Limit Cycles

Definition: $\boxed{\text{A limit cycle is an isolated closed orbit}}$

nearby trajectories are not closed: spiral toward or away from limit cycle

- an inherently nonlinear phenomenon:

No limit cycles in linear systems $\dot{\vec{x}} = A\vec{x}$:

if $\vec{x}(t)$ is a periodic solution, so is $c\vec{x}(t)$, $c \neq 0$

- $\vec{x} = \vec{0}$ is a centre, one-parameter family of closed orbits
- amplitude of oscillations around centre set by initial conditio

Amplitude of limit cycle oscillations dictated by system.

stable limit cycle (attracts nearby trajectories) — half-stable limit cycle — unstable limit cycle

Stable limit cycles - model systems exhibiting self-sustained oscillations

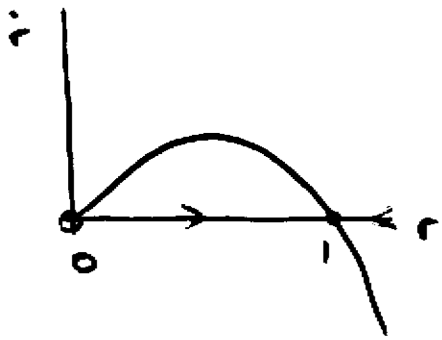
eg biological oscillations - heartbeat, periodic firing of pacemaker neuron, rhythms in body temperature, hormonal secretion, ...

oscillating chemical reactions

Examples: (easiest in polar coordinates)

$r \geq 0$

$$\text{eg} \quad \left.\begin{aligned} \dot r &= r(1-r^2) \\ \dot\theta &= 1 \end{aligned}\right\}$$



$r^* = 0$ unstable fixed point
$r^* = 1$ stable fixed point
} for 1-d system

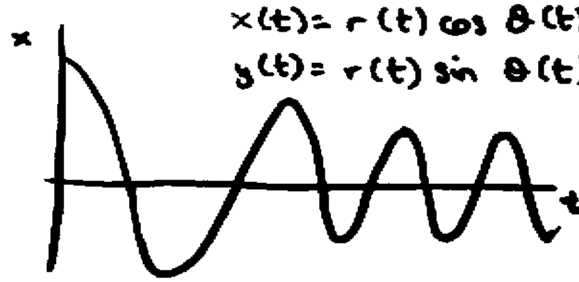
$r(0) > 0 \Rightarrow r(t) \to 1$ as $t \to \infty$

$\theta(t) = \theta_0 + t$ - rotation with constant angular velocity

Limit cycle:

circle at $r=1$

$x(t) = r(t)\cos\theta(t)$
$y(t) = r(t)\sin\theta(t)$



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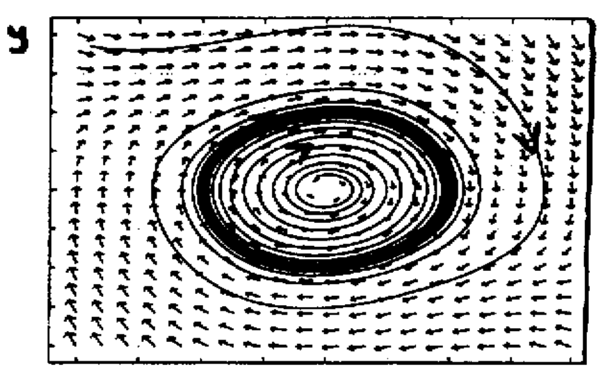
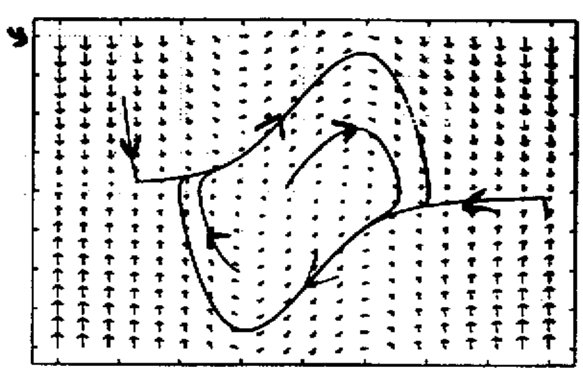
eg <u>van der Pol oscillator</u>

$$\ddot x + \mu(x^2-1)\dot x + x = 0 \quad \Rightarrow \quad \left.\begin{aligned} \dot x &= y \\ \dot y &= -x - \mu(x^2-1)y \end{aligned}\right\}$$

$\mu > 0$



$\mu = 0.1$



$\mu = 1.8$

- like a harmonic oscillator, with nonlinear damping term $\underbrace{\mu(x^2-1)\dot x}_{b}$

positive damping for $|x|>1$ — large-amplitude oscillations decay
negative damping for $|x|<1$ — pumps up small oscillations

- stable self-sustained oscillations for each $\mu > 0$.

istence of Closed Orbits

C is a limit cycle -
necessary condition: sum of the indices at equilibria enclosed by C is +1

- limit cycle cannot surround a region with no equilibrium points, or only a saddle.

