Two-dimensional flows

General vector field on the phase plane:

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\} \Rightarrow \dot{\vec{x}} = \vec{f}(\vec{x})$$

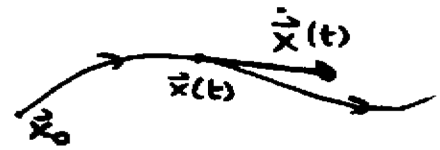
velocity vector

point in phase plane

$\vec{x} = (x_1, x_2)$

$\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}))$

$\vec{x}(t)$: trajectory in phase plane

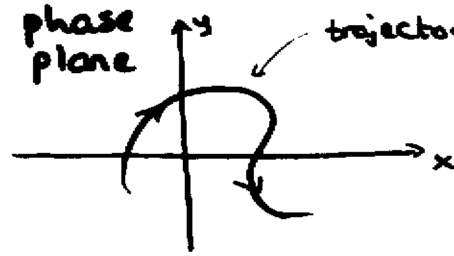


flow in direction of $\dot{\vec{x}}$:

at each point $\vec{x}$ in phase plane, the trajectory is parallel to the vector field $\vec{f}(\vec{x}) = \dot{\vec{x}}$ at that point.

Usually write:

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\}$$



Trajectory: a curve satisfying

$$\frac{dy}{dx} = \dot{y}/\dot{x} = \frac{g(x,y)}{f(x,y)}$$

Direction field: - plot short line segments at a representative grid of points, indicating the local direction of flow.

Many important examples (eg from mechanics) have the form

$$\ddot{x} = F(x, \dot{x})\ ;$$

these can be written as a flow in the phase plane via

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= F(x,y)\end{aligned}\right\}$$

For general nonlinear systems $\dot{\vec{x}} = \vec{f}(\vec{x})$, we cannot expect to find trajectories analytically:

Goal: seek qualitative behaviour directly from $\vec{f}(\vec{x})$

Objects of study:

- Fixed points (steady states, equilibria): $\vec{f}(\vec{x}^*) = \vec{0}$
- Closed orbits (periodic solutions)
  $\vec{x}(t+T) = \vec{x}(t)$ for some $T > 0$, all t
- Stability / instability of fixed points and closed orbits
- Local arrangement of trajectories
  near fixed points and closed orbits

Quantitative aspects:

- Numerical methods:
  Methods for 1-dimensional ODEs $\dot{x} = f(x)$
  eg Euler, Runge-Kutta methods carry over directly for systems in the form $\dot{\vec{x}} = \vec{f}(\vec{x})$
- Solvers in packages:
  eg Maple: dsolve(... , numeric)
  DEplot (in package DEtools)
  Matlab: ode45, ode15s (stiff solver), ...
- Direction field and phase portrait.
  eg Maple: dfieldplot, DEplot (DEtools)
  Matlab: pplane 6

Linear Systems

Two-dimensional linear system

$$\left.\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}\right\} \quad \text{or} \quad \dot{\vec{x}} = A\vec{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>Linear</u>: if $\vec{x}_1(t)$, $\vec{x}_2(t)$ are solutions, so is any linear combination $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$\vec{x}^* = \vec{0}$ is always a fixed point, for any A $(A\cdot\vec{0} = \vec{0})$

eg <u>Simple harmonic oscillator</u>

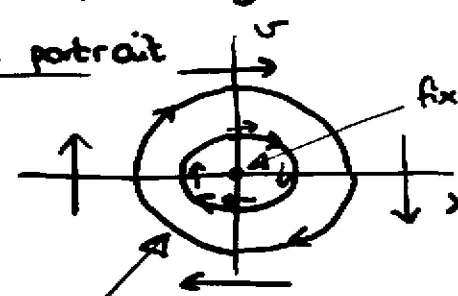
$$m\ddot{x} + kx = 0$$
$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

m mass
k spring constant
$\omega = \sqrt{k/m}$ natural frequency

State is characterized by position x, velocity $v = \dot{x}$

$$\Rightarrow \left.\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega^2 x\end{aligned}\right\}$$

