Bifurcations

Bifurcation : A change in the topological structure of the phase portrait as a parameter is varied continuously

At a bifurcation point, the system is structurally unstable

Solutions in 2-d :
- fixed points
- limit cycles , closed orbits
- saddle connections

Bifurcations – these are created / destroyed / change stability

Local bifurcations of fixed points :

One dimension: $\dot{x} = f(x,r)$

Bifurcation when $\left.\begin{aligned} f(x^*, r) &= 0 \\ \lambda = \frac{\partial f}{\partial x}(x^*, r) &= 0 \end{aligned}\right\}$ double zero

Higher dimensions: $\dot{\vec{x}} = \vec{f}(\vec{x}, r)$ Consider for now one parameter $r \in \mathbb{R}$ "codimension 1 bifurcations"

Bifurcation when $\mathrm{Re}\,\lambda_i = 0$ for some eigenvalue(s) λ_i of the Jacobian matrix $D\vec{f}(\vec{x}^*)$

ie $\vec{x}^*$ is a non-hyperbolic fixed point.

Bifurcations of Fixed Points: Zero eigenvalue bifurcations

- qualitatively similar to 1-d case
- "interesting" dynamics confined to a one-dimensional subspace: centre manifold

Prototypical equations (bifurcation behaviour on x-axis)

Saddle-Node
$$\left.\begin{aligned} \dot{x} &= \mu - x^2 \\ \dot{y} &= -y \end{aligned}\right\}$$

Transcritical
$$\left.\begin{aligned} \dot{x} &= \mu x - x^2 \\ \dot{y} &= -y \end{aligned}\right\}$$

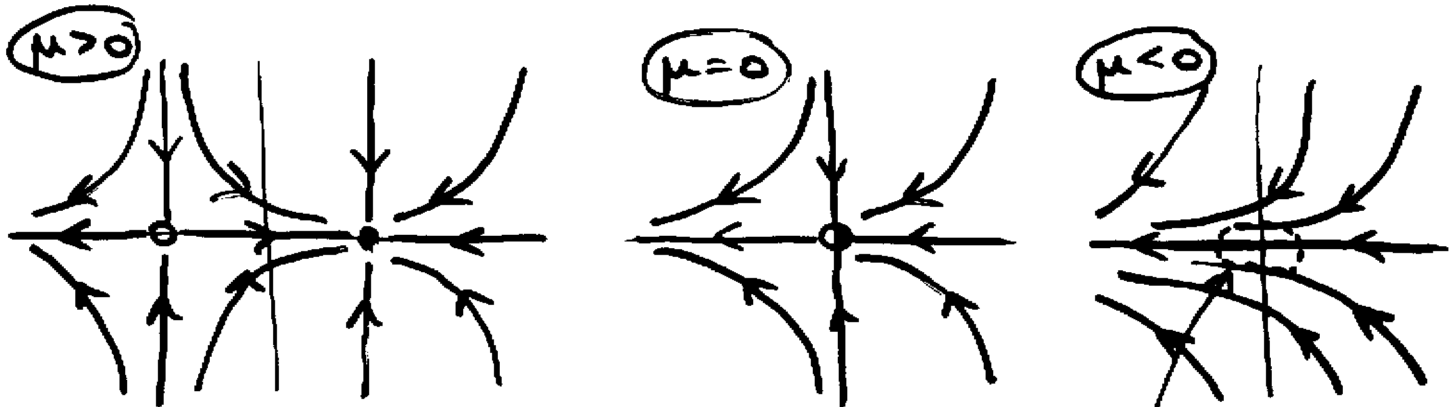
Supercritical Pitchfork
$$\left.\begin{aligned} \dot{x} &= \mu x - x^3 \\ \dot{y} &= -y \end{aligned}\right\}$$

Subcritical Pitchfork
$$\left.\begin{aligned} \dot{x} &= \mu x + x^3 \\ \dot{y} &= -y \end{aligned}\right\}$$

eg. Saddle-Node Bifurcation

$$\left.\begin{aligned}\dot{x} &= \mu - x^2 \\ \dot{y} &= -y\end{aligned}\right\}\quad \begin{aligned}&\leftarrow \text{1-d bifurcation in } x \quad (x,y \text{ uncoupled}) \\ &\leftarrow \text{exponential decay in } y\end{aligned}$$

Saddle point and node collide and annihilate ...



