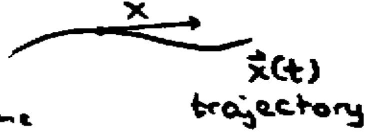
Phase Portraits

$$\left.\begin{array}{l}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{array}\right\} \quad \text{or} \quad \dot{\vec{x}} = \vec{f}(\vec{x})$$



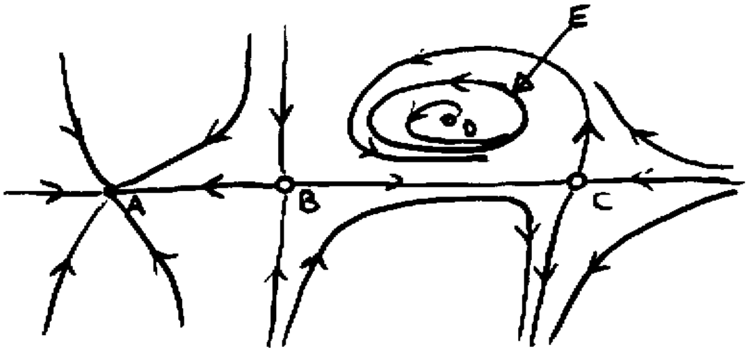
Goal: qualitative information based on the location and stability of fixed points and periodic orbits.

Fixed points: solve simultaneously

$$\left.\begin{array}{l}\dot{x} = 0\\ \dot{y} = 0\end{array}\right\} \text{ ie } \left.\begin{array}{l}f(x,y) = 0\\ g(x,y) = 0\end{array}\right\} \qquad (\vec{f}(\vec{x}) = \vec{0})$$

Closed orbits: $\vec{x}(t+T) = \vec{x}(t)$ for all t, some $T > 0$

$\Rightarrow$ periodic solutions



Fixed points A, B, C, D

Closed orbit E
(limit cycle)

A, E stable

B, C, D unstable

A: node, D: spiral

B, C: saddle point

(qualitatively similar local flows)

Nullclines: x-nullcline: curve where $\dot{x} = 0$ ie $f(x,y) = 0$

$\uparrow$ in x-y plane (phase plane)

Along the x-nullcline, the flow is purely vertical

The x-nullcline separates regions with flow to the right ($\dot{x} > 0$) and flow to the left ($\dot{x} < 0$)

y-nullcline: curve where $\dot{y} = 0$ ie $g(x,y) = 0$

– flow is horizontal along a y-nullcline.

Note: Fixed points are at intersections of an x-nullcline and a y-nullcline (since then $\dot{x} = \dot{y} = 0$)

Existence and Uniqueness

Theorem: Consider the initial-value problem
$$\dot{\vec{x}} = \vec{f}(\vec{x}), \quad \vec{x}(0) = \vec{x}_0 \qquad (\vec{x} \in \mathbb{R}^n, \ \vec{f} : \mathbb{R}^n \to \mathbb{R}^n)$$
Suppose $\vec{f}$ and all its partial derivatives $\frac{\partial f_i}{\partial x_j}$ $(i,j = 1..n)$ exist and are continuous for all $\vec{x} \in D$, where $D \subset \mathbb{R}^n$ is an open, connected set containing $\vec{x}_0$.

Then there exists $\tau > 0$ s.t. the initial-value problem has a unique solution $\vec{x}(t)$ on the time interval $(-\tau, \tau)$.

(Existence, uniqueness guaranteed if $\vec{f}$ is continuously differentiable : not the strongest result, sufficient for our purposes)

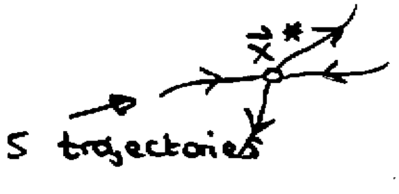
Corollary: Different trajectories never intersect

- a trajectory cannot move in two directions at once

(suppose trajectories intersect at $\vec{x}_0$: then there would be two solutions with initial data at $\vec{x}_0$, violating uniqueness)

No!

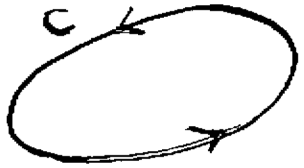
⌈ What about fixed points?



$\vec{x}(t) = \vec{x}^*$ is a trajectory

other trajectories approach / leave $\vec{x}^*$ asymptotically : no intersections in finite time

Consequence in 2-d:



A closed orbit C separates space

(a trajectory starting at $\vec{x}_0$ inside C is trapped there

$\Rightarrow$ trajectory approaches C, approaches a fixed point, is / approaches a periodic orbit)

(Poincaré - Bendixson Theorem)

Linearization

$$\left.\begin{aligned}\dot x &= f(x,y)\\ \dot y &= g(x,y)\end{aligned}\right\}$$

Suppose (x^*, y^*) is a fixed point : $f(x^*,y^*) = g(x^*,y^*) = 0$

Small disturbances: $u = x - x^*$, $v = y - y^*$

Then
$$\dot u = \dot x = f(x,y) = f(x^*+u, y^*+v)$$

Taylor expand $\cong \underbrace{\cancel{f(x^*,y^*)}}_{=0} + u \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)}$

$$+ \tfrac{1}{2}u^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,y^*} + uv \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x^*,y^*)} + \tfrac{1}{2}v^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x^*,y^*)} + \dots$$

$$\Rightarrow \dot u = \underset{\uparrow}{\frac{\partial f}{\partial x}} u + \underset{\uparrow}{\frac{\partial f}{\partial y}} v + \underset{\uparrow}{O}(u^2, uv, v^2)$$

partial derivatives evaluated at (x^*,y^*)

quadratic terms in u, v (and higher-order) $\big[O(2)$

Similarly
$$\dot v = \frac{\partial g}{\partial x} u + \frac{\partial g}{\partial y} v + O(u^2, uv, v^2)$$

$$\Rightarrow \begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = \underbrace{\left.\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\right|_{(x^*,y^*)}}_{A\ :\ \text{Jacobian matrix}\ (J)} \begin{pmatrix}u\\ v\end{pmatrix} + \text{quadratic terms}$$

$A = \left.\dfrac{\partial(f,g)}{\partial(x,y)}\right|_{(x^*,y^*)}$ (Derivative matrix) evaluated at fixed point (x^*,y^*)

$A = D\vec f(\vec x^*)$ (analogue of $f'(x^*)$ in 1-d)

Small disturbances: neglect nonlinear (quadratic and higher-order) terms :

<u>Linearized system</u> about (x^*,y^*):

$$\boxed{\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = A\begin{pmatrix}u\\ v\end{pmatrix}}$$

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\}\qquad \text{Fixed point } (x^*,y^*) : \left.\begin{aligned} f(x^*,y^*)&=0\\ g(x^*,y^*)&=0\end{aligned}\right\}$$

Linearization near (x^*,y^*) :

$$\boxed{\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = A\begin{pmatrix}u\\ v\end{pmatrix}, \quad A = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\Bigg|_{(x^*,y^*)}}$$

Near the fixed point, for small u,v, we might expect that the nonlinear terms are extremely small, and may be neglected to obtain the qualitative dynamics near (x^*,y^*).

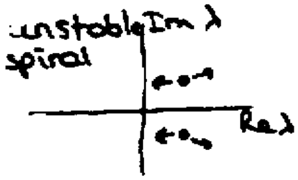
Is this valid ?

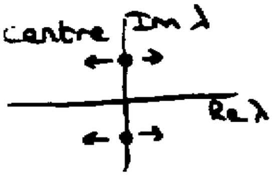
Yes, provided the fixed point of the linearized system is not a "borderline case" ...