Phase <u>portrait</u>



$(0,0)$: fixed point
Other trajectories:
closed orbits (periodic solutions)

(Note: x is increasing when $v > 0 \Rightarrow$ clock<u>wise</u> rotation)

Shape of orbits:

$$\omega^2 x\dot{x} + v\dot{v} = 0 \qquad \left(\text{or: } \frac{dv}{dx} = \frac{\dot{v}}{\dot{x}} = -\frac{\omega^2 x}{v} \Rightarrow \ldots\right)$$

$$\Rightarrow \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}v^2 = \text{const} = C \geq 0$$

trajectories are ell<u>ipses</u>

$(\Rightarrow \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2 = \text{const}$: Conservation of Energy)

since $\omega^2 = k/m$

Example: $\dot{\vec{x}} = A\vec{x}$, $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$: x, y equations decoupled

$$\Rightarrow \left.\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned}\right\} \Rightarrow \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} \end{aligned}$$

$x_0 e^{at}$ ← decays exponentially if $a<0$, grows if $a>0$

$y_0 e^{-t}$ ← decays exponentially

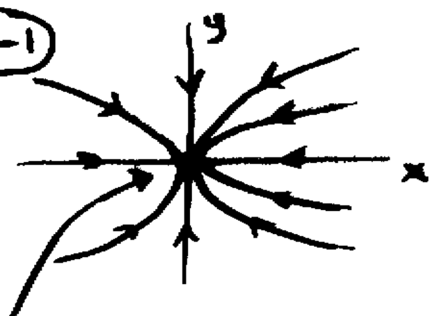
Trajectories: $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{y}{ax} \Rightarrow \ln|y| = -\frac{1}{a}\ln|x| + \tilde{c}$

$\Rightarrow$ trajectories have the form $y = c|x|^{-1/a}$

We can write the solution as $\vec{x}(t) = x_0 e^{at}\begin{pmatrix}1\\0\end{pmatrix} + y_0 e^{-t}\begin{pmatrix}0\\1\end{pmatrix}$

Phase portraits:

$a<-1$



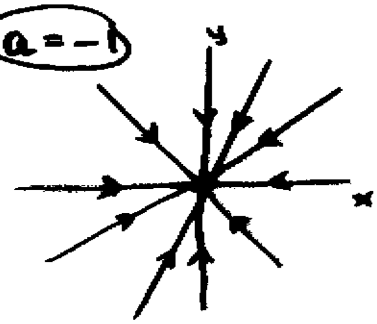
(0,0) stable node

$x(t)$ decays more rapidly than $y(t)$

$\Rightarrow$ trajectories approach origin tangent to y-axis (slower direction)

In backwards time, $t \to -\infty$, trajectories parallel to x-axis

$a=-1$

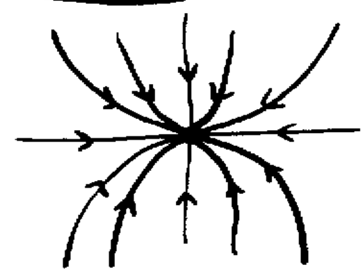


(0,0) star

$x(t)$, $y(t)$ decay at equal rates

$\Rightarrow$ trajectories straight lines

$-1<a<0$

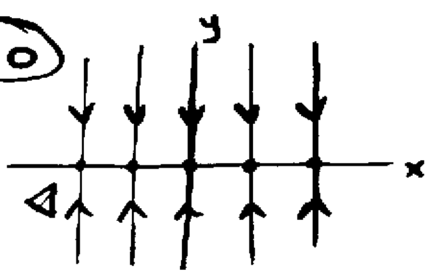


(0,0) stable node

$y(t)$ decays more rapidly than $x(t)$

Trajectories approach $\vec{x}^*$ tangent to x-axis

$a=0$



line of fixed points

$x(t) = x_0 =$ const.