Saddle Node $\longrightarrow$ collide "ghost" of fixed point (bottleneck)

Fixed points
$(-\sqrt{\mu},0)$ $(\sqrt{\mu},0)$
Eigenvalues
$2\sqrt{\mu},\ -1$ $-2\sqrt{\mu},\ -1$

Time in bottleneck
$T \sim \dfrac{1}{\sqrt{\mu-\mu_c}}$ (bifurcation at $\mu=\mu_c$)

(similar analysis for prototypical transcritical, supercritical and subcritical pitchfork bifurcations)

Above example - idealized

- bifurcation occurs on a straight (linear) subspace - centre manifold
- manifolds intersect at right angles

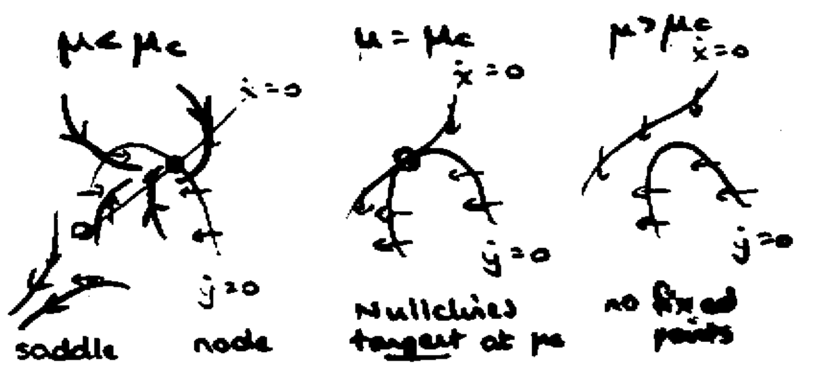
More generally: curved manifolds, arbitrary angles

Typical: $\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\}$

- look at Nullclines

and their intersections

qualitative picture for all saddle-node bifurcations in 2d $\longrightarrow$



eg **Model for a Genetic Control System**

$$\left.\begin{aligned}\dot{x} &= y - ax \\ \dot{y} &= \frac{x^2}{1+x^2} - by\end{aligned}\right\}$$

DNA → mRNA → protein

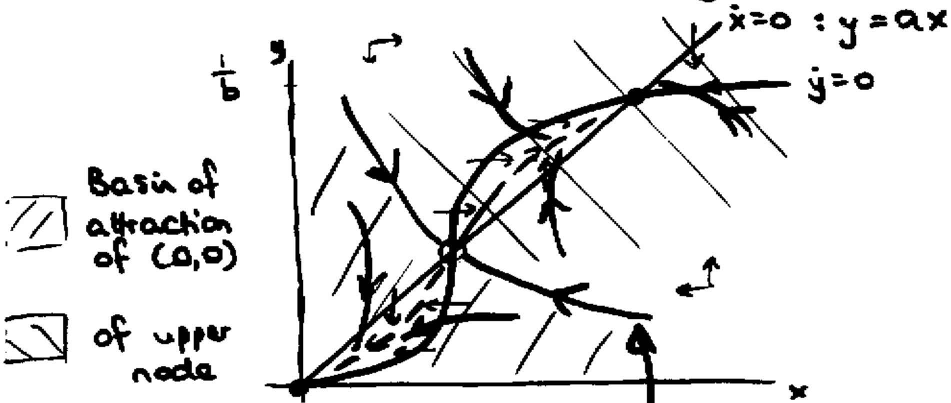
Gene activity stimulated by its product (protein for which it codes): autocatalytic feedback

$a, b > 0$: degradation of x, y

x: concentration of protein
y: concentration of messenger RNA

Nullclines: $\dot{x} = 0 \Rightarrow y = ax$
$\dot{y} = 0 \Rightarrow y = \frac{1}{b}\frac{x^2}{1+x^2}$

Fix b, vary a.



stable manifold of saddle — separates plane into basins of attraction for sinks

Small a: 3 fixed points
Large a: 1 fixed point

$ab < \frac{1}{2}$: biochemical switch
2 stable steady states

Fixed points: $\dot{x} = \dot{y} = 0 \Rightarrow y = ax = \frac{1}{b}\frac{x^2}{1+x^2}$

$$\Rightarrow x = 0 \quad \text{or} \quad ab = \frac{x}{1+x^2} \Rightarrow abx^2 - x + ab = 0$$

$$\Rightarrow x^*_{1,2} = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}$$

$2ab > 1$: one fixed point $(0,0)$

$2ab < 1$: three fixed points

Saddle-node bifurcation at $a_c = \frac{1}{2b}$: Bifurcation occurs at $x^*_c = 1$, $y^*_c = a$