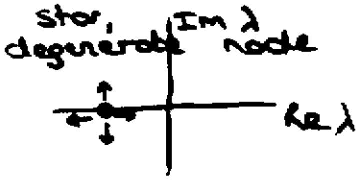
Eigenvalues	Prediction of linearized system	Qualitative dynamics near fixed point of nonlinear system	
Real, opp. sign	Saddle	saddle	Stability the same for linearized, nonlinear system
Real, same sign	Node	node	
Complex conjugate $Re(\lambda) \neq 0$	Spiral	spiral	
Pure imaginary	Centre	centre, stable spiral or unstable spiral	Linearization does not predict stability
One or two zero eigenvalues	Nonisolated fixed point	various ...	
Real, equal	Star, degenerate node	node or spiral	Linearization predicts stability type (qualitative dynamics) affected by small nonlinear terms

Consider the effect of nonlinearities as small perturbations to eigenvalues in the complex plane

eg







eg $\left.\begin{aligned} \dot{x} &= 3y - y^2 \\ \dot{y} &= x - 2y \end{aligned}\right\}$

x-nullclines $\dot{x} = 0 \Rightarrow y(3-y) = 0 \Rightarrow y = 0$ or $y = 3$

y-nullcline $\dot{y} = 0 \Rightarrow x = 2y$

Fixed points: $\dot{x} = 0$ and $\dot{y} = 0$: $(0,0), (6,3)$

Linearization: $A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 3-2y \\ 1 & -2 \end{pmatrix}$

Near $(0,0)$: $A\big|_{(0,0)} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$

eigenvalues: $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow (\lambda+3)(\lambda-1) = 0 \Rightarrow \lambda = 1, -3$

$(0,0)$ is a saddle point

eigenvectors:

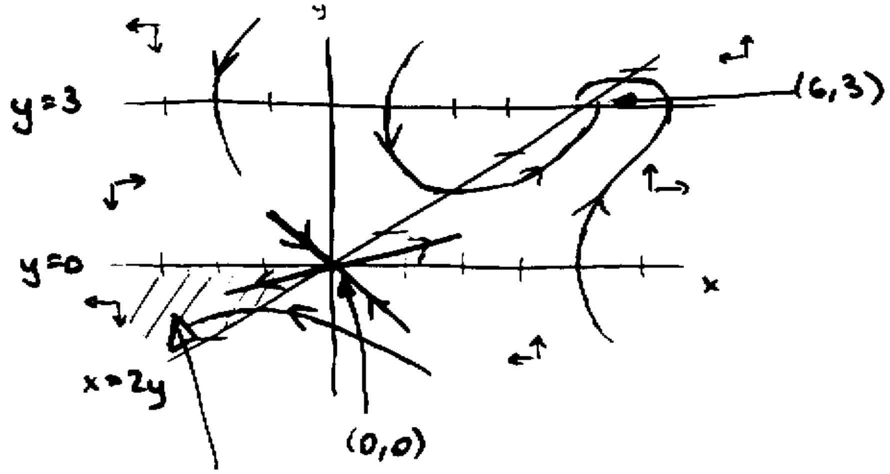
$\lambda_1 = 1$: $\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $\lambda_2 = -3$: $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\qquad$ $\Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Near $(6,3)$: $A\big|_{(6,3)} = \begin{pmatrix} 0 & -3 \\ 1 & -2 \end{pmatrix}$

eigenvalues: $\lambda^2 + 2\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = -1 \pm i\sqrt{2}$

$(6,3)$ is a stable spiral



Defn: A trapping region $R \subset \mathbb{R}^2$ is a closed connected set so that the vector field points "inward everywhere on ∂R (bounda

$\vec{f} \cdot \hat{n} \leq 0$, $\vec{x} \in \partial R$

Once a trajectory enters R, it can never leave (in positive time)

Example: linear stability analysis fails

(non-hyperbolic fixed point)

eg $$\left.\begin{aligned}\dot{x} &= -y + ax(x^2+y^2)\\ \dot{y} &= x + ay(x^2+y^2)\end{aligned}\right\}$$ near fixed point $(0,0)$

Linearization : $\dot{x} = -y$, $\dot{y} = x$ (drop nonlinear terms)

or : $$A = \left.\begin{pmatrix} a(3x^2+y^2) & -1+2axy \\ 1+2axy & a(3y^2+x^2)\end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\tau = 0$, $\Delta = 1 > 0$

eigenvalues $\lambda_{1,2} = \pm i$:

$(0,0)$ is a centre of the linearized system (all a)

Full nonlinear system : Change to polar coordinates

$\boxed{x = r\cos\theta,\ y = r\sin\theta} \Leftrightarrow r^2 = x^2+y^2$, $\tan\theta = y/x$

$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \Rightarrow$ $\boxed{\begin{aligned}\dot{r} &= \frac{x\dot{x}+y\dot{y}}{r}\\ \dot{\theta} &= \frac{x\dot{y}-y\dot{x}}{r^2}\end{aligned}}$

and $\underbrace{\sec^2\theta}_{(r/x)^2}\,\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{x^2} \Rightarrow$

Substitute:

$$\left.\begin{aligned} r\dot{r} &= x\dot{x}+y\dot{y} = a(x^2+y^2)(x^2+y^2) = ar^4 &&\Rightarrow \dot{r} = ar^3\\ \dot{\theta} &= \frac{x\dot{y}-y\dot{x}}{r^2} = \frac{x^2+y^2}{r^2} &&\Rightarrow \dot{\theta} = 1\end{aligned}\right\}$$

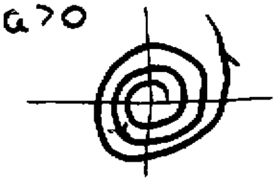
So: $r=0$ is stable if $a<0$, unstable if $a>0$

$a<0$: $r(t) \to 0$ monotonically : stable spiral

$a=0$: $\dot{r} = 0 \Rightarrow r(t) = \text{const} = r_0$: centre

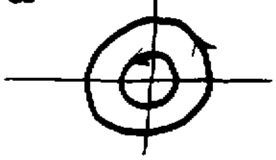
$a>0$: $\dot{r} > 0 \Rightarrow r(t)$ grows : unstable spiral at $(0,0)$

$a>0$



unstable spiral

$a=0$



centre

$a<0$

stable spiral

Centre: trajectories must "close perfectly" after one cycle - else spiral

Hyperbolicity and Structural Stability

$\dot{\vec{x}} = \vec{f}(\vec{x})$: $\vec{x}^*$ is a fixed point : $\vec{f}(\vec{x}^*) = \vec{0}$

ummary:
Re(λ_i) < 0, all i : stable
Re(λ_i) > 0, some i : unstable

Linearization: $\dot{\vec{u}} = A\vec{u}$,
where the linearized matrix $A = D\vec{f}(\vec{x}^*)$
has components $A_{ij} = \frac{\partial f_i}{\partial x_j}(\vec{x}^*)$. eigenvalues λ_i

Defn: The fixed point $\vec{x}^*$ is hyperbolic if Re(λ_i) ≠ 0, for all $i = 1, \ldots, n$ ie all eigenvalues of the linearization (Jacobian matrix A evaluated at $\vec{x}^*$) have nonzero real parts.

eigenvalues off imaginary axis

[nonhyperbolic: Re(λ_i) = 0, some i]

Hyperbolic fixed points are "robust" (structurally stable): stability is unaffected by small perturbations/nonlinearities ⇒ stability predicted by linearization.