② Gradient Systems: $\dot{\vec{x}} = -\underset{\text{grad}}{\nabla} V(\vec{x})$

$V(\vec{x})$ potential function $\quad V \in C^1$

- Closed orbits are impossible in gradient systems

Proof: suppose to the contrary, C is a closed orbit:

$$\underbrace{0 = \Delta V}_{\text{change in V after one circuit = 0 (V single-valued)}} = \int_0^T \frac{dV}{dt}\,dt = \int_0^T \underbrace{(\nabla V)}_{-\dot{\vec{x}}}\cdot\dot{\vec{x}}\,dt = -\int_0^T |\dot{\vec{x}}|^2\,dt \le 0$$

Equality only if $\dot{\vec{x}} \equiv \vec{0} \Rightarrow \vec{x}(t) = \vec{x}^*$ fixed point. ∎

Note: • All 1-d systems are gradient systems $\quad \dot{x} = f(x) = -\frac{dV}{dx}$
$\Rightarrow$ no oscillations in 1-d

• Most 2-d systems are not gradient systems

Criterion: $\left.\begin{array}{l}\dot{x} = -\frac{\partial V}{\partial x}\\ \dot{y} = -\frac{\partial V}{\partial y}\end{array}\right\}$ $\quad \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} \Rightarrow$ need $\quad \frac{\partial \dot{x}}{\partial y} = \frac{\partial \dot{y}}{\partial x}$

eg $\left.\begin{array}{l}\dot{x} = \sin y\\ \dot{y} = x\cos y\end{array}\right\}$ $\quad \frac{\partial \dot{x}}{\partial y} = \cos y = \frac{\partial \dot{y}}{\partial x}$ ✓ $\quad$ Gradient system, $V(x,y) = -x\sin$
$\Rightarrow$ no closed orbits

(3) Liapunov functions $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x}^*$: fixed point

Definition: $V(\vec{x})$ is a Liapunov function $V \in C^1$ real-valued

if

- $V(\vec{x}) > 0$ for all $\vec{x} \neq \vec{x}^*$, $V(\vec{x}^*) = 0$
  V is positive definite
- $\dfrac{dV}{dt} < 0$ for all $\vec{x} \neq \vec{x}^*$
  dV/dt negative definite

trajectories "flow downhill" towards $\vec{x}^*$

$\Rightarrow$ no closed orbits; $\vec{x}^*$ is (globally) asymptotically stable

[If $\dfrac{dV}{dt}$ is negative semidefinite, $\dfrac{dV}{dt} \leq 0$ for $\vec{x} \neq \vec{x}^*$, then $\vec{x}^*$ is stable

Difficulty: No systematic way to construct $V(\vec{x})$
(often try sums of squares)

eg $\left.\begin{aligned} \dot{x} &= -x - 2y^2 \\ \dot{y} &= xy - y^3 \end{aligned}\right\}$ unique fixed point $(0,0)$

Try $V(\vec{x}) = V(x,y) = x^2 + ay^2$ a to be chosen (positive definite for $a > 0$)

$$\Rightarrow \frac{dV}{dt} = 2x\dot{x} + 2ay\dot{y} = 2x(-x - 2y^2) + 2ay(xy - y^3)$$
$$= -2x^2 + (2a-4)xy^2 - 2ay^4$$

Choose $\underline{a = 2}$ $\Rightarrow \dfrac{dV}{dt} = -(2x^2 + 4y^4) < 0$ unless $x = y = 0$

ie $\dfrac{dV}{dt}$ is negative definite. Liapunov function $V = x^2 + 2y^2$.

eg $\ddot{x} + (\dot{x})^3 + x = 0$

$E(x,\dot{x}) = x^2 + \dot{x}^2 \Rightarrow \dot{E} = 2x\dot{x} + 2\dot{x}\ddot{x} = 2x\dot{x} + 2\dot{x}(-x - \dot{x}^3) = -2(\dot{x})^4 \leq 0$

E is not a true Liapunov function ($\dot{E} = 0$ for $\dot{x} = 0$: negative semidefinite)

but no closed orbits [suppose otherwise:

$0 = \Delta E = \int_0^T \dot{E}\, dt = -\int_0^T (\dot{x})^4 dt \leq 0$, equality only at a fixed point $\dot{x} \equiv 0$.]