Jacobian $\begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$

$\tau = -(a+b) < 0$
$\Delta = ab - \frac{2x}{(1+x^2)^2}$

$(0,0)$: eigenvalues $-a, -b$: stable node ← gene silent

other fixed points: $\Delta = ab - \frac{2}{1+x^2}\frac{x}{1+x^2} = ab\left[1 - \frac{2}{1+x^{*2}}\right] = ab\left[\frac{x^{*2}-1}{x^{*2}+1}\right]$

Middle: $x^* < 1 \Rightarrow \Delta < 0$ saddle

Upper: $x^* > 1 \Rightarrow \Delta > 0$ stable node ← gene active eg $\left.\begin{array}{l}\dot{x} = \mu x + y + \sin x \\ \dot{y} = x - y\end{array}\right\}$ Symmetry $x \to -x$, $y \to -y$, $t \to -t$ (reflection through origin)

- expect pitchfork bifurcation

$(0,0)$: fixed point for all μ

Jacobian $A\Big|_{(0,0)} = \begin{pmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix}$ $\tau = \mu$

$\Delta = -(\mu+2)$

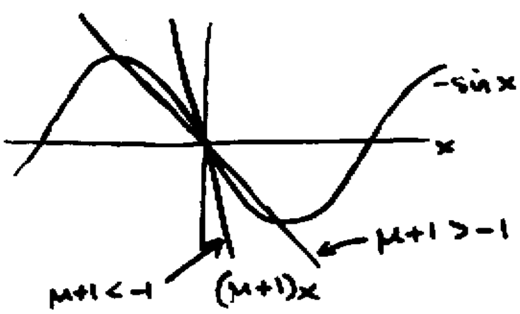
$\mu < -2$: $(0,0)$ is a stable node

$\mu > -2$: $(0,0)$ is a saddle point

$\Rightarrow$ $\mu_c = -2$ Super- or subcritical pitchfork bifurcation?

Fixed points: $\left.\begin{array}{l}\mu x + y + \sin x = 0 \\ x - y = 0 \Rightarrow y^* = x^*\end{array}\right\} \Rightarrow (\mu+1)x + \sin x = 0$

$\Rightarrow (\mu+1)x = -\sin x$



$\mu+1 < -1$ - one fixed point

$\mu+1 > -1$ - three fixed points $\left(\begin{array}{c}0<\mu+2 \\ \text{small}\end{array}\right)$

Nontrivial fixed points appear for $\mu > -2$, where $(0,0)$ is unstable

$\Rightarrow$ supercritical pitchfork bifurcation

$\Rightarrow$ new fixed points are stable

Near bifurcation point:

Expand near $x = 0$:

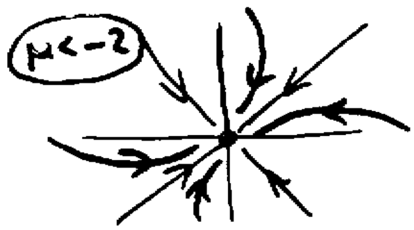
$(\mu+1)x + \sin x = (\mu+1)x + (x - \frac{1}{6}x^3 + ..) \simeq (\mu+2)x - \frac{1}{6}x^3 = 0$

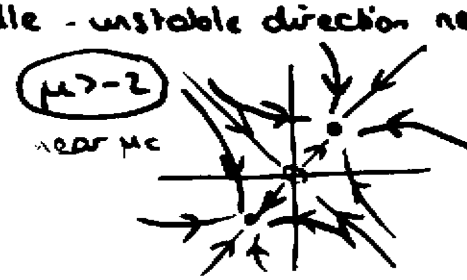
$\Rightarrow$ $x^* = 0$ (all μ) $x^* = \pm\sqrt{6(\mu+2)}$ $(\mu > -2)$ supercritical

For $\mu \simeq -2$, $A \approx \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$: $\begin{array}{ll}\lambda_1 \simeq -2 & \vec{v}_1 \simeq \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \lambda_2 \simeq 0 & \vec{v}_2 \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{array}$

$0 < \mu+2 \ll 1$: $(0,0)$ is a saddle - unstable direction near $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Phase portrait near bifurcation point :

