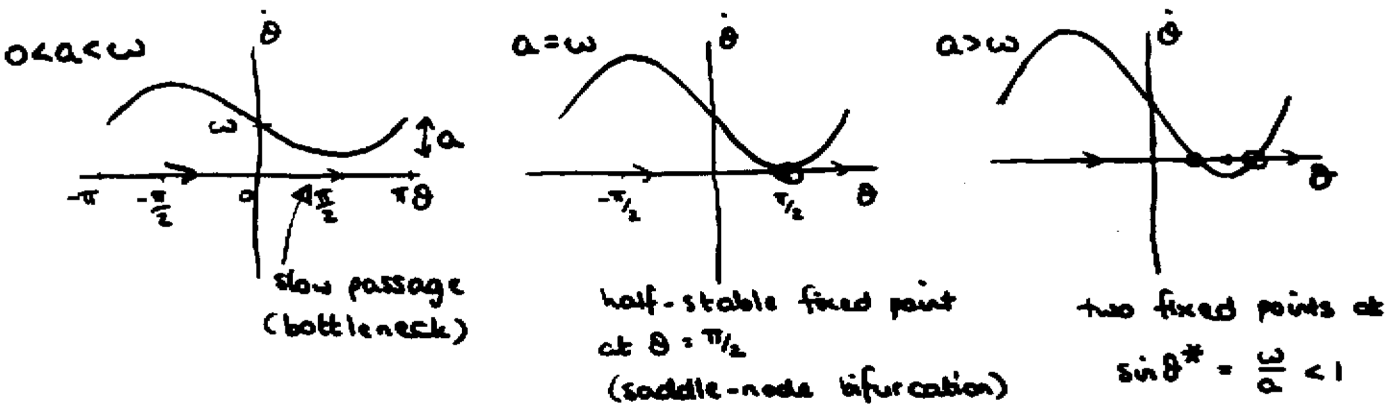
(ϕ increases by 2π)

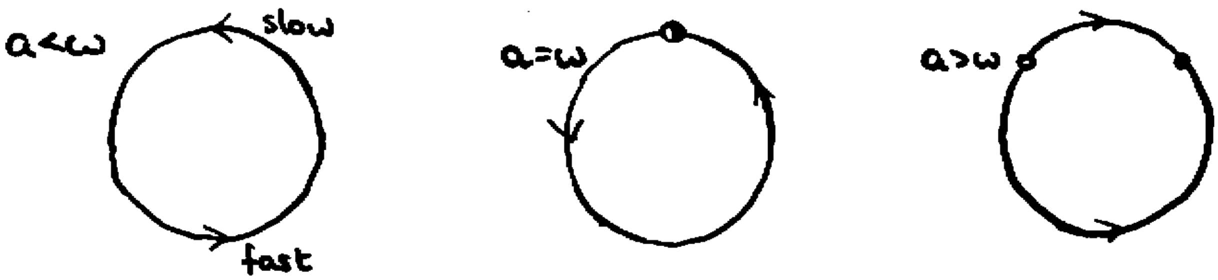
Nonuniform Oscillator

$$\dot\theta = \omega - a\sin\theta \qquad \left(\begin{matrix}\text{assume}\\ \omega>0,\ a\geq 0\end{matrix}\right)$$

$a=0$: uniform oscillator

$a>0$: nonuniformity – $\dot\theta$ maximum at $\theta = -\pi/2$ ← fastest
$\dot\theta$ minimum at $\theta = \pi/2$ ← slowest





(check stability of fixed points by linear stability analysis)