④ Bendixson's & Dulac's Criterion

$\dot{\vec{x}} = \vec{f}(\vec{x})$, $f \in C^1$ (continuously differentiable)

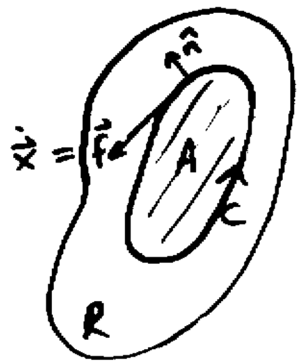
on a simply connected subset $R \subset \mathbb{R}^2$

Bendixson's Negative Criterion

If $\underbrace{\nabla \cdot}_{\text{div}} \vec{f} = \dfrac{\partial \dot{x}}{\partial x} + \dfrac{\partial \dot{y}}{\partial y}$ has one sign in R $\left[\begin{array}{l}\text{either } \nabla\cdot\vec{f} > 0 \text{ in } R \\ \text{or } \nabla\cdot\vec{f} < 0 \text{ in } R\end{array}\right.$

then there are no closed orbits lying entirely in R.

Proof: Suppose to the contrary, C is a closed orbit, enclosing area $A \subset R$.



$$\underbrace{\iint_A \nabla\cdot\vec{f}\, dA}_{\neq 0} = \underbrace{\oint_C \vec{f}\cdot\hat{n}\, dl}_{=0}$$

$\hat{n}$: outward normal

dl: element of arclength along C

Green's theorem (divergence theorem in 2-d)

$\neq 0$: $\nabla\cdot\vec{f}$ has constant sign

$= 0$: $\dot{\vec{x}}\cdot\hat{n} = \vec{f}\cdot\hat{n} = 0$: C is a trajectory

$\Rightarrow$ tangent of C is in direction $\dot{\vec{x}} = \vec{f}$

$\Rightarrow$ normal is orthogonal to $\vec{f}$

– Contradiction. ∎

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eg $\ddot{x} + f(x)\dot{x} + g(x) = 0$

– No periodic solutions in any region in which f is of one sign

(pure positive or negative damping $\Rightarrow$ no oscillations)

$$\left.\begin{array}{l}\dot{x} = y \\ \dot{y} = -g(x) - f(x)y\end{array}\right\}$$

$\Rightarrow \nabla\cdot\vec{f} = \dfrac{\partial \dot{x}}{\partial x} + \dfrac{\partial \dot{y}}{\partial y} = -f(x)$ ← of one sign whenever f is.

Generalization:

<u>Dulac's Criterion</u> $\quad \dot{\vec{x}} = \vec{f}(\vec{x}),\ \vec{f} \in C^1$

$R \subset \mathbb{R}^2$ simply connected

If there exists a real-valued function $g(\vec{x})$, $g \in C^1$ so that $\nabla \cdot (g\,\dot{\vec{x}}) = \nabla \cdot (g\vec{f})$ has one sign in R, then there are no closed orbits in R.

<u>Proof</u>: As before – for any closed trajectory C,

$$\underbrace{\iint_A \nabla \cdot (g\,\dot{\vec{x}})\, dA}_{\neq 0} = \oint_C g(\vec{x})\, \underbrace{\dot{\vec{x}} \cdot \hat{n}}_{=0}\, d\ell$$

∎

eg
$$\left.\begin{aligned} \dot{x} &= x(2-x-y) \\ \dot{y} &= y(4x-x^2-3) \end{aligned}\right\}$$