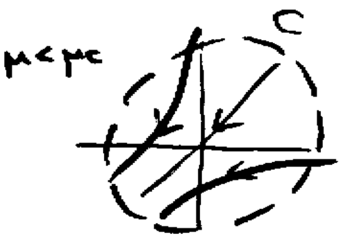




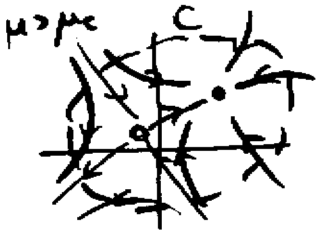
Note: Index theory

– constrains possibilities for local bifurcations of isolated fixed points

eg Saddle-node

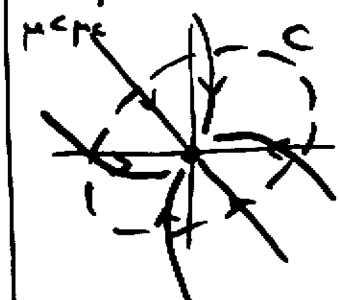


C contains no fixed points: index 0

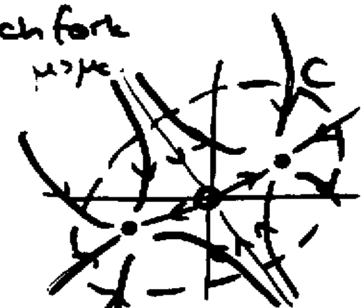


Saddle (−1) + node (+1) index 0

Supercritical Pitchfork



Node (+1) index +1

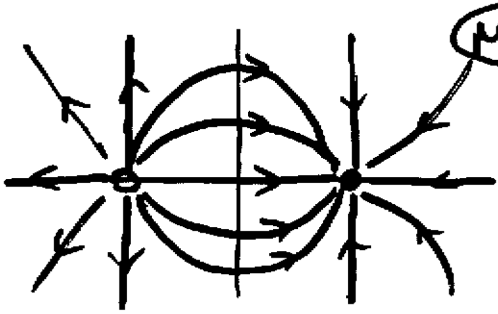


Two nodes (2×+1) + Saddle (−1) index +1

Can there be a node-node bifurcation? Yes, but degenerate "codimension 2"

$$\left.\begin{aligned}\dot{x} &= \mu - x^2\\ \dot{y} &= -xy\end{aligned}\right\}$$

Symmetry $x \to -x$, $y \to -y$, $t \to -t$

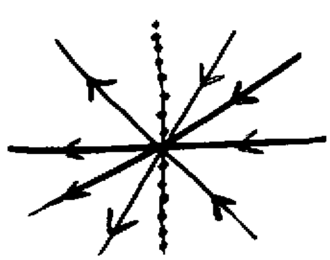


$\mu > 0$

Fixed points $x^* = \pm\sqrt{\mu}$, $y^* = 0$

Jacobian $A\big|_{(x^*,y^*)} = \begin{pmatrix} -2x^* & 0 \\ 0 & -x^* \end{pmatrix}$

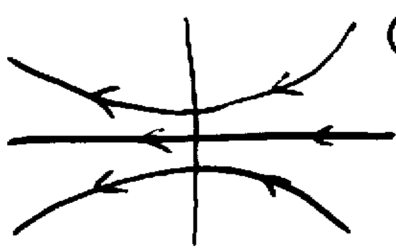
$x^* > 0$ – stable node
$x^* < 0$ – unstable node



$\mu = 0$

Fixed points $x=0$, $y=y_0$: line of fixed points

Trajectories $\frac{dy}{dx} = \frac{-xy}{-x^2} = \frac{y}{x} \Rightarrow y = ax$ straight lines



$\mu > 0$

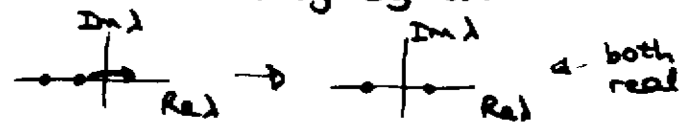
No fixed points

[No contradiction with index theory: Any closed curve surrounding the origin passes through fixed points at $\mu = 0$]

Hopf Bifurcation

Saddle-node
Transcritical
Pitchfork
} fixed points ⟷ fixed points

a real eigenvalue crosses the imaginary axis

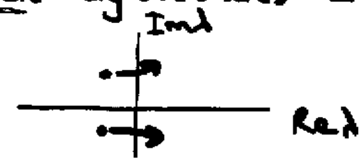


Hopf bifurcation :

fixed point ⟷ limit cycle

a pair of complex conjugate eigenvalues crosses the imaginary axis

eigenvalues λ pure imaginary at bifurcation



Supercritical Hopf bifurcation

– as bifurcation parameter μ increases through μ_c

$\mu < \mu_c$

damped oscillations
(stable spiral in phase space)
– decay rate μ-dependent

$\mu > \mu_c$

equilibrium is unstable (spiral)
– small-amplitude steady oscillations

Prototypical example (Normal form) supercritical Hopf

$$\left.\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2\end{aligned}\right\}$$