Structural Stability: the topology of the phase portrait is unchanged by an arbitrarily small perturbation to the vector field $\vec{f}(\vec{x})$.

continuous deformation, bending of trajectories allowed, not ripping.

eg a saddle point is structurally stable (hyperbolic)
a centre is not (nonhyperbolic)
(small damping ⇒ spiral)

Hartman-Grobman Theorem:

If $A = D\vec{f}(\vec{x}^*)$ has no zero or purely imaginary eigenvalues (ie $\vec{x}^*$ is a hyperbolic fixed point) then there is a homeomorphism (one-to-one, continuous map with a continuous inverse) defined on some neighbourhood $U \subset \mathbb{R}^n$ of $\vec{x}^*$, which locally takes trajectories of the nonlinear system $\dot{\vec{x}} = \vec{f}(\vec{x})$ to those of the linearized system $\dot{\vec{u}} = A\vec{u}$, and preserves the sense of time.

⇒ The local phase portrait near a hyperbolic fixed point is topologically equivalent to that of its linearization.

Population Dynamics : Competition Model :

Lotka-Volterra model of competition:
two species competing for some limited resource

eg $x(t)$ rabbits, $y(t)$ sheep, $x, y \geq 0$

- consume the same resource (grass)
- rabbits reproduce faster, sheep eat more

} Assumptions

$$\dot{x} = \underbrace{x(3-x)}_{\text{logistic growth in absence of sheep}} - 2xy$$

competition term: rabbits inhibited by interactions with sheep – proportional to size of each population

$$\dot{y} = y(2-y) - xy$$

sheep: lower natural growth rate, less inhibition

$$\left.\begin{aligned} \dot{x} &= x(3-x-2y) \\ \dot{y} &= y(2-x-y) \end{aligned}\right\}$$

Fixed points: (0, 0) both species extinct
(3, 0) rabbits, no sheep
(0, 2) sheep, no rabbits
(1, 1) coexistence: both rabbits and sheep

Nullclines: x-nullclines $x = 0$, $y = \frac{1}{2}(3-x)$
y-nullclines $y = 0$, $y = 2-x$

Jacobian:

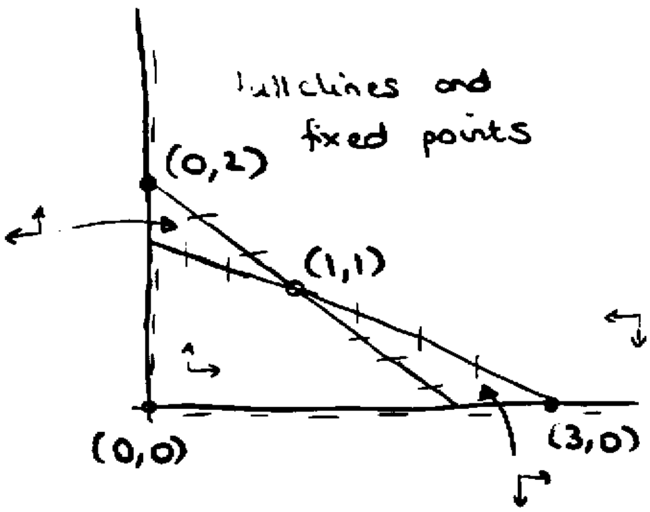
$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-2y-x \end{pmatrix}$$

$(0,0)$: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$: $\tau = 5$, $\Delta = 6$: $\lambda_1 = 3$, $\vec{v}_1 = \binom{1}{0}$; $\lambda_2 = 2$, $\vec{v}_2 = \binom{0}{1}$ unstable node

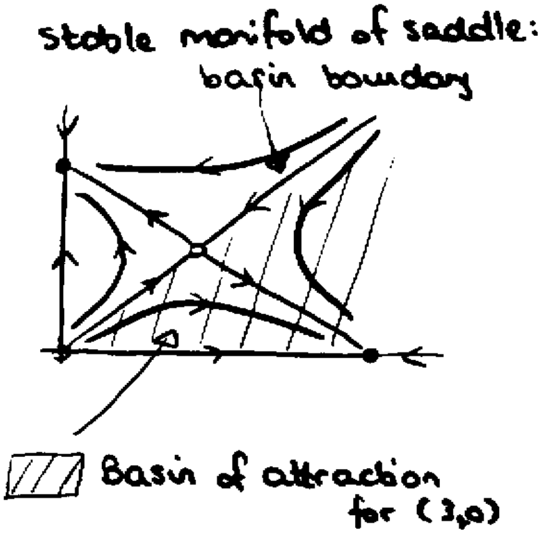
$(3,0)$: $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$: $\tau = -4$, $\Delta = 3$: $\lambda_1 = -3$, $\vec{v}_1 = \binom{1}{0}$; $\lambda_2 = -1$, $\vec{v}_2 = \binom{3}{-1}$ stable node

$(0,2)$: $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$: $\tau = -3$, $\Delta = 2$: $\lambda_1 = -1$, $\vec{v}_1 = \binom{1}{-2}$; $\lambda_2 = -2$, $\vec{v}_2 = \binom{0}{1}$ stable node

$(1,1)$: $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$: $\tau = -2$, $\Delta = -1$: $\lambda_{1,2} = -1 \pm \sqrt{2}$ saddle point



Phase portrait



Defn: <u>Basin of attraction</u> of a fixed point $\vec{x}^* \in \mathbb{R}^n$:

the set of initial conditions $\vec{x}_0$ s.t. $\vec{x}(t) \to \vec{x}^*$ as $t \to \infty$

$\vec{x}(0) = \vec{x}_0$

orbits which form the boundary of the basin of attraction:

basin boundary, separatrix

- here: stable manifold of saddle.

This example demonstrates:

<u>Principle of Competitive Exclusion</u>:

> Two species competing for the same limited resource typically cannot coexist.

General Competition Model:

$$\begin{rcases} \dot{x} = a_1 x - b_1 x^2 - c_1 x y \\ \dot{y} = a_2 y - b_2 y^2 - c_2 x y \end{rcases} \qquad a_i, b_i, c_i > 0.$$

[Note: $c_1, c_2 > 0$: <u>Competition</u> - both species suffer from interaction

Also
- $c_1 = 0, c_2 < 0$: <u>Commensalism</u> - a relationship between two species in which one (y) profits, the other (x) is unaffected
- $c_1 < 0, c_2 < 0$: <u>Mutualism</u> - both species profit]

$$\left.\begin{aligned}\dot{x} &= x\,(a_1 - b_1 x - c_1 y)\\ \dot{y} &= y\,(a_2 - c_2 x - b_2 y)\end{aligned}\right\}$$

(assume $a_i, b_i, c_i > 0$
$b_1 b_2 \neq c_1 c_2$)

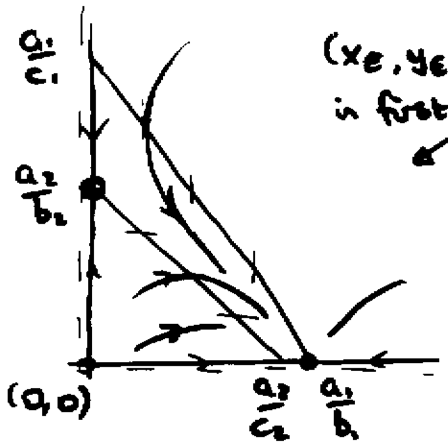
Fixed points: $(0,0)$, $(0, a_2/b_2)$, $(a_1/b_1, 0)$, (x_E, y_E).