Difficulty:
No general algorithm for finding a suitable $g(\vec{x})$

Choose $g(\vec{x}) = g(x,y) = \dfrac{1}{xy}$:

$$\nabla \cdot (g\dot{\vec{x}}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{xy}x(2-x-y)\right) + \frac{\partial}{\partial y}\left(\frac{1}{xy}y(4x-x^2-3)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right) = \frac{\partial}{\partial x}\left(\frac{2}{y} - 1 - \frac{x}{y}\right) + \frac{\partial}{\partial y}\left(4 - x - 3/x\right)$$

$$= -\frac{1}{y}$$

< 0 for $y > 0$ $\qquad$ need $g = \dfrac{1}{xy} \in C^1$

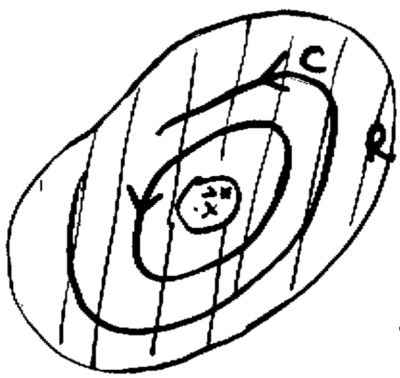
$\Rightarrow$ no closed orbits in the positive quadrant $x>0$, $y>0$
(or in any other quadrant: $\nabla \cdot (g\dot{\vec{x}})$ is of one sign)

Poincaré - Bendixson Theorem

- a fundamental result in two dimensions
- no counterpart in higher dimensions

Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ be a smooth dynamical system $(\vec{f} \in C^1)$ on (an open subset U of) $\mathbb{R}^2$.

Theorem: Suppose that



1) $R \subset U$ is a $\underbrace{\text{closed, bounded}}_{\text{compact}}$ subset of $\mathbb{R}^2$
2) R contains no fixed points
3) There is a trajectory $C: \vec{x}(t)$ that is confined in R : $\vec{x}(t) \in R$ for all $t \geq 0$

Then either C is a closed orbit, or it spirals towards a closed orbit as $t \to \infty$

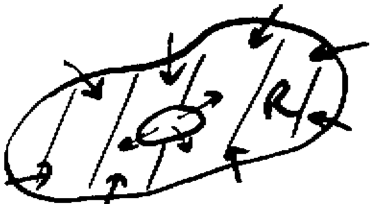
(fixed point $\vec{x}^* \notin R$: any closed orbit must enclose a fixed point) $\Rightarrow$ R contains a closed orbit.

Proof - subtle, needs ideas from topology.

"a compact, positively invariant region containing no fixed points contains a limit cycle"

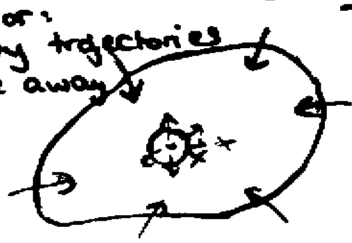
How do we find C? Usually construct R as a trapping region : R closed, connected, so that the vector field points inward on the boundary ∂R of R

$\Rightarrow$ any trajectory that enters R stays in R.



Corollary: If R is as above, but contains a finite number of isolated repelling fixed points (unstable nodes or spirals) then R contains a closed orbit (limit cycle).

repellor: nearby trajectories move away



Proof: define a new trapping region - exclude circles

Alternative statement of Poincaré-Bendixson Theorem:

If a trajectory $\vec{x}(t)$ is trapped in a compact, positively invariant region $R \subset \mathbb{R}^2$, then it must approach (or be) a fixed point, or a limit cycle, or a cycle graph

← a finite number of fixed points connected by a finite number of trajectories, with uniform orientation

Example
$$\left.\begin{aligned} \dot{r} &= r(1-r^2) + \mu r \cos\theta \\ \dot{\theta} &= 1 \end{aligned}\right\}$$
only fixed point at origin $r = 0$

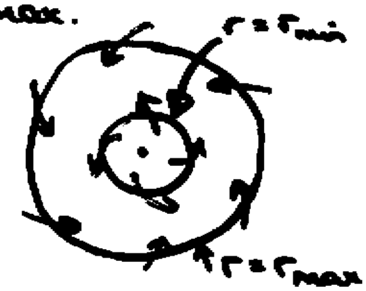
$\mu = 0$: stable limit cycle at $r = 1$

$\mu > 0$? Assume $\mu < 1$. $(-1 < \mu < 0$ similar)

Seek concentric circles $r = r_{min}$ and $r = r_{max}$, s.t.

$\dot{r} > 0$ on $r = r_{min}$, $\dot{r} < 0$ on $r = r_{max}$.