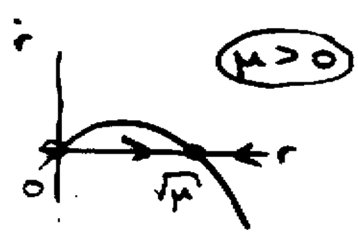
$\omega \neq 0$
rotation in θ

μ : bifurcation parameter (gives stability of $r=0$)

ω : frequency of infinitesimal oscillations

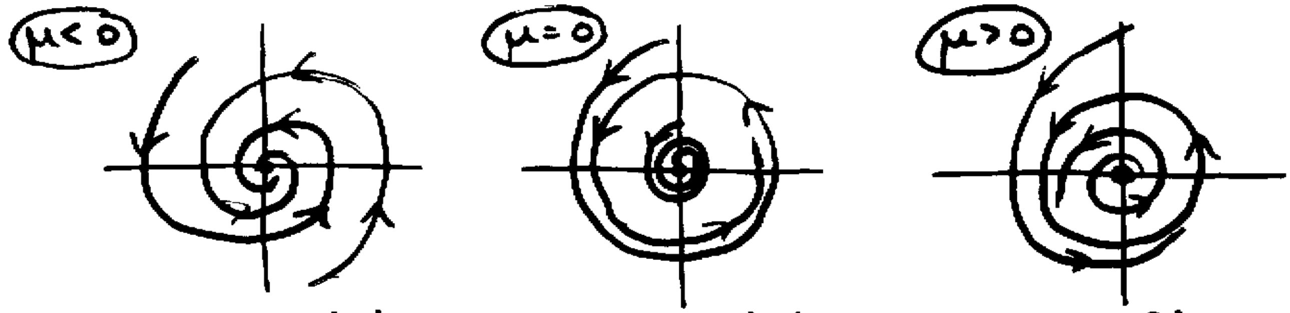
b : dependence of frequency on amplitude

Note: supercritical pitchfork bifurcation in radial direction



Supercritical Hopf bifurcation $\left.\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2\end{aligned}\right\}$

Phase portrait



$r=0$: stable spiral weak spiral stable limit cycle radius $\sqrt{\mu}$

Cartesian coordinates:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = (\mu r - r^3)\cos\theta - (r\sin\theta)(\omega + br^2)$$

$$= \mu x - r^2 x - \omega y - br^2 y = \mu x - \omega y - (x+by)(x^2+y^2)$$

Similarly $\dot{y} = \ldots$

$$\Rightarrow \left.\begin{aligned} \dot{x} &= \mu x - \omega y + \text{cubic terms} \\ \dot{y} &= \omega x + \mu y + \text{cubic terms} \end{aligned}\right\}$$

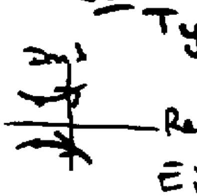
Jacobian $A = D\vec{f}(\vec{0})\Big|_{(0,0)} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$ Eigenvalues $\boxed{\lambda = \mu \pm i\omega}$

– eigenvalues cross the imaginary axis at $\mu = \mu_c = 0$

Note: (generic properties for supercritical Hopf bifurcation near μ_c)

- Radius of limit cycle $r \sim \sqrt{\mu - \mu_c}$
  (growth of small-amplitude oscillations)
- Frequency at bifurcation μ_c is $\omega = \operatorname{Im}\lambda\big|_{\mu=\mu_c}$
  $\Rightarrow$ period of orbit $T = \dfrac{2\pi}{\operatorname{Im}\lambda} + O(\mu - \mu_c)$

but Typically
- elliptical limit cycle
- eigenvalues cross imaginary axis with nonzero slope



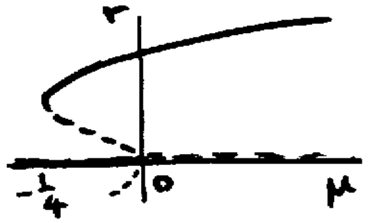
Eigenvalues $\lambda(\mu) = \alpha(\mu) \pm i\omega(\mu)$ Hopf at μ_c: $\alpha(\mu_c) = 0$, $\alpha'(\mu_c) \neq 0$, $\omega(\mu_c) \neq 0$