$a>0$

(0,0) saddle point (unstable)

exponential growth in x direction

- most trajectories move away from $\vec{x}^*$, unless $x_0 = 0$

Saddle point $\vec{x}^*$:

one stable, one unstable direction.

Stable manifold of $\vec{x}^*$: Set of initial conditions $\vec{x}_0$ s.t. $\vec{x}(t) \to \vec{x}^*$ as $t \to \infty$ (here: y-axis)

Unstable manifold: set of $\vec{x}_0$ s.t. $\vec{x}(t) \to \vec{x}^*$ as $\boxed{t \to -\infty}$

(think: reverse direction of arrows) (here: x-axis)

A typical trajectory asymptotically approaches the unstable manifold as $t \to +\infty$, and the stable manifold as $t \to -\infty$. !

Stability: • $\vec{x}^*$ is an attracting fixed point:

all trajectories starting sufficiently nearby approach $\vec{x}^*$ as $t \to \infty$

$$\left(\exists\, \delta > 0 \text{ s.t. } \|\vec{x}_0 - \vec{x}^*\| < \delta \Rightarrow \lim_{t\to\infty} \|\vec{x}(t) - \vec{x}^*\| = 0 \right)$$

some norm in $\mathbb{R}^2$
eg Euclidean distance

Globally attracting: $\vec{x}(t) \to \vec{x}^*$ for all $\vec{x}_0$ $(\delta = \infty)$

• $\vec{x}^*$ is Liapunov stable: all trajectories starting close enough to $\vec{x}^*$ remain close for all $t > 0$

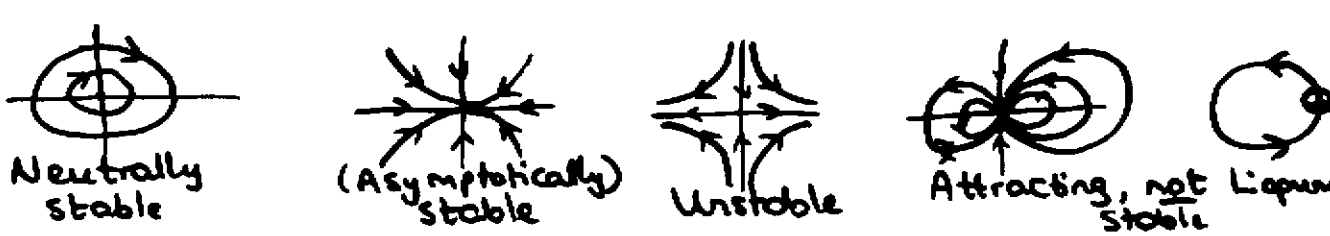
$$\left(\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ s.t. } \|\vec{x}_0 - \vec{x}^*\| < \delta \Rightarrow \|\vec{x}(t) - \vec{x}^*\| < \varepsilon \ \forall t \geq 0 \right)$$

• Neutrally stable: Liapunov stable, not attracting

• (Asymptotically) stable: Liapunov stable and attracting

• Unstable: neither Liapunov stable nor attracting

eg



Classification of Linear Systems

$$\dot{\vec{x}} = A\vec{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d \in \mathbb{R}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{ie} \quad \begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

Seek solutions of the form $\boxed{\vec{x}(t) = e^{\lambda t}\vec{v}}$

for a fixed vector $\vec{v} \neq \vec{0}$, and (complex) scalar λ to be determined.