$-1 \leq \cos\theta \leq 1$.



$$\dot{r} = r\left[1 - r^2 + \mu\cos\theta\right] \geq r\left[1 - r^2 - \mu\right]$$

$\Rightarrow \dot{r} > 0$ for $r^2 < 1 - \mu$, $r \neq 0$

ie $0 < r < \sqrt{1-\mu}$ $(\mu < 1)$

eg choose $r_{min} = 0.99\sqrt{1-\mu}$ ← then $\dot{r} > 0$ at $r = r_{min}$

$$\dot{r} = r\left[1 - r^2 + \mu\cos\theta\right] \leq r\left[1 - r^2 + \mu\right]$$

$\Rightarrow \dot{r} < 0$ for $r^2 > 1 + \mu$ ie $r > \sqrt{1+\mu}$

Choose $r_{max} = 1.01\sqrt{1+\mu}$ ← $\dot{r} < 0$ for $r = r_{max}$

$\Rightarrow$ a closed orbit exists for all $|\mu| < 1$ ← in fact also for (some) $|\mu| \geq 1$

- limit cycle lies in the annulus

$$0.99\sqrt{1-\mu} < r < 1.01\sqrt{1+\mu}$$

Glycolytic Oscillations

a simplified model:

Glycolysis: break down glucose (sugar) to provide energy for cellular metabolism in living cells

Sel'kov model (dimensionless)

$$\left.\begin{aligned}\dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y\end{aligned}\right\}$$

$x \sim [ADP]$ adenosine diphosphate

$y \sim [F6P]$ fructose-6-phosphate

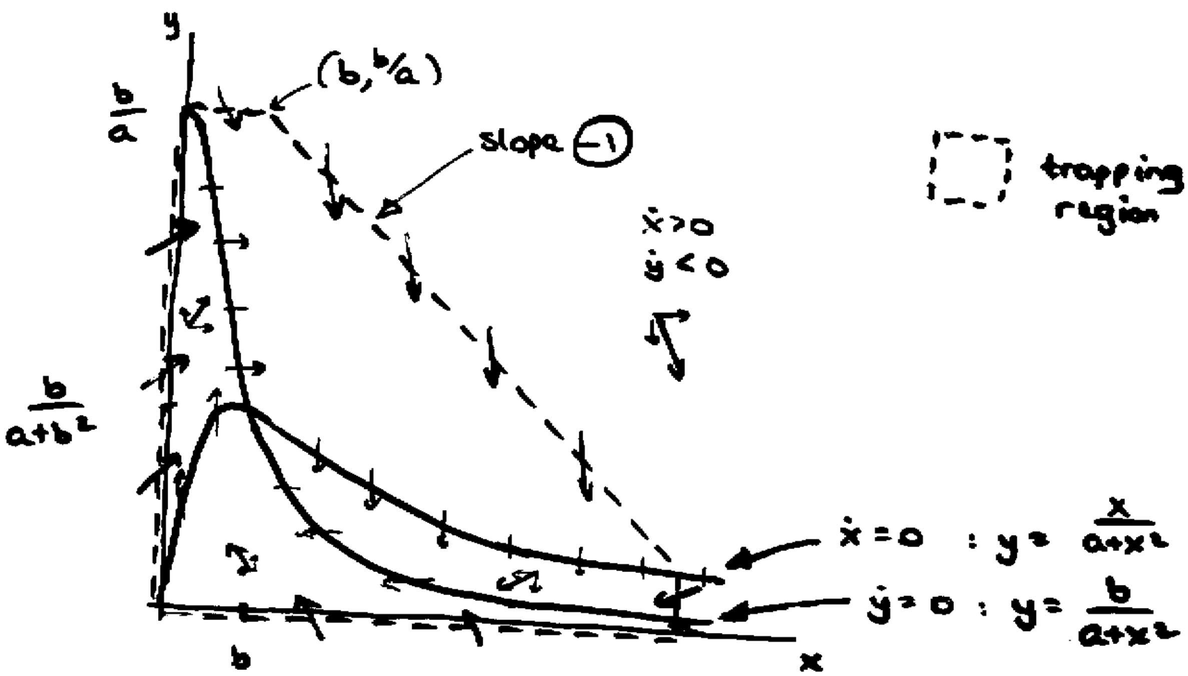
$a, b > 0$

Nullclines:

$$\left.\begin{aligned}\dot{x} = 0 \;&\Rightarrow\; y = \frac{x}{a+x^2} \\ \dot{y} = 0 \;&\Rightarrow\; y = \frac{b}{a+x^2}\end{aligned}\right\}$$