Nullclines: x-nullclines $x = 0$, & $a_1 - b_1 x - c_1 y = 0$

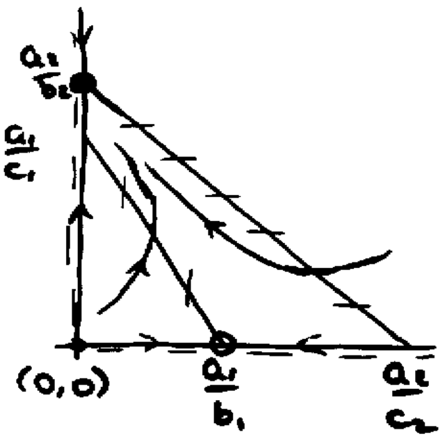
y-nullclines $y = 0$, & $a_2 - c_2 x - b_2 y = 0$

Four cases:

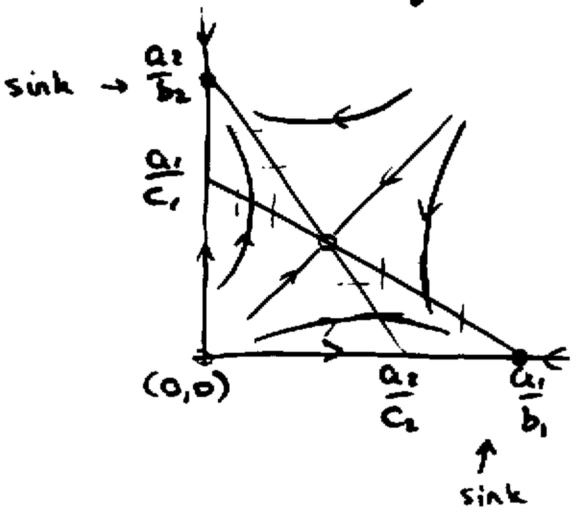
$$x_E = \frac{a_1 b_2 - c_1 a_2}{b_1 b_2 - c_1 c_2}, \quad y_E = \frac{b_1 a_2 - a_1 c_2}{b_1 b_2 - c_1 c_2}$$

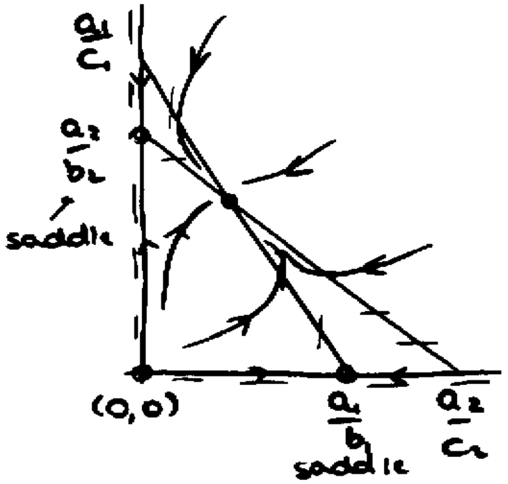


(x_E, y_E) not in first quadrant



Competitive Exclusion





Coexistence!

(x_E, y_E) is a stable node

Conservative Systems

Newton's Law $F = ma = m\ddot{x}$

Assume force $F = F(x)$: F independent of $\dot{x}$
(no damping / friction)
F independent of t (autonomous)

$$\Rightarrow \quad m\ddot{x} = F(x)$$

Define $F(x) = -\frac{dV}{dx}$: $V(x)$: potential energy

$$\Rightarrow \quad m\ddot{x} + \frac{dV}{dx} = 0 \qquad \left[\frac{d}{dt}\left(V(\overset{x(t)}{\overset{\downarrow}{x}})\right) = \frac{dV}{dx}\frac{dx}{dt} \right]$$

multiply by $\dot{x}$

$$\Rightarrow \quad \dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m\dot{x}^2 + V(x)}_{E}\Big) = 0$$

$$\Rightarrow \quad \boxed{E = \tfrac{1}{2}m\dot{x}^2 + V(x)} \text{ is constant in time}$$

$E = E(x, \dot{x})$: total energy = kinetic + potential
- a conserved quantity
(first integral)

Defn: $\dot{\vec{x}} = \vec{f}(\vec{x})$

A <u>conserved quantity</u> $E(\vec{x})$ is a real-valued, continuous function, non-constant on every open set*, which is constant on trajectories, $\frac{dE}{dt} = \frac{d}{dt}E(\vec{x}(t)) = 0$

* to avoid trivial cases eg $E \equiv 1$.

A <u>conservative system</u> is one that has a conserved quantity.

Orbits of $\dot{\vec{x}} = \vec{f}(\vec{x})$ lie on level sets of E

(Note: E is not unique:
if E is conserved, so is any function of E
eg E^2, $\sin(3E)$ etc.)

A conservative system cannot have any attracting fixed points (or repelling)

Suppose $\vec{x}^*$ is an attracting fixed point, basin of attraction D.

Nearby trajectory $\vec{x}(t)$ $[\vec{x}_0 \in D]$

$$\lim_{t\to\infty} \vec{x}(t) = \vec{x}^* \Rightarrow \lim_{t\to\infty} E(\vec{x}(t)) = E(\vec{x}^*)$$

E continuous

E constant on trajectories $\Rightarrow E(\vec{x}) = E(\vec{x}^*) \quad \forall\, \vec{x} \in D$

– contradicts E non-constant on any open set. ■

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eg Double-well potential $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$ $\boxed{m=1}$

$$\Rightarrow F(x) = -\frac{dV}{dx} = x - x^3 \qquad \Rightarrow \ddot{x} = x - x^3$$

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= \underbrace{x - x^3}_{-V'(x)} \end{aligned}\right\}$$

Equilibria: $\dot{x} = \dot{y} = 0$ $\Rightarrow$ 3 fixed points $(-1,0), (0,0), (1,0)$

Nullclines: x $y=0$

y $x = 0, \pm 1$

Jacobian:

$$A = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$$

$$A\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} :$$

$\lambda = \pm 1$ $(0,0)$ is a saddle point

$\underset{\text{unstable}}{\lambda_1 = 1, \vec{v} = \binom{1}{1}}$, $\underset{\text{stable}}{\lambda_2 = -1, \vec{v}_2 = \binom{1}{-1}}$

$$A\big|_{(\pm 1,0)} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} : \quad \lambda = \pm i\sqrt{2}$$

Linear analysis predicts $(\pm 1, 0)$ are centres

Nonlinear terms?

Energy conservation! $\frac{d}{dt}\big(\underbrace{\tfrac{1}{2}y^2 + V(x)}_{E}\big) = 0$

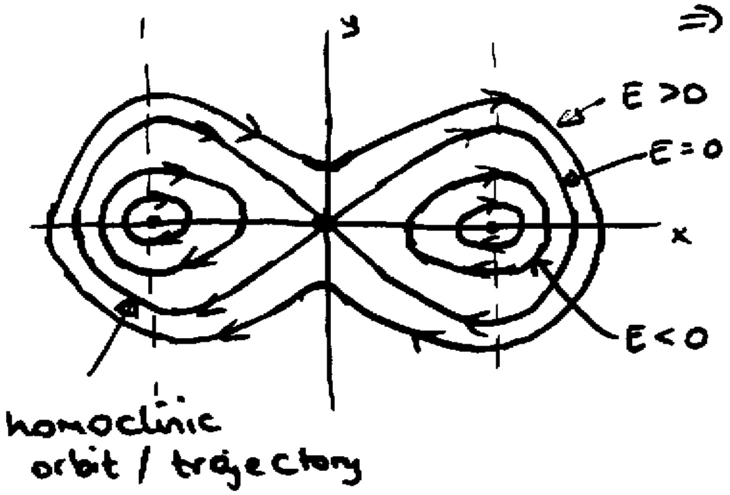
$$\left[\frac{d}{dt}\left(\tfrac{1}{2}y^2 + V(x)\right) = y\dot{y} + V'(x)\dot{x} = y(-V'(x)) + V'(x)y = 0 \right]$$