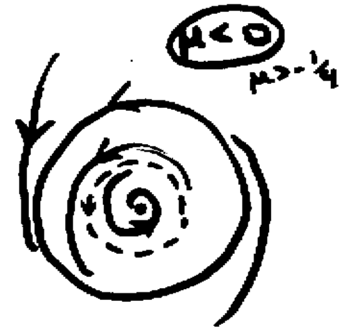
Subcritical Hopf bifurcation

eg
$$\left.\begin{aligned}\dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + b r^2\end{aligned}\right\}$$

cubic term destabilizing
subcritical pitchfork in r
+ rotation

Subcritical pitchfork bifurcation diagram ($r \geq 0$)





bistability: coexistence of stable fixed point and stable limit cycle: basins of attraction separated by unstable limit cycle

stable large-amplitude oscillation – only attractor

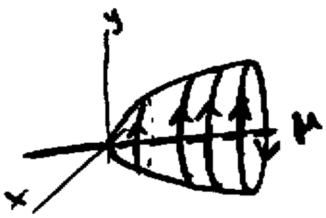
$\mu \to 0^-$: unstable cycle shrinks

Assume:

fixed point $\vec{x}^*$ undergoes a Hopf bifurcation at $\mu = \mu_c$

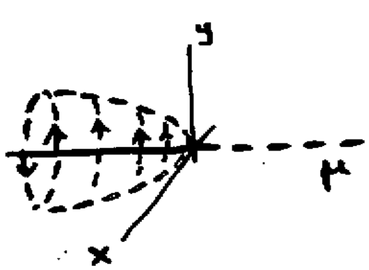
$\vec{x}^*$ is stable for $\mu < \mu_c$, unstable for $\mu > \mu_c$.

Supercritical Hopf bifurcation: "soft"



- small-amplitude stable oscillations for $\mu > \mu_c$
- amplitude of limit cycle $\to 0$ as $\mu \to \mu_c^+$, no hysteresis
- $\vec{x}^*$ is stable at $\mu = \mu_c$

Subcritical Hopf bifurcation: "hard"



- large-amplitude stable oscillations for $\mu > \mu_c$
- as μ increases through μ_c, rapid jump from steady state to oscillations of finite amplitude
- hysteresis: if μ decreases again below μ_c, oscillations remain
- $\vec{x}^*$ is unstable at $\mu = \mu_c$

Subcritical or supercritical Hopf bifurcation?

<u>Normal form</u>

$$\left.\begin{aligned}\dot r &= d\mu r + a r^3 + O(r^5, \mu r^3, \ldots)\\ \dot\theta &= \omega + d\mu + b r^2 + O(r^4, \mu r^2, \ldots)\end{aligned}\right\} \quad d \neq 0$$

can rescale time to set $\boxed{d=1}$

Hopf bifurcation at $\mu = 0$:

$d=1$ $\quad a < 0$: supercritical Hopf $\quad$ limit cycle $r \sim \sqrt{-a\mu}$

$\quad a > 0$: <u>sub</u>critical Hopf $\quad$ for $a\mu < 0$

$\Rightarrow$ sign of a in normal form determines type of bifurcation

Analytical criterion:

Suppose the fixed point at (x^*, y^*) becomes unstable at $\mu = \mu_c$

what is a? $\quad \tilde{x} = x - x^*$, $\tilde{y} = y - y^*$, $\tilde{\mu} = \pm(\mu - \mu_c)$

- change coordinates (translation + rescaling) so in the new coordinates, the origin becomes unstable in a Hopf bifurcation as μ increases through 0

$\Rightarrow$ At the Hopf bifurcation ($\mu = 0$)

$$\left.\begin{aligned}\dot x &= -\omega y + f(x,y)\\ \dot y &= \omega x + g(x,y)\end{aligned}\right\} \quad f, g \text{ quadratic}$$

- then the formula for a is:

$$a = \frac{1}{16}\left[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right] + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx} g_{xx} + f_{yy} g_{yy}\right]$$

!

<u>Degenerate Hopf bifurcation</u> eg "damped" pendulum

$$\ddot x + \mu \dot x + \sin x = 0$$

$\mu > 0$: origin is <u>stable</u> spiral $\qquad \mu < 0$: un<u>stable</u> spiral

- but <u>no</u> limit cycles for $\mu > 0$ or $\mu < 0$!

<u>$\mu = 0$</u> : <u>conservative</u> <u>system</u> (nonlinear centre) <u>degenerate</u>

- continuous band of closed orbits. 