Fixed point $\dot{x} = \dot{y} = 0$

$\Rightarrow x^* = b,\; y^* = \dfrac{b}{a+b^2}$



Slope of trajectories:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b - ay - x^2 y}{-x + ay + x^2 y} = \frac{+x - ay - x^2 y + b - x}{-x + ay + x^2 y} = -1 + \frac{b - x}{-x + ay + x^2 y}$$

$x > b \Rightarrow b - x < 0$; $y > \frac{x}{a+x^2}$ (above nullcline $\dot{x} = 0$) $\Rightarrow -x + ay + x^2 y >$

$\Rightarrow$ on diagonal line $\qquad \dfrac{dy}{dx} < -1$

$\Rightarrow$ trajectories point inward: $\;$ is a trapping region

Sel'kov model

$$\left.\begin{aligned} \dot{x} &= -x + ay + x^2 y \\ \dot{y} &= b - ay - x^2 y \end{aligned}\right\}$$

We have a trapping region (positively invariant); but we cannot yet use the Poincaré-Bendixson theorem to deduce the existence of a limit cycle: there is a fixed point inside the trapping region.

- but if the fixed is a <u>repeller</u>, then we <u>can</u> conclude there is a closed orbit.

fixed point $(x^*, y^*) = \left(b, \frac{b}{a+b^2}\right)$. Jacobian $A = \begin{pmatrix} -1+2xy & a+x^2 \\ -2xy & -a-x^2 \end{pmatrix}$

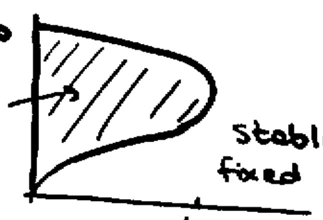
$$\Rightarrow A\Big|_{(x^*,y^*)} = \begin{pmatrix} -1+2by^* & a+b^2 \\ -2by^* & -a-b^2 \end{pmatrix}$$: determinant $\Delta = a+b^2 > 0$ (not saddle)

trace $\tau = -1 - a - b^2 + 2by^*$ ← $\frac{b}{a+b^2}$ $= \frac{\{-1-a-b^2\}(a+b^2)+2b^2}{a+b^2}$

$$= \frac{-a-b^2-a^2-2ab^2-b^4+2b^2}{a+b^2} = -\frac{b^4+b^2(2a-1)+a^2+a}{a+b^2}$$

fixed point is a repeller if $\tau > 0$: $\tau = 0 \Rightarrow b_\pm^2 = \frac{1-2a \pm \sqrt{1-8a}}{2}$

ie if $b_- < b < b_+$



$\Rightarrow$ system has a <u>closed orbit</u> (stable limit cycle).

<u>Implications</u>: Poincaré-Bendixson Theorem $\Rightarrow$ <u>No Chaos in the Phase Plane</u>

Dimension $n=2$ - a bounded trajectory approaches a fixed point or a periodic orbit (nothing more complicated)

Dimension $n \geq 3$ - bounded trajectories can wander around forever without settling down to a fixed point or closed orbit

eg trajectories may be attracted to a strange attractor ← CHAOS!

[complicated geometry, aperiodic motion, sensitive dependence

Liénard Systems

- criteria for existence of nonlinear oscillations

(many results exist) eg vacuum tubes, oscillating circuits

eg **Liénard's equation**

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

nonlinear damping $-f(x)\dot{x}$; nonlinear restoring force $-g(x)$

Define $F(x) = \int_0^x f(s)\,ds$

Equivalent system:

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= -g(x) - f(x)y\end{aligned}\right\} \quad \text{or}$$

(let $y = \dot{x} + F(x)$
$\Rightarrow \dot{y} = \ddot{x} + f(x)\dot{x} = -g(x)$)

$$\left.\begin{aligned}\dot{x} &= y - F(x)\\ \dot{y} &= -g(x)\end{aligned}\right\} \text{ system in Liénard plane}$$

Theorem Suppose f, g, $F = \int_0^x f$ satisfy

1) $f, g \in C^1$
2) g is odd: $g(-x) = -g(x)$ $\forall x$
3) $g(x) > 0$ for $x > 0$
4) f is even: $f(-x) = f(x)$ $\forall x$
5) $F(x)$ (odd) has exactly one positive zero at $x = a$:
   $F(x) < 0$, $0 < x < a$, $F(x) > 0$, $x > a$,
   $F(x) \to \infty$ as $x \to \infty$ $(F'(x) \geq 0,\ x > a)$
   (ie $f(x) < 0$ for $|x|$ small)