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= -V'(x) = x - x^3\end{aligned}\right\}$$ Trajectories are closed curves

Level sets (contours of constant energy)

$$E = \tfrac{1}{2}y^2 + V(x) = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 = \text{const}$$

$$\Rightarrow \quad y = \pm\sqrt{2E + x^2 - \tfrac{1}{2}x^4}$$

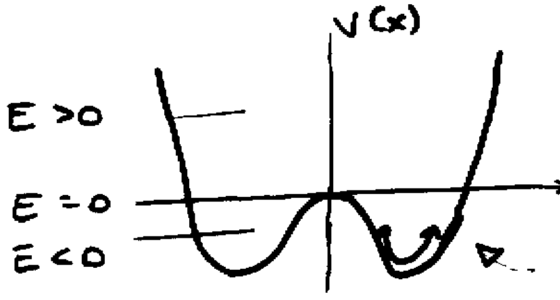


Three fixed points, two trajectories which approach the origin as t → ±∞ ; all other solutions are periodic

Homoclinic orbit : A trajectory that starts and ends at the same fixed point

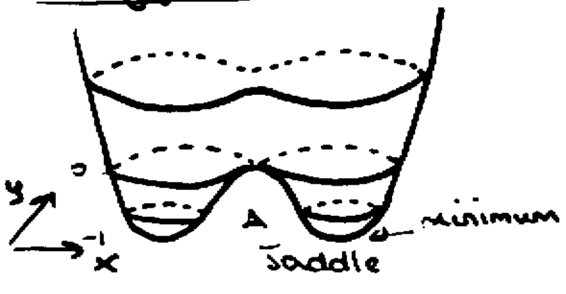
Heteroclinic trajectory (saddle connection) connects two different saddle points.

Homoclinic / heteroclinic orbits – common in conservative systems, or in presence of symmetries, else rare



E<0 : small oscillations about neutrally stable equilibria

Energy surface



$$E(x,y) = \underbrace{\tfrac{1}{2}y^2 - \tfrac{1}{2}x^2}_{\text{saddle}} + \underset{\text{higher order correction}}{\tfrac{1}{4}x^4}$$

Flow around energy surface, maintaining E = const : 2nd order, no dissipation

Nonlinear Centres

- robust for conservative systems
- centres occur at local minima of energy function

(neutrally stable equilibria at bottom of potential well)

Theorem:

$\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in \mathbb{R}^2$, $f \in C^1$, conserved quantity $E(\vec{x})$

> If $\vec{x}^*$ is an isolated local minimum (or maximum) of E, then $\vec{x}^*$ is an isolated fixed point, and all trajectories sufficiently near $\vec{x}^*$ are closed.

⌈ Idea: Trajectories are contained in level contours of E.

$\vec{x} \neq \vec{x}^*$ (sufficiently close) $\Rightarrow E(\vec{x}) \neq E(\vec{x}^*)$,

so $\vec{x}_0 = \vec{x}^* \Rightarrow \vec{x}(t) = \vec{x}^*$: fixed point

("nowhere else to go")

Level sets of E are closed near an extremum,

no nearby fixed points

$\Rightarrow$ nearby trajectories are closed orbits ⌋

Symmetries and Reversible Systems

eg Time-reversal symmetry (in undamped, frictionless mechanical systems)

$\ddot{x} = F(x)$: invariant under $t \to -t$

(velocity $\dot{x}$ is reversed)

$$\Rightarrow \quad \left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= F(x) \end{aligned}\right\} \quad \text{invariant under } t \to -t,\ y \to -y$$

if $(x(t), y(t))$ is a solution, so is $(x(-t), -y(-t))$

All conservative systems are reversible

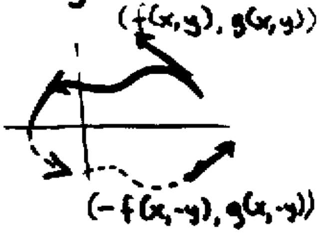
Reversible system

invariant under $t \to -t$, $y \to -y$
(or under a more general symmetry)

$$\left.\begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{array}\right\} \qquad f(x,y) = -f(x,-y), \; g(x,y) = g(x,-y)$$

f odd in y, g even in y

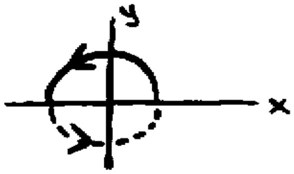
$\Rightarrow$ phase portrait is symmetric w.r.t. $y \to -y$
(change direction of arrows)



Nonlinear centres for reversible systems

Reversible system, origin $\vec{x}^* = \vec{0}$ is a linear centre

$\Rightarrow$ all trajectories sufficiently near the origin are closed curves



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eg
$$\left.\begin{array}{l} \dot{x} = y \\ \dot{y} = x - x^2 \end{array}\right\}$$

Fixed points $(0,0)$, $(1,0)$

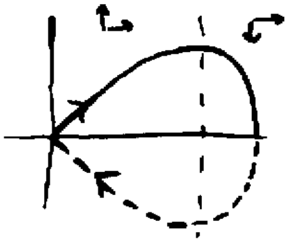
Jacobian $A = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$

$$A\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{l} \lambda_1 = 1, \; \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = -1, \; \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}$$

(0,0) is a saddle point

also: (1,0) is a centre

System is reversible: invariant under $t \to -t$, $y \to -y$



Unstable manifold of saddle leaves origin in direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y > 0 \Rightarrow x$ increasing : eventually $x > 1$

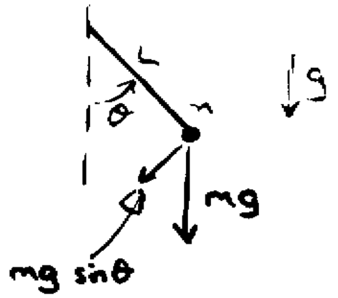
$\Rightarrow \dot{y} < 0 \Rightarrow y(t)$ decreases :

eventually $y(t) = 0$

By reversibility, there is a twin trajectory reflected in x-axis ($y \to -y$) with arrows reversed ($t \to -t$)

$\Rightarrow$ homoclinic orbit for $x > 0$.

Pendulum



No damping, no external driving

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Nondimensionalize: $\tau = \frac{t}{T} = \omega t, \quad \omega = \sqrt{g/L}$

$$\Rightarrow \quad \ddot{\theta} + \sin\theta = 0 \qquad \left(\dot{\theta} = \frac{d\theta}{d\tau}\right)$$

Flow in phase plane

$$\left.\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta \end{aligned}\right\}$$

v: dimensionless angular velocity

Fixed points $(n\pi, 0)$, $n \in \mathbb{Z}$

Jacobian $A = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$

"down" : $A\big|_{(0,0)} = A\big|_{(2k\pi,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \tau = 0, \ \Delta = 1$

$\lambda = \pm i$

$\Rightarrow$ centre (linear)

(0,0) is a nonlinear centre :

- system is reversible (symmetry $t \to -t$, $v \to -v$)
- system is conservative

$$\ddot{\theta} + \sin\theta = 0 \Rightarrow \dot{\theta}(\ddot{\theta} + \sin\theta) = \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\dot{\theta}^2 - \cos\theta}_{E(\theta,\dot{\theta})}\Big) = 0$$

$\Rightarrow \frac{1}{2}\dot{\theta}^2 - \cos\theta$: constant of motion

$$E = \tfrac{1}{2}v^2 - \cos\theta = \tfrac{1}{2}v^2 - (1 - \tfrac{1}{2}\theta^2 + \dots) \approx \tfrac{1}{2}(\theta^2 + v^2) - 1$$

near (0,0)
(small θ)

$\Rightarrow$ E has a local minimum at (0,0)

$\Rightarrow$ origin is a nonlinear centre.