<u>Degenerate Hopf bifurcation</u> : $\mu \neq \mu_c$ non-conservative

At μ_c, nonlinear centre, not spiral $\longrightarrow$ $\mu = \mu_c$ conservative

eg
$$\left.\begin{aligned}\dot{x} &= \mu x - y + xy^2 \\ \dot{y} &= x + \mu y + y^3\end{aligned}\right\}$$
Fixed point $(x^*, y^*) = (0,0)$

Jacobian $A|_{(0,0)} = \begin{pmatrix} \mu + y^2 & -1 + 2xy \\ 1 & \mu + 3y^2 \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

Eigenvalues $\underline{\lambda = \mu \pm i}$.

$\mu < 0$ ⟶ $\mu = 0$ ⟶ $\mu > 0$

(0,0) stable spiral — expect Hopf bifurcation — (0,0) unstable spiral

<u>Type</u> - supercritical, subcritical or degenerate ?

Polar coordinates : $r\dot{r} = x\dot{x} + y\dot{y} = \mu(x^2 + y^2) + y^2(x^2 + y^2)$

$$\Rightarrow \dot{r} = \mu r + y^2 r \quad \geq \mu r \qquad (\dot{\theta} = 1)$$

⇒ For $\mu > 0$, $r(t)$ grows <u>at least as fast</u> as the solution of $\dot{r} = \mu r$ ie $r(t) \geq r_0 e^{\mu t}$ $\quad r(0) = r_0$

$\Big[$ $\dot{r} \geq \mu r \Rightarrow \dot{r} - \mu r \geq 0 \Rightarrow e^{-\mu t}(\dot{r} - \mu r) = \frac{d}{dt}\left(e^{-\mu t} r\right) \geq 0$
$\Rightarrow e^{-\mu t} r(t)$ is increasing with $t \Rightarrow e^{-\mu t} r(t) \geq r_0$ $\Big]$

$\mu > 0$: $\quad r(t) \geq r_0 e^{\mu t} \longrightarrow \infty$ for all $r_0 > 0$

⇒ no closed orbits for $\mu > 0$ (no limit cycles)

⇒ <u>not</u> supercritical Hopf.

$\mu = 0$: $\quad \dot{r} = y^2 r \geq 0$, $\dot{r} > 0$ for $y \neq 0, r \neq 0$

⇒ origin cannot be a nonlinear centre for $\mu = 0$

⇒ <u>not</u> degenerate Hopf.

<u>Thus</u> :

<u>Sub</u> critical Hopf bifurcation at $\mu = 0$

(unstable limit cycle for $\mu < 0$)

Oscillating Chemical Reactions

eg Belousov - Zhabotinsky reaction

oxidation of citric acid by bromate in acid medium, catalyzed by cerium

- observed oscillations in $[Br^-]$ and $[Ce^{4+}]/[Ce^{3+}]$ (yellow / colourless)

Chlorine dioxide - Iodine - Malonic Acid "CIMA" reaction

ClO_2 I_2 MA : $HOOC-CH_2-COOH$

oscillations in intermediates I^- and ClO_2^-

3 basic reaction steps

$$\left.\begin{array}{l} MA + I_2 \to IMA + I^- + H^+ \\ ClO_2 + I^- \to ClO_2^- + \tfrac{1}{2} I_2 \\ ClO_2^- + 4I^- + 4H^+ \to Cl^- + 2I_2 + 2H_2O \end{array}\right\}$$

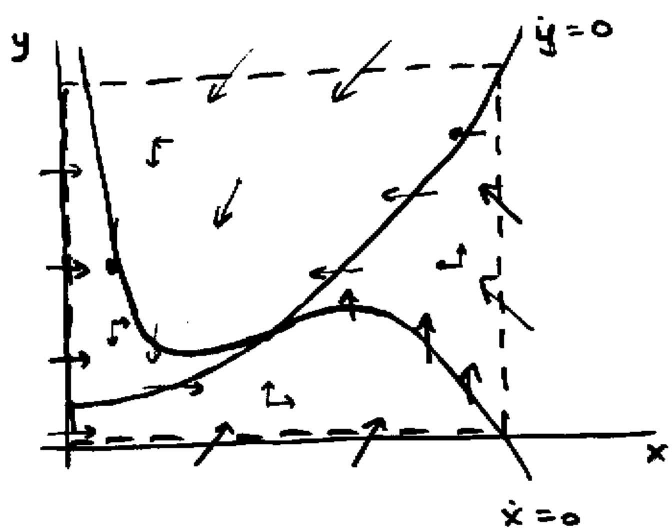
reactants MA, I_2, ClO_2 vary slowly over time scale of oscillations of I^-, ClO_2^- : assume reactants have (approximately) constant concentration

$\Rightarrow$ 2 ODE model, nondimensionalized

$$\left.\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \end{aligned}\right\}$$