Period of oscillation : $(a<\omega)$

For $\frac{d\theta}{dt} = f(\theta)$, time for θ to change by 2π is

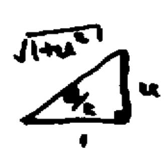
$$T = \int_0^T dt = \int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{f(\theta)} = \int_0^{2\pi} \frac{d\theta}{f(\theta)}$$

f 2π-periodic

In this case: $\dot\theta = \omega - a\sin\theta$

$$\Rightarrow T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

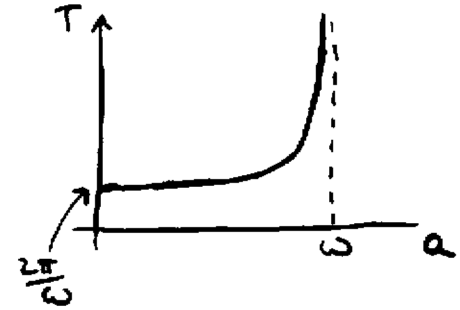
$u = \tan\theta/2$



$$\left(\Rightarrow \sin\theta = \frac{2u}{1+u^2},\ \cos\theta = \frac{1-u^2}{1+u^2},\ d\theta = \frac{2}{1+u^2}du\right)$$

$\dot{\theta} = \omega - a\sin\theta, \qquad a<\omega$

Period of oscillation $T = \dfrac{2\pi}{\sqrt{\omega^2-a^2}}$



Period increases with a,

diverges as $a \to \omega^-$

Order of divergence: $\sqrt{\omega^2-a^2} = \sqrt{\omega+a}\,\sqrt{\omega-a} \approx \sqrt{2\omega}\,\sqrt{\omega-a}$

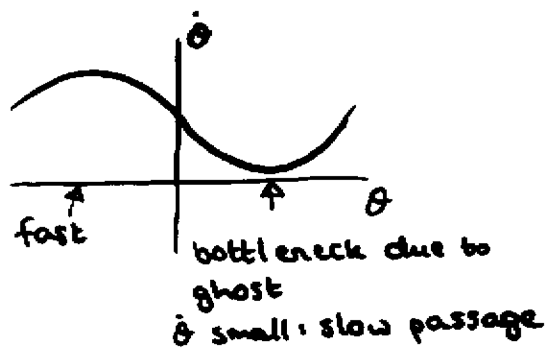
$$\Rightarrow \quad T \approx \frac{\pi\sqrt{2}}{\sqrt{\omega}}\,\frac{1}{\sqrt{\omega-a}}, \quad a\to\omega^-$$

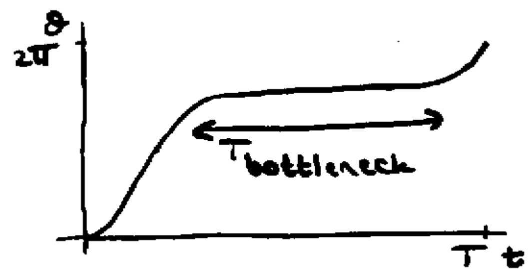
ie $\quad T \sim (a_c - a)^{-1/2}$, $a_c = \omega$: square root scaling

Ghosts and bottlenecks

Near a saddle-node bifurcation:

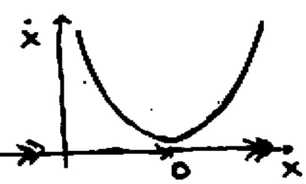
Slow passage through a bottleneck due to "ghosts" of fixed point





Scaling Law: $\dot{\theta}$ locally parabolic

– use normal form for saddle-node $\dot{x} = r + x^2$



dominated by $x \approx 0$ $\to$ $T_{\text{bottleneck}} \approx \displaystyle\int_{-\infty}^{\infty} \frac{dx}{r+x^2} \underset{x=\sqrt{r}\,x'}{=} \frac{1}{\sqrt{r}} \arctan x \Big|_{-\infty}^{\infty} = \pi/\sqrt{r}$

square root scaling in general

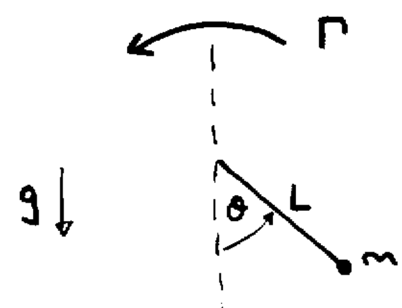
eg $\quad \dot{\theta} = \omega - a\sin\theta$: expand near $\pi/2$: $\theta = \pi/2 + \phi$