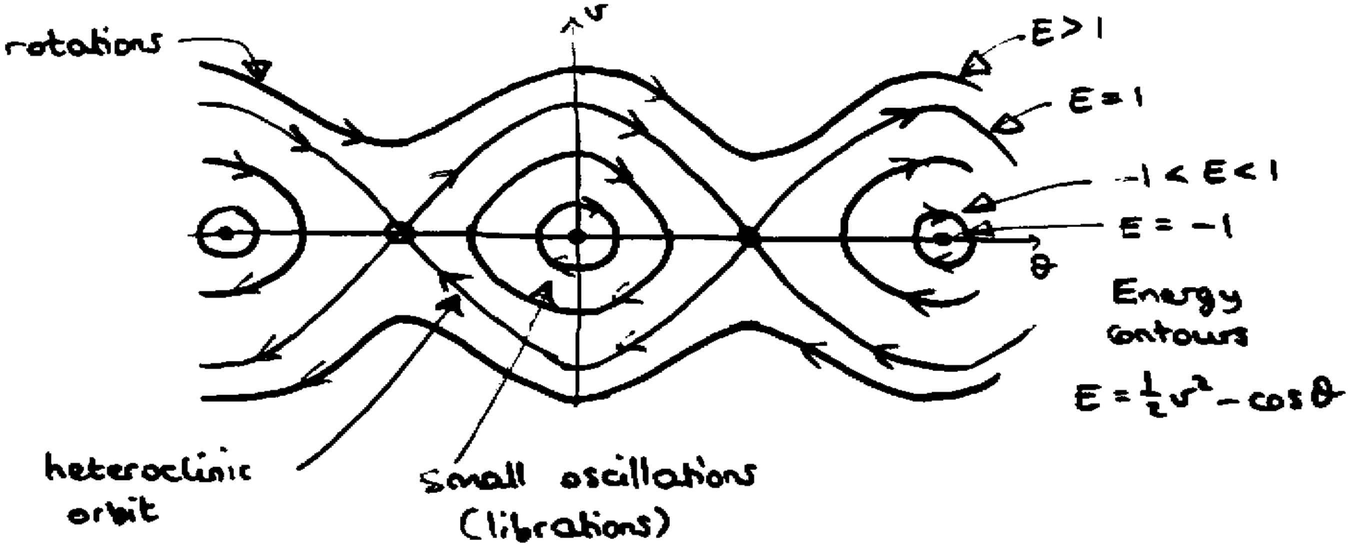
small amplitude solutions approx. circular $\theta^2 + v^2 \approx 2(E+1)$

"up" $A|_{(\pi,0)} = A|_{((2k+1)\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau = 0,\ \Delta = -1$

$\lambda = \pm 1 \Rightarrow$ saddle

$\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Phase portrait : undamped pendulum



$E = -1$: lowest energy state : pendulum hanging down

$(\theta, v) = (2k\pi, 0), \quad k \in \mathbb{Z}$

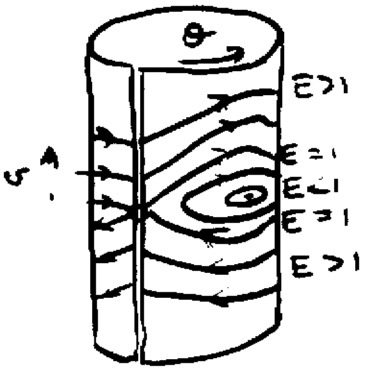
neutrally stable equilibrium, nonlinear centre

$-1 < E < 1$: closed orbits surrounding centres : oscillations about equilibrium librations

$E = 1$: heteroclinic trajectories connecting saddles $(\theta, v) = ((2k+1)\pi, 0)$

$E > 1$: whirling modes rotations

Cylindrical phase space



θ is an angle , v is a real number

$\theta \in S^1$ (circle) $\quad v \in \mathbb{R}$

so $(\theta, v) \in S^1 \times \mathbb{R}$ natural phase space

Saddle connections: heteroclinic orbits in plane

homoclinic orbits on cylinder

Pendulum with damping

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0 \Rightarrow \left.\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta - bv \end{aligned}\right\}$$

$b > 0$ damping strength

Fixed points: $(\theta, v) = (n\pi, 0) \quad n \in \mathbb{Z}$

Linearization
$$A = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -b \end{pmatrix}$$

"down"
$$A\big|_{(0,0)} = A\big|_{(2k\pi,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} \quad \lambda = \frac{-b \pm \sqrt{b^2-4}}{2} = -\frac{b}{2} \pm \frac{i}{2}\sqrt{4-b^2}$$

$\Rightarrow$ stable spiral $(0<b<2)$

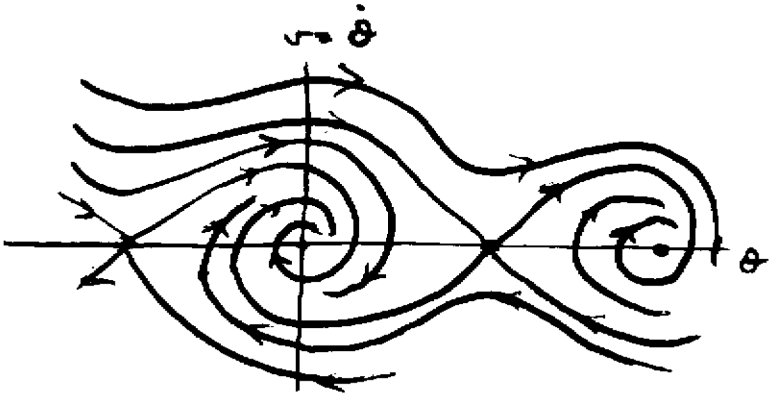
"up"
$$A\big|_{(\pi,0)} = A\big|_{((2k+1)\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix} \quad \lambda = \frac{-b \pm \sqrt{b^2+4}}{2}$$

$\Rightarrow$ saddle

Change in energy along a trajectory:

$$\frac{dE}{d\tau} = \frac{d}{d\tau}\left(\tfrac{1}{2}v^2 - \cos\theta\right) = v\dot{v} + (\sin\theta)\dot{\theta}$$
$$= v(-\sin\theta - bv) + v\sin\theta$$
$$= -bv^2 = -b\dot{\theta}^2 \leq 0$$

$\Rightarrow$ E decreases monotonically along trajectories, except at fixed points (where $v = \dot{\theta} = 0$)



Dissipation:
energy is not conserved
damped oscillations

System is no longer conservative or reversible

centre is not structurally stable (in absence of symmetries)

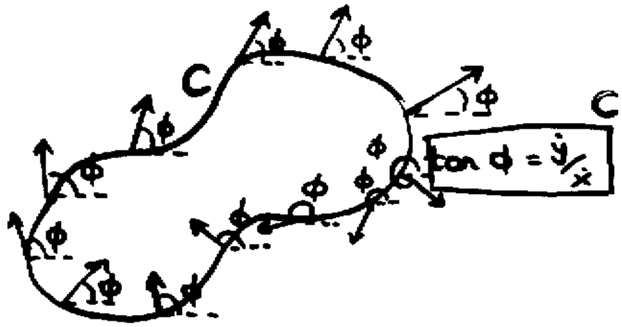
Index Theory

Linearization - local information
Index - global information about phase portrait

Index of a closed curve C:

an integer that measures the winding of the vector field on C.

$\dot{\vec{x}} = \vec{f}(\vec{x}) \Rightarrow \left.\begin{array}{l}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{array}\right\}$ Smooth vector field in phase plane $\Rightarrow$ direction field $\quad \vec{f} \in C^1$



C : closed curve in plane (not necessarily trajectory)

- no self-intersections (simple closed curve)
- C does not pass through fixed points

Angle $\boxed{\phi = \tan^{-1} \dot{y}/\dot{x}} = \tan^{-1} \dfrac{g(x,y)}{f(x,y)}$ - angle with positive x-axis

[Note: at a fixed point, $\dot{x} = \dot{y} = 0 \Rightarrow \phi$ is not defined]

As $\vec{x}$ moves counterclockwise on C, ϕ varies continuously ($\vec{f}$ smooth)

After one complete loop, $\vec{x}$ in original direction
$\Rightarrow \phi$ changes by a multiple of 2π

$[\phi]_C$: Change in ϕ over one counterclockwise circuit on C

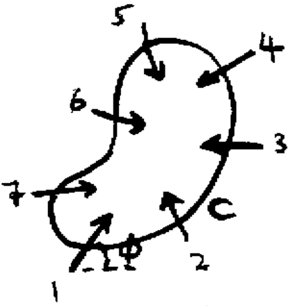
Index of C (w.r.t $\vec{f}$) : $\boxed{I_C = \frac{1}{2\pi}[\phi]_C}$ (integer)

⌈ Curve C parametized by s: $C : \vec{x}(s) = (x(s), y(s)), \; s_0 \le s \le s_1$

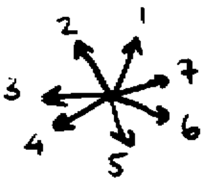
$$\frac{d}{ds}\tan\phi = \frac{d}{ds}\left(\frac{g(x(s),y(s))}{f(x(s),y(s))}\right) \Rightarrow \frac{d\phi}{ds} = \frac{f\dot{g} - g\dot{f}}{f^2+g^2} \qquad \dot{g} = \frac{dg}{ds}$$

$$I_C = \frac{1}{2\pi}\oint_C d\phi = \frac{1}{2\pi}\int_{s_0}^{s_1}\frac{d\phi}{ds}ds = \frac{1}{2\pi}\int_{s_0}^{s_1}\frac{f\dot{g}-g\dot{f}}{f^2+g^2}ds = \frac{1}{2\pi}\oint\frac{f\,dg - g\,df}{f^2+g^2}$$ ⌋

eg

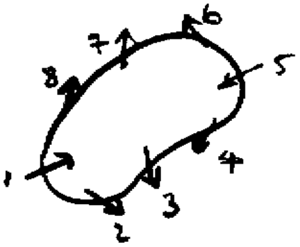


⇔



$I_C = \frac{1}{2\pi}\cdot 2\pi = +1$

(stable) node



⇔

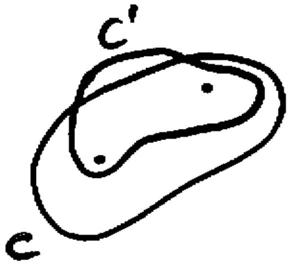
$I_c = \frac{1}{2\pi}(-2\pi) = -1$

saddle

Properties

1. Deform curve C continuously to C' without passing through a fixed point

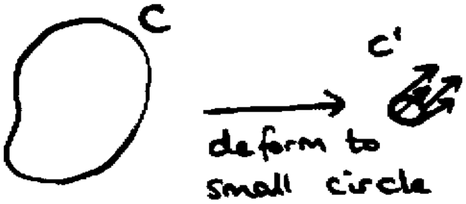
$\Rightarrow I_C = I_{C'}$



⌈ $[\phi]_C$, I_c varies continuously under deformation

I_c is integer-valued ⇒ constant ⌋

2. C contains no fixed points ⇒ $I_C = 0$



⌈ smooth vector field ⇒ $\vec{f}$, ϕ approx. constant on a small circle C'

$\Rightarrow [\phi]_{C'} = 0 \Rightarrow I_C = I_{C'} = 0$. ⌋

(or use Green's theorem on integral formula)

3. Index is invariant under t → −t (reverse arrows)

⌈ since all angles change as $\phi \to \phi + \pi \Rightarrow [\phi]_C$ unchanged ⌋

4. If C is a trajectory (closed orbit) ⇒ $I_C = +1$.

vector field tangent to C

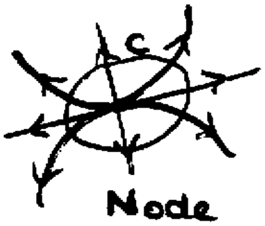
Index of a point $\vec{x}^*$: isolated fixed point

Define the index of $\vec{x}^*$ as $I = I_{\vec{x}^*} = I_C$,

where C is any closed curve enclosing $\vec{x}^*$ and no other fixed points

(by (1), I_C is independent of the specific curve C subject to this condition ⇒ property of only $\vec{x}^*$.)

eg



Node

$I = +1$

Spiral

$I = +1$

Saddle

$I = -1$

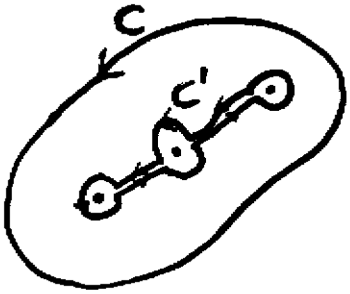
By (3), index does not depend on stability of fixed point, only on type

Also I = +1 for degenerate node, star, centre

Theorem. If C is a closed curve containing n isolated fixed points $\vec{x}_1^*, \vec{x}_2^*, \dots, \vec{x}_n^*$, then

$$I_C = \sum_{i=1}^{n} I_i = I_1 + I_2 + \dots + I_n$$

where $I_i = I_{\vec{x}_i^*}$



⌈ $I_C = I_{C'}$: no fixed points between C and C'.
from $[\phi]_{C'}$

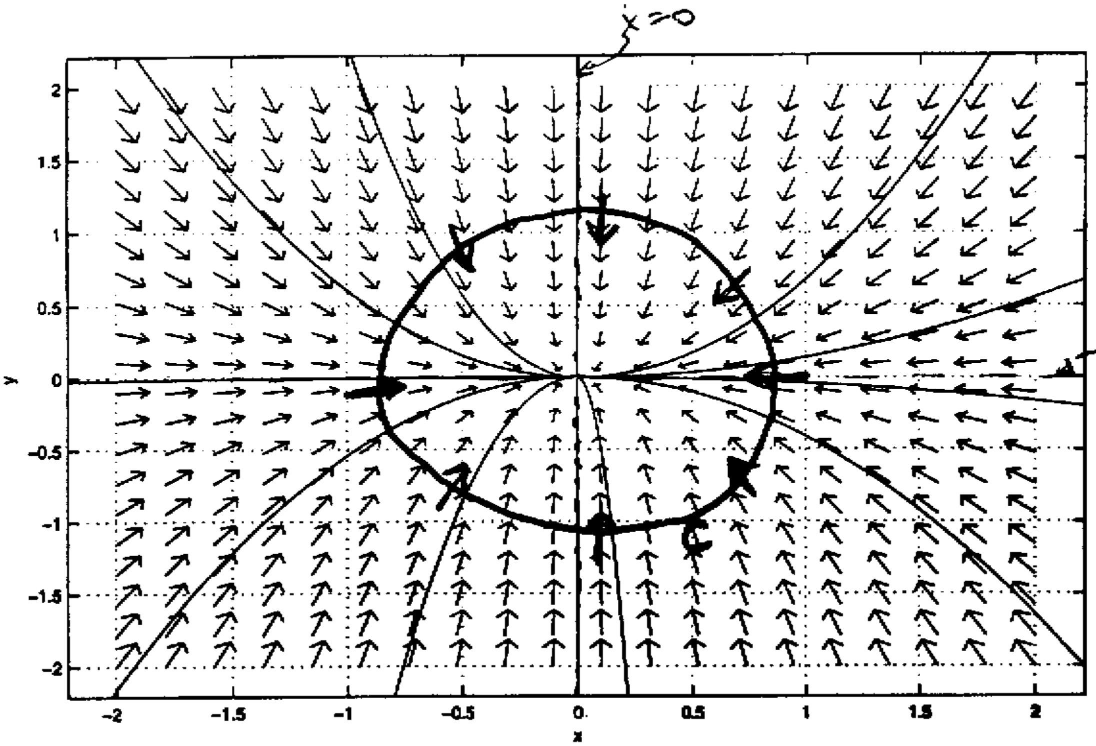
Compute $I_{C'}$: contribution from small circles ⇒ I_i

Narrow bridges, traversed once in each direction ⇒ contributions to $[\phi]_{C'}$ cancel out. ⌋

Corollary Any closed orbit in the plane must enclose fixed points whose indices sum to +1.

⇒ every periodic orbit must enclose at least one fixed point. If exactly one, the fixed point cannot be a saddle point.

6.22

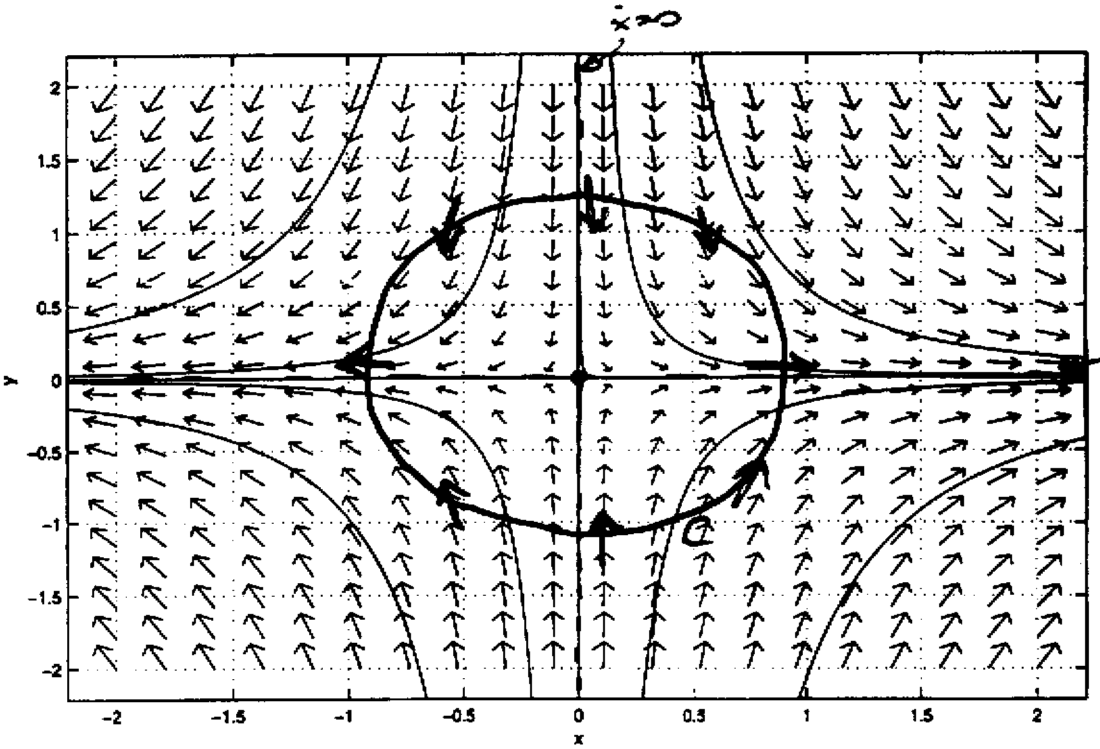


$\dot{x} = -x$
$\dot{y} = -2y$

(0,0)
stable node

$[\phi]_C = 2\pi$

index
$I_c = +1$



$\dot{x} = x$
$\dot{y} = -2y$

(0,0)
saddle
point

$[\phi]_c = -2\pi$

index
$I_c = -1$

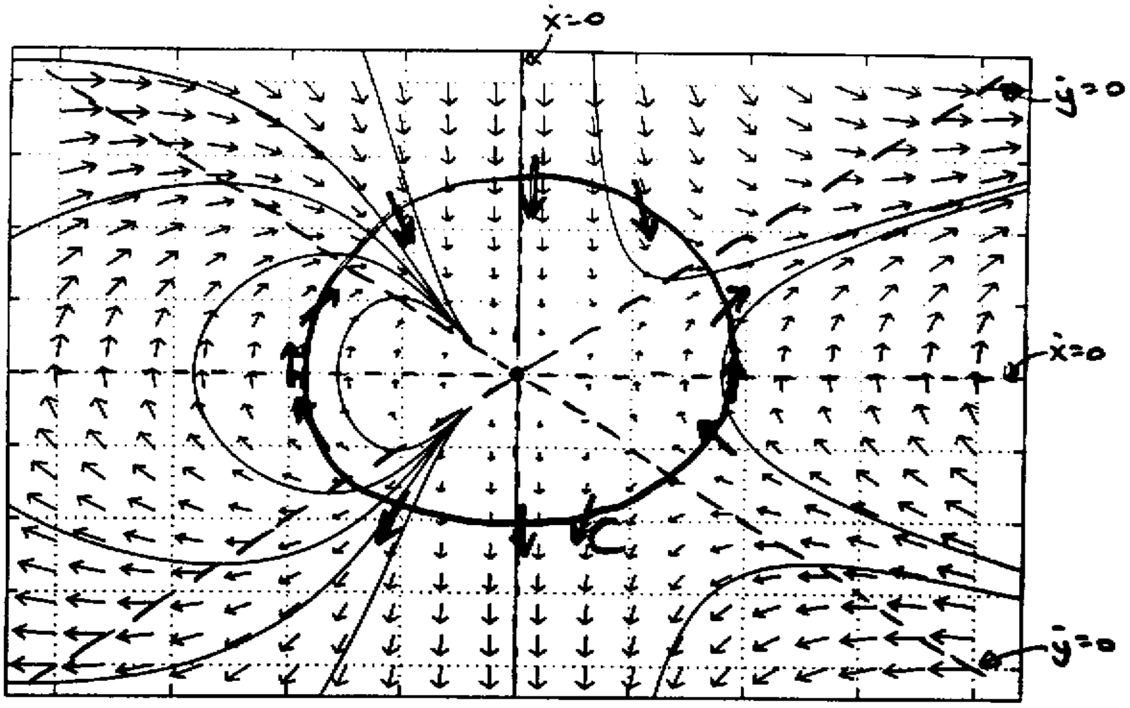
$$\left.\begin{array}{l}\dot{x} = x^2 y \\ \dot{y} = x^2 - y^2\end{array}\right\}$$

fixed point (0,0)

$[\phi]_C = 0$

index

$I_C = 0$



$$\left.\begin{array}{l}\dot{x} = x^2 - y^2 \\ \dot{y} = 2xy\end{array}\right\}$$

$$\text{ie}\quad \left.\begin{array}{l}\dot{x} = r^2 \cos 2\theta \\ \dot{y} = r^2 \sin 2\theta\end{array}\right\}$$

fixed point (0,0)

$[\phi]_C = 4\pi$

index

$I_C = +2$