$x \sim [I^-]$ $a, b > 0$

$y \sim [ClO_2^-]$



Nullclines:

$\dot{x} = 0 \Rightarrow y = \dfrac{(a-x)(1+x^2)}{4x}$

$\dot{y} = 0 \Rightarrow x = 0$ or $y = 1+x^2$

Trapping region

Fixed point: $\dot{x} = 0$ and $\dot{y} = 0$

$\Rightarrow 1+x^2 = \dfrac{(a-x)(1+x^2)}{4x} \Rightarrow a - x = 4x$

$\Rightarrow \boxed{x^* = a/5, \quad y^* = 1 + a^2/25}$

$$\left.\begin{aligned} \dot{x} &= a - x - \frac{4xy}{1+x^2} \\ \dot{y} &= bx\left(1 - \frac{y}{1+x^2}\right) \end{aligned}\right\} \qquad (x^*, y^*) = \left(\frac{a}{5},\ 1 + \frac{a^2}{25}\right)$$

If the fixed point is a <u>repeller</u>, there is a closed orbit inside the trapping region (Poincaré – Bendixson theorem)

Jacobian

$$A\Big|_{(x^*,y^*)} = \begin{pmatrix} -1 - 4y\,\dfrac{1-x^2}{(1+x^2)^2} & \dfrac{-4x}{1+x^2} \\ b\left(1 - y\,\dfrac{1-x^2}{(1+x^2)^2}\right) & -\dfrac{bx}{1+x^2} \end{pmatrix} = \frac{1}{1+(x^*)^2}\begin{pmatrix} 3(x^*)^2 - 5 & -4x^* \\ 2b(x^*)^2 & -bx^* \end{pmatrix}$$

(using $y^* = 1 + (x^*)^2$)

determinant

$$\Delta = \frac{-bx^*(3(x^*)^2 - 5) + 8b(x^*)^3}{(1+(x^*)^2)^2} = \frac{5bx^*}{1+(x^*)^2} = \frac{ab}{1 + a^2/25} > 0$$

Not saddle

trace

$$\tau = \frac{3(x^*)^2 - 5 - bx^*}{1+(x^*)^2} = \frac{3\frac{a^2}{25} - 5 - b\cdot\frac{a}{5}}{1 + a^2/25} = \frac{3a^2 - 5ab - 125}{25 + a^2}$$

Eigenvalues

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \qquad \tau = 0 : \boxed{b = b_c = \frac{3a}{5} - \frac{25}{a}}$$

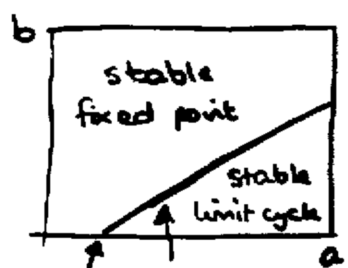
$\Rightarrow$ <u>pure</u> <u>imaginary</u> eigenvalues (<u>Hopf</u> bifurcation) if $\tau = 0$, $\Delta > 0$

ie at $b = b_c = \frac{3a}{5} - \frac{25}{a}$

$$(x^*, y^*) \text{ is } \begin{cases} \underline{\text{stable}} \text{ spiral} & \text{if } \tau < 0 \Rightarrow b > b_c \\ \underline{\text{unstable}} \text{ spiral} & \text{if } \tau > 0 \Rightarrow b < b_c \end{cases} \Bigg\}\ \text{assuming } \tau \text{ small, } \tau^2 < 4\Delta$$

$b < b_c = \frac{3a}{5} - \frac{25}{a}$: (x^*, y^*) is a repeller (unstable spiral) $\Rightarrow$ by P-B theorem trapping region contains a closed orbit : stable limit cycle

supercritical Hopf bifurcation



$a_c = \sqrt{\frac{125}{3}}$ $\quad b = \frac{3a}{5} - \frac{25}{a}$

Hopf bif. curve

Approximate period of oscillation :

At $b = b_c$, $\tau = 0$, $\Delta = \dfrac{a\left(\frac{3a}{5} - \frac{25}{a}\right)}{1 + a^2/25} = \dfrac{15a^2 - 625}{a^2 + 25}$: $\lambda = \pm i\sqrt{\Delta} = \pm i\omega$ $\quad \omega = \sqrt{\Delta} = \operatorname{Im}\lambda$: frequency

<u>Period</u> $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\operatorname{Im}\lambda} = 2\pi\left(\dfrac{a^2 + 25}{15a^2 - 625}\right)^{1/2} + O(|b - b_c|)$

$\sqrt{125}$

Global Bifurcations of