(solution corresponds to growth/decay along line in direction $\vec{v}$) — for $\lambda \in \mathbb{R}$

Substitute: $\dot{\vec{x}} = \lambda e^{\lambda t}\vec{v}$, so $\lambda \cancel{e^{\lambda t}}\vec{v} = A\cancel{e^{\lambda t}}\vec{v}$

$$\Rightarrow \boxed{A\vec{v} = \lambda\vec{v}}$$

eigenvector of A $\vec{v} \neq \vec{0}$ (defined up to multiplicative constant)

eigenvalue λ corresponding to $\vec{v}$

$$\Rightarrow (A - \lambda I)\vec{v} = \vec{0}$$

identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Nontrivial solutions $\vec{v} \neq \vec{0}$ exist provided $(A - \lambda I)$ is singular (noninvertible)

$$\Rightarrow \boxed{\det(A - \lambda I) = 0}$$ **characteristic equation** of A

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - \underbrace{(a+d)}_{\tau = \operatorname{tr} A}\lambda + \underbrace{(ad - bc)}_{\Delta = \det A} = 0$$

$\tau = \operatorname{tr} A$ — trace A (sum of diagonal entries)

$\Delta = \det A$ — determinant of A

$$\lambda^2 - \tau\lambda + \Delta = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

If $\lambda_1 \neq \lambda_2$, then $\vec{v}_1, \vec{v}_2$ are linearly independent, so: 2 indep. solns $\begin{cases} \vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1 \\ \vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 \end{cases}$

General solution of $\dot{\vec{x}} = A\vec{x}$:

$$\boxed{\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2}$$

c_1, c_2 from initial conditions $\vec{x}(0) = \vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

Example: $\begin{cases} \dot{x} = -3x+4y \\ \dot{y} = -2x+3y \end{cases}$, $\begin{matrix} x(0)=5 \\ y(0)=2 \end{matrix}$ $\Rightarrow$ $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \dot{\vec{x}} = A\vec{x}$, $A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Find eigenvalues and eigenvectors of A:

Characteristic equation $\det(A - \lambda I) = 0$

$$\Rightarrow \det\begin{pmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{pmatrix} = (-3-\lambda)(3-\lambda)+8 = \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda_1 = +1, \quad \lambda_2 = -1$$

$\lambda_1 = 1$: corresponding eigenvector $\vec{v}_1$:

$A\vec{v}_1 = \lambda_1\vec{v}_1 \Rightarrow (A - \lambda_1 I)\vec{v}_1 = \vec{0}$ let $\vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -4a+4b=0 \\ -2a+2b=0 \end{cases}$ (redundant system – needed for nontrivial solutions)

$\vec{v}_1$ is determined only up to a multiplicative constant

$\Rightarrow$ can choose a (or b) arbitrarily.

Choose $a=1 \Rightarrow b=1$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ← or any scalar multiple

$\Rightarrow \vec{x}_1(t) = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution.

$\lambda_2 = -1$: corresponding eigenvector $\vec{v}_2$:

$(A - \lambda_2 I)\vec{v}_2 = \vec{0} \Rightarrow \begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2a+4b=0$

Choose $b=1 \Rightarrow a=2$: $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{x}_2(t) = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

General solution: $\vec{x}(t) = c_1 e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

exponential growth in the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\lambda_1 > 0$
unstable direction

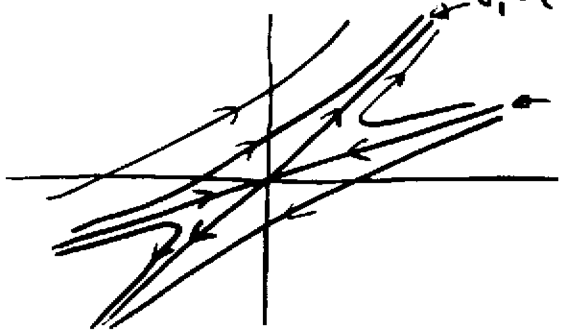
exponential decay in the direction $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$: $\lambda_2 < 0$
stable direction

Initial condition: $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{rcases} c_1 + 2c_2 = 5 \\ c_1 + c_2 = 2 \end{rcases} \Rightarrow c_1 = -1, \ c_2 = 3.$