Then the Liénard system has a unique, stable limit cycle in the phase plane, surrounding the origin.

eg von der Pol oscillator $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ $\mu > 0$

$f(x) = \mu(x^2-1)$, $F(x) = \mu\int_0^x (s^2-1)\,ds = \mu\left(\frac{x^3}{3} - x\right)$ $a = 3$

$g(x) = x$

Conditions 1) - 5) satisfied $\Rightarrow$ unique, stable limit cycle

[$\mu = 0$: centre, simple harmonic ; $\mu < 0$: $t \to -t \Rightarrow$ unstable cycle]

Relaxation Oscillations

- seek quantitative information
  eg shape, period of closed orbits

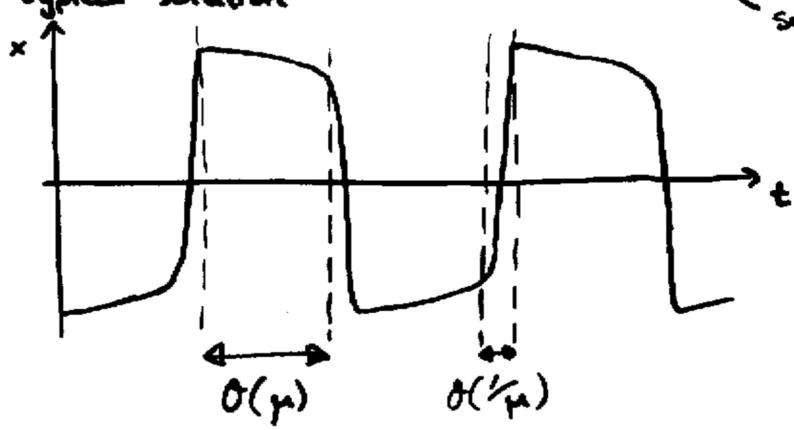
Perturbation theory: seek approximate solutions if a parameter is large or small

eg von der Pol equation

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0 \qquad \mu \gg 1 \text{ - strongly nonlinear}$$

Liénard theorem ⇒ limit cycle: slow buildup (time $O(\mu)$) → sudden discharge ($O(1/\mu)$)

typical solution



Relaxation oscillation

- fast and slow time scales operating sequentially
- singular perturbation problem

$$\ddot{x} + \mu \underbrace{F'(x)}_{f(x)}\dot{x} + x = 0 \qquad F(x) = \int_0^x (s^2-1)\,ds = \tfrac{1}{3}x^3 - x$$

Note: $F(0)=0$, F is "S-shaped", odd

In Liénard plane, equivalent system:

$$\left.\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x \end{aligned}\right\} \qquad (\Rightarrow \ddot{x} = \dot{w} - \mu F'(x)\dot{x} = -x - \mu F'(x)\dot{x}) \checkmark$$

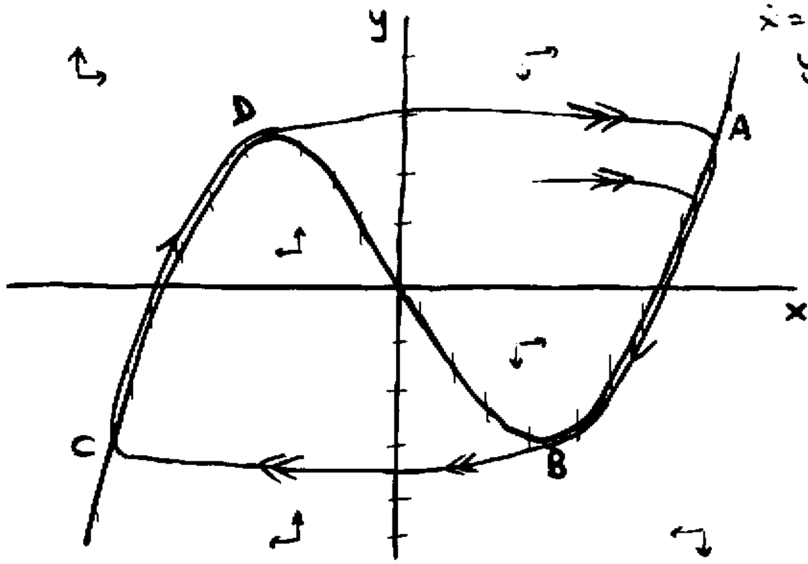
Let $y = \frac{w}{\mu}$ $\quad (= \varepsilon w, \quad \varepsilon = \mu^{-1} \ll 1)$

$$\Rightarrow \left.\begin{aligned} \dot{x} &= \mu(y - F(x)) \\ \dot{y} &= -x/\mu \end{aligned}\right\} \quad \text{ie} \quad \left.\begin{aligned} \dot{x} &= \frac{y-F(x)}{\varepsilon} \\ \dot{y} &= -\varepsilon x \end{aligned}\right\} \quad \varepsilon \text{ small}$$