$\Rightarrow \dot{\phi} = \omega - a\sin(\pi/2+\phi) = \omega - a\cos\phi = \omega - a(1-\tfrac{1}{2}\phi^2 + \ldots) = (\omega-a) + \tfrac{1}{2}a\phi^2 + \ldots$

Rescale: $x = \tfrac{1}{2}a\phi$, $r = \tfrac{1}{2}a(\omega-a) \quad \Rightarrow \quad \dot{x} = r + x^2 + O(x^4)$

Period: $\quad T \approx \pi/\sqrt{r} = \dfrac{\pi\sqrt{2}}{\sqrt{a}}\,\dfrac{1}{\sqrt{\omega-a}} \underset{a\approx\omega}{\propto} \pi\sqrt{\dfrac{2}{\omega}}\,\dfrac{1}{\sqrt{\omega-a}}$, as before.

Overdamped Pendulum



θ: angle between pendulum and vertical

L: length, m: mass, g: gravity

b: viscous damping

Γ: constant torque

Balance of torques (Newton's Law)

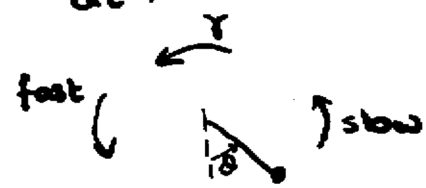
$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

Overdamped limit – large b: $mL^2\ddot{\theta}$ negligible (after transient)

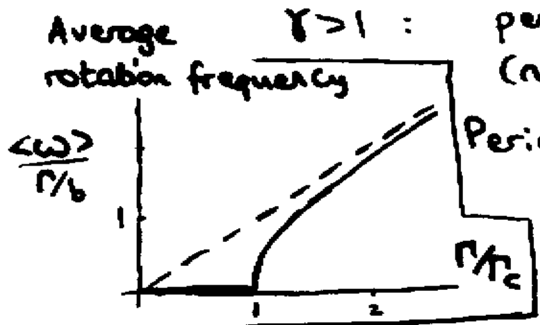
$$\Rightarrow \quad b\dot{\theta} = \Gamma - mgL\sin\theta \Rightarrow \underbrace{\left(\frac{b}{mgL}\right)}_{t_0}\dot{\theta} = \underbrace{\left(\frac{\Gamma}{mgL}\right)}_{\gamma} - \sin\theta$$

Nondimensionalize: $\tau = t/t_0 = \dfrac{mgL}{b}t$, $\gamma = \dfrac{\Gamma}{mgL}$ $\dfrac{\text{applied torque}}{\text{max. gravit. torque}}$

$$\Rightarrow \boxed{\theta' = \gamma - \sin\theta} \qquad \left(\theta' = \frac{d\theta}{d\tau}\right)$$



$\gamma > 1$: pendulum overturns constantly (non-uniform rotation rate)



Period (in τ): $\dfrac{2\pi}{\sqrt{\gamma^2-1}}$, average frequency $\sqrt{\gamma^2-1}$

(in dimensional units: $\langle\omega\rangle = \dfrac{mgL}{b}\sqrt{\gamma^2-1} = \dfrac{\sqrt{\Gamma^2-(mgL)^2}}{b}$

$\gamma = 1$: torques balance

saddle-node bifurcation at $\theta^* = \pi/2$. Critical torque $\Gamma_c = mg$

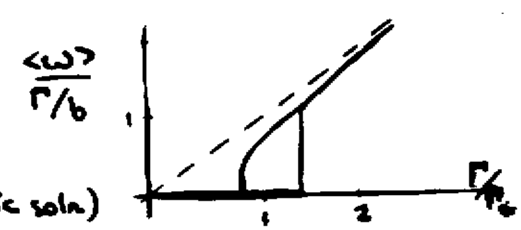
$\gamma < 1$: two fixed points (lower is stable)

no net rotation: $\langle\omega\rangle = 0$

$\gamma = 0$: no applied torque: free damped pendulum

Non-negligible inertia:

Need to increase Γ beyond Γ_c to get rotation; inertia keeps pendulum whirling even for $\Gamma < \Gamma_c$: hysteresis: bistability (stable fixed pt / periodic soln)



Fireflies

Fireflies influence each other ⇒ <u>synchronization</u>

Response to a flashing stimulus: a firefly speeds up or slows down to flash more nearly in phase with stimulus.