Solution: $\vec{x}(t) = -e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$: $\begin{cases} x(t) = -e^t + 6e^{-t} \\ y(t) = -e^t + 3e^{-t} \end{cases}$

Phase portrait: $\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$; $\lambda_2 = -1,\ \vec{v}_2 = \begin{pmatrix}2\\1\end{pmatrix}$



$\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ unstable manifold

$\vec{v}_2 = \begin{pmatrix}2\\1\end{pmatrix}$ stable manifold

$\lambda_2 < 0 < \lambda_1$

Saddle point.

In general:

Real, distinct eigenvalues $\quad \lambda_1, \lambda_2$ real, $\lambda_1 \neq \lambda_2$

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

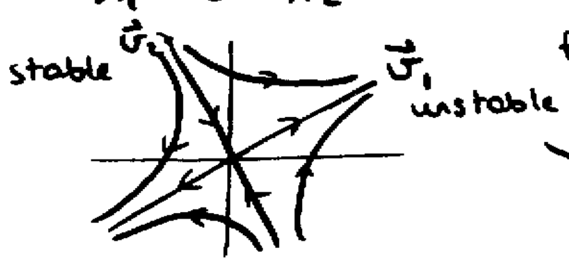
If $\vec{x}_0$ is in the direction $\vec{v}_1$ $(c_1 \neq 0, c_2 = 0)$, solution is $c_1 e^{\lambda_1 t}\vec{v}_1$
exponential growth / decay in direction $\vec{v}_1$:
trajectory is a straight line. Similarly if $\vec{x}_0 = c_2\vec{v}_2$.

In general the solution is a linear combination.
If $\lambda_1 > \lambda_2$, long-time behaviour is in direction $\vec{v}_1$:

$$\vec{x}(t) = e^{\lambda_1 t}\big(c_1\vec{v}_1 + c_2 \underbrace{e^{(\lambda_2-\lambda_1)t}}_{\to 0}\vec{v}_2\big)$$

Phase portraits:

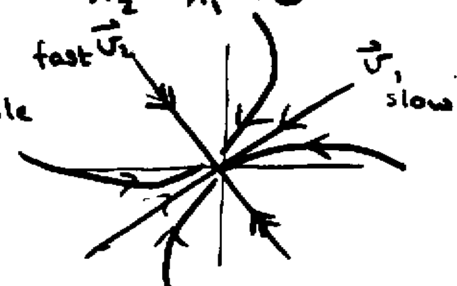
$\lambda_1 > 0 > \lambda_2$



saddle point

typical trajectories approach unstable manifold as $t \to +\infty$, approach stable manifold as $t \to -\infty$

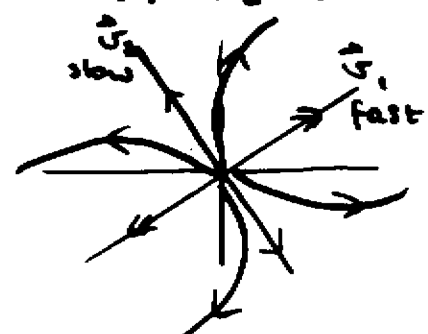
$\lambda_2 < \lambda_1 < 0$



stable node

trajectories approach origin tangent to slower eigendirection

$\lambda_1 > \lambda_2 > 0$



unstable node

trajectories diverge from origin parallel to faster eigendirection

Complex eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

$\tau^2 < 4\Delta \Rightarrow$ eigenvalues are complex conjugates

Write $\lambda_{1,2} = \alpha \pm i\omega$, $\quad \alpha = \frac{\tau}{2}$, $\omega = \frac{1}{2}\sqrt{4\Delta - \tau^2} \neq 0$

[Note: λ complex $\Rightarrow$ corresponding eigenvector $\vec{v}$ has complex components

$A\vec{v} = \lambda\vec{v} \underset{\uparrow}{\Rightarrow} A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$ ie $\bar{\vec{v}}$ is the eigenvector

take complex conjugate: A is real $\quad$ corresponding to the eigenvalue $\bar{\lambda}$]

General solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

but $e^{\lambda_1 t} = e^{(\alpha + i\omega)t} = e^{\alpha t} e^{i\omega t} = e^{\alpha t}(\cos\omega t + i\sin\omega t)$

$|e^{i\omega t}| = 1 \qquad e^{\lambda_2 t} = e^{(\alpha - i\omega)t} = e^{\alpha t} e^{-i\omega t} = e^{\alpha t}(\cos\omega t - i\sin\omega t)$.

The general solution is a linear combination of

$e^{\alpha t}\cos\omega t$ and $e^{\alpha t}\sin\omega t$

$\mathrm{Re}\,\lambda = \alpha < 0$: decaying oscillations $\quad (e^{\alpha t} \to 0$ as $t \to \infty)$

$\vec{x}^* = \vec{0}$ is a stable spiral

$\alpha > 0$: growing oscillations : unstable spiral

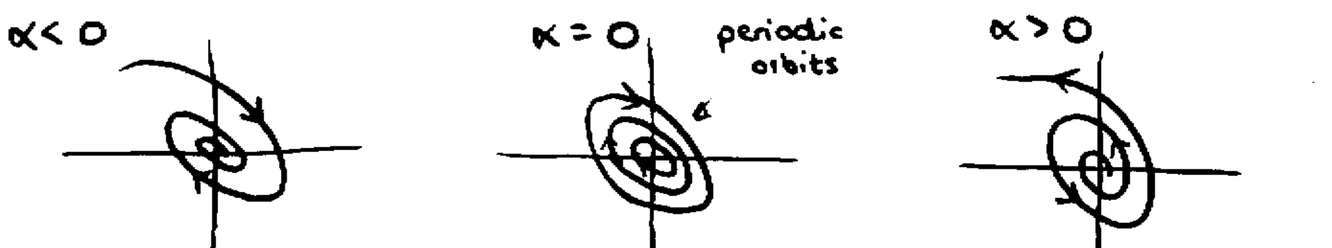
$\alpha = 0$: (pure imaginary eigenvalues)

all solutions periodic, period $T = 2\pi/\omega$. centre

(Liapunov stable, not attracting)

(determine the sense of rotation by computing $\dot{\vec{x}}$ at some points eg on axes: clockwise / counterclockwise

Phase portraits:



stable spiral $\qquad$ centre $\qquad$ unstable spiral

Equal eigenvalues

$$\lambda_1 = \lambda_2 = \lambda = \tau/2 \qquad (\tau^2 = 4\Delta)$$

Case 1: Two independent eigenvectors $\vec{v}_1, \vec{v}_2$ $\quad A\vec{v}_1 = \lambda\vec{v}_1$, $A\vec{v}_2 = \lambda\vec{v}_2$

$\Rightarrow$ any vector in the plane is an eigenvector:

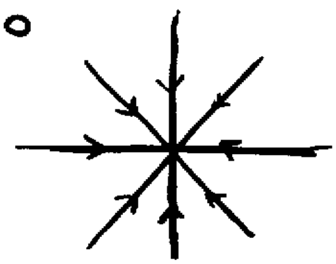
$\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$: $\vec{x}_0 \in \mathbb{R}^2 \Rightarrow \vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

$$\Rightarrow A\vec{x}_0 = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \lambda\vec{x}_0$$

$\Rightarrow$ multiplication by A stretches every vector by the factor λ

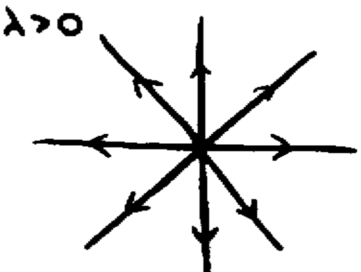
$$\Rightarrow \quad A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$\lambda < 0$



stable star node

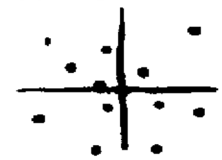
$\lambda > 0$



unstable star node

$\lambda = 0 \Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\dot{\vec{x}} = \vec{0}$



$\lambda = 0$: every point is a fixed point.

Case 2: Only one (independent) eigenvector $\vec{v}_1$ $\quad A\vec{v}_1 = \lambda\vec{v}_1$

(the eigenspace of A corresponding to λ is one-dimensional

One solution of $\dot{\vec{x}} = A\vec{x}$: $e^{\lambda t}\vec{v}_1$ need a second solution

Example:

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Let $\vec{v}_2$ be a generalized eigenvector :

$$(A - \lambda I)^2 \vec{v}_2 = \vec{0} \qquad \text{ie} \quad (A - \lambda I)\big[\underbrace{(A - \lambda I)\vec{v}_2}_{= \alpha\vec{v}_1}\big] = \vec{0}$$

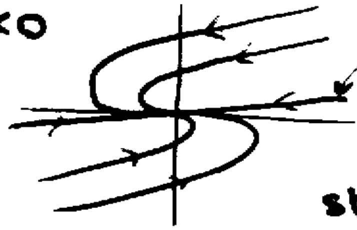
General solution:

$$\vec{x}(t) = e^{\lambda t}\big(c_1\vec{v}_1 + c_2\vec{v}_2 + c_2 t \underbrace{(A - \lambda I)\vec{v}_2}_{\alpha\vec{v}_1}\big)$$

all trajectories become parallel to $\vec{v}_1$ as $t \to +\infty$ and $t \to -\infty$

Phase portrait:

$\lambda < 0$



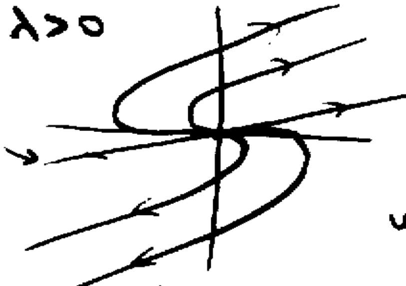
$\vec{v}_1$

stable

improper node/
degenerate node

$\lambda > 0$

$\vec{v}_1$

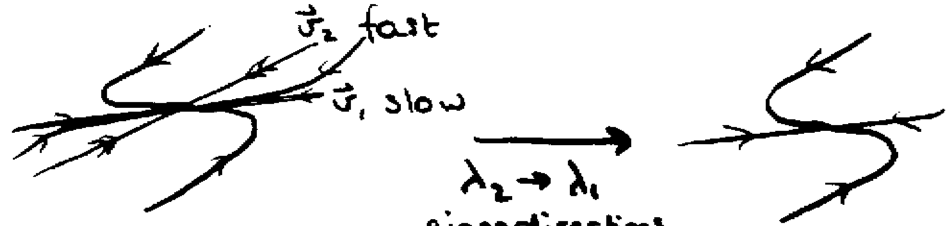


unstable

improper/degenerate node

Degenerate node: obtained by deforming an ordinary node

eg $\lambda_2 < \lambda_1 < 0$: 2 independent eigendirections $\vec{v}_1, \vec{v}_2$

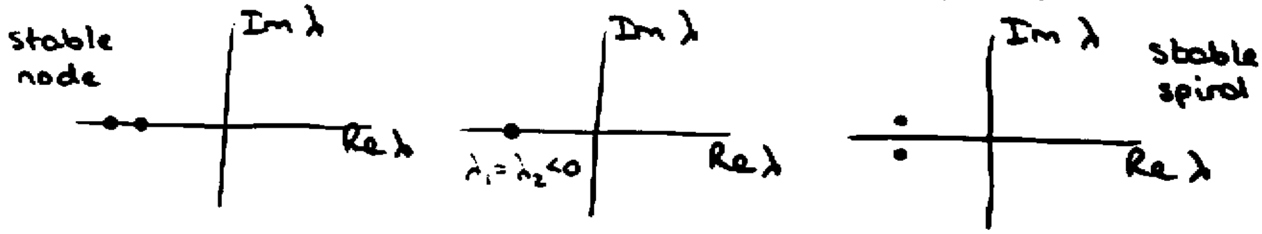


stable node

eigendirections $\vec{v}_2, \vec{v}_1$ approach each other

degenerate node

Alternative interpretation: eigenvalues in complex plane



degenerate node: intermediate between spiral and node (trajectories "almost" spiral)

Classification

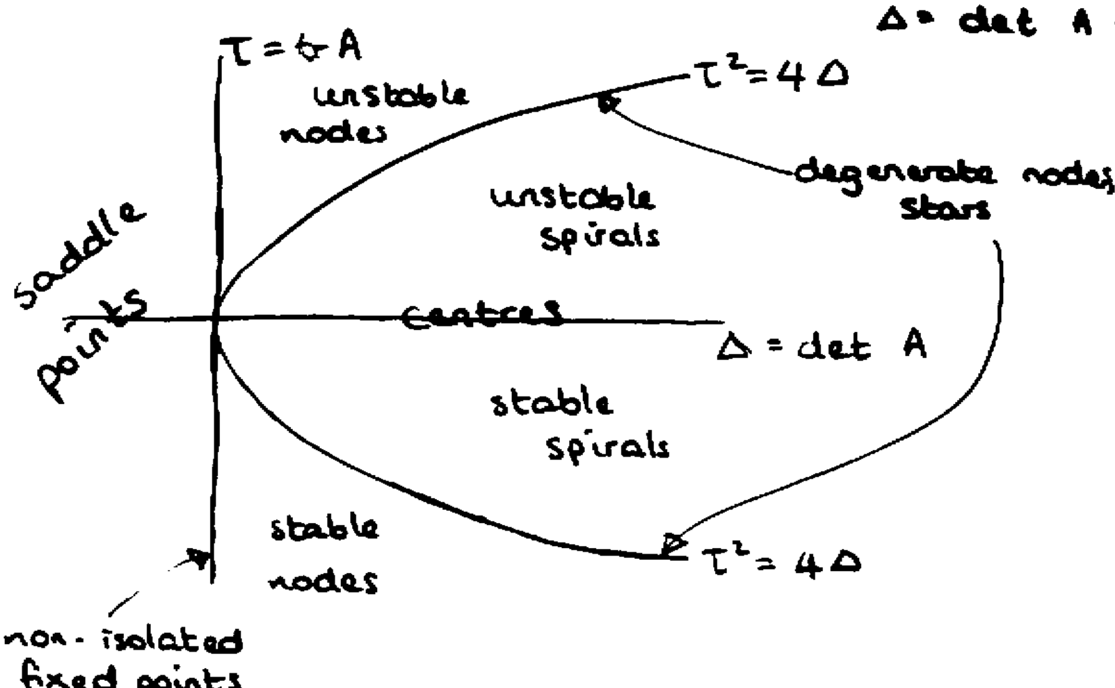
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A - \lambda I)$

$= \lambda^2 - \underbrace{(a+d)}_{\tau = \operatorname{tr} A}\lambda + \underbrace{(ad-bc)}_{\Delta = \det A} = 0$

$$\Rightarrow \quad \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

$\tau = \text{trace } A = \lambda_1 + \lambda_2$

$\Delta = \det A = \lambda_1 \lambda_2$



* $\Delta = \det A = 0 \Rightarrow \vec{x}^* = \vec{0}$ is not an isolated fixed point: line or plane of fixed points