Two timescales:

Fast: $\tau = t/\varepsilon = \mu t$ $\Rightarrow \underbrace{\frac{dx}{d\tau} = y - F(x), \ \frac{dy}{d\tau} = -\varepsilon^2 x}$ Appropriate if $y - F(x) = O(1)$

Slow: $\tau = \varepsilon t = t/\mu$ $\Rightarrow \underbrace{\frac{dx}{d\tau} = \frac{y-F(x)}{\varepsilon^2}, \ \frac{dy}{d\tau} = -x}$ Appropriate when $y - F(x) = O(\varepsilon^2)$



$\dot{x}=0$: x-nullcline
$y = F(x)$
$= \frac{1}{3}x^3 - x$

$\dot{y}=0$: y-nullcline
$x=0$

B, D at local extrema of $F(x)$:
$F'(x) = f(x) = x^2 - 1 = 0$
$\Rightarrow x = \pm 1$

$B: (1, -\tfrac{2}{3})$
$D: (-1, \tfrac{2}{3})$
$A: (2, \tfrac{2}{3})$
$C: (-2, -\tfrac{2}{3})$

$y - F(x) = \mathcal{O}(1) \Rightarrow \left.\begin{array}{l} \dot{x} = \mathcal{O}(1/\varepsilon) \\ \dot{y} = \mathcal{O}(\varepsilon) \end{array}\right\} \Rightarrow$ x changes much more rapidly than y

$\Rightarrow$ trajectory approximately horizontal — FAST : time $\sim \varepsilon$

until $y - F(x) = \mathcal{O}(\varepsilon^2)$ (near nullcline): $\left.\begin{array}{l} \dot{x} = \mathcal{O}(\varepsilon) \\ \dot{y} = \mathcal{O}(\varepsilon) \end{array}\right\}$ - at A, C

$\Rightarrow$ SLOW motion along nullcline $y \approx F(x)$ time $\sim 1/\varepsilon$

until trajectory reaches knee of nullcline - at B, D

- then trajectory jumps sideways to other branch, ...

Fixed point: $(0,0)$

$$A\big|_{(0,0)} = \begin{pmatrix} -\frac{F'(0)}{\varepsilon} & \frac{1}{\varepsilon} \\ -\varepsilon & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} & \frac{1}{\varepsilon} \\ -\varepsilon & 0 \end{pmatrix}$$

$\tau = 1/\varepsilon > 0$, $\Delta = 1$: unstable node

Period of oscillation

T_{AB}, T_{CD} slow $\mathcal{O}(1/\varepsilon)$, T_{BC}, T_{DA} fast $\mathcal{O}(\varepsilon)$

$\Rightarrow$ period $\approx T_{AB} + T_{CD} \underset{\text{symmetry } x \to -x,\ y \to -y}{=} 2T_{AB}$ - period dominated by slow branches

On the branch AB: $y \approx F(x) \Rightarrow \dot{y} \approx F'(x)\dot{x}$ and $\dot{y} = -\varepsilon x$

$$\Rightarrow \frac{dx}{dt} = \dot{x} \approx -\frac{\varepsilon x}{F'(x)}$$

$$\Rightarrow \text{period} \approx 2\int_{t_A}^{t_B} dt = 2\int_{x_A}^{x_B} \underbrace{\frac{dt}{dx}}_{=1/\dot{x}} dx = 2\int_2^1 -\frac{F'(x)}{\varepsilon x}\,dx = \frac{2}{\varepsilon}\int_1^2 \frac{x^2-1}{x}\,dx$$

$$= \frac{2}{\varepsilon}\left[\tfrac{1}{2}x^2 - \ln x\right]\Big|_1^2$$

$$\Rightarrow \boxed{\text{period} \approx \frac{1}{\varepsilon}(3 - 2\ln 2)} = \mathcal{O}(\mu)$$

[First correction term: estimate time to turn corner $= \mathcal{O}(\varepsilon^{1/3})$]