<u>Experiment and Model</u>:

Periodic stimulus, phase Θ, frequency Ω, flash at $\Theta = 0$

$$\dot{\Theta} = \Omega$$

Firefly: phase $\theta(t)$, natural frequency ω - in absence of stimuli

<u>Observations</u>:
- If period of stimulus $\frac{2\pi}{\Omega}$ is close to firefly's natural period, firefly matches frequency → <u>entrainment</u>
- If stimulus is too fast or slow, phase differences increases nonuniformly → <u>phase drift</u>

<u>Model</u>:

$$\dot{\theta} = \omega + \underset{\text{resetting strength}}{A} \sin(\Theta - \theta) \quad \begin{cases} \text{speed up if behind} \\ \text{slow down if ahead} \end{cases}$$

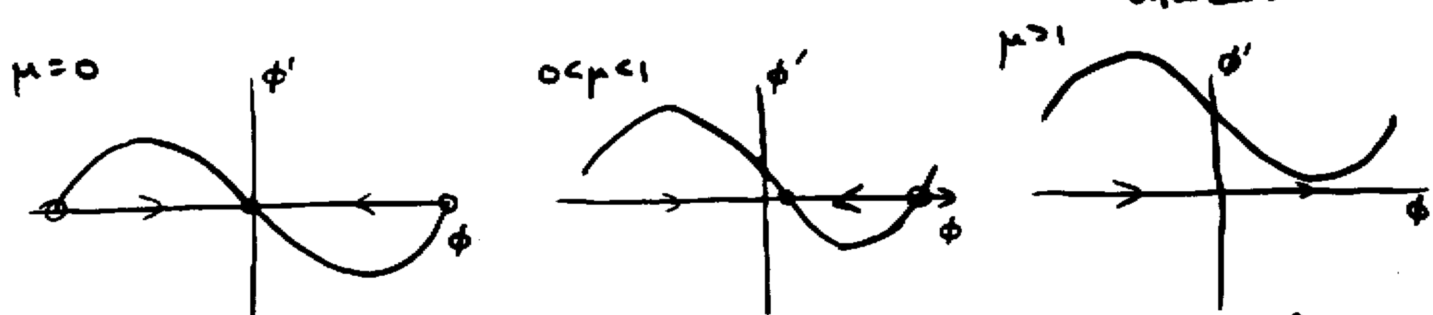
Phase difference $\phi = \Theta - \theta$:

$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$$

Nondimensionalize: $\tau = At$, $\mu = \frac{\Omega - \omega}{A}$ $\quad \frac{\text{freq. difference}}{\text{resetting strength}}$

$$\Rightarrow \frac{d\phi}{d\tau} = \mu - \sin\phi$$

$\mu = 1$: saddle-node bifurcation



Entrainment
Zero phase difference
- simultaneous flashes

Phase-locking
$\Omega > \omega \Rightarrow \mu > 0 \Rightarrow \phi^* > 0$
Constant phase lag
- stimulus flashes before firefly

Phase drift
No phase locking

<u>Entrainment</u> is possible for $-1 \le \mu \le 1$ ie $-1 \le \frac{\Omega - \omega}{A} \le 1$

$$\Rightarrow \underbrace{\omega - A \le \Omega \le \omega + A}_{\text{range of entrainment}}$$

(measure experimentally: find resetting strength A. since ω, Ω are known)

Phase difference during entrainment: $\sin\phi^* = \frac{\Omega - \omega}{A}$

<u>stable</u> fixed point: $-\pi/2 \le \phi^* \le \pi/2$

for $\mu > 1$: Period of phase drift

$$T_{drift} = \int_0^{2\pi} \frac{d\phi}{(\Omega - \omega) - A\sin\phi} = \frac{2\pi}{\sqrt{(\Omega-\omega)^2 - A^2}}$$

biological reality:

- Model is reasonable for species with fixed A, ω
- poor agreement for species which can vary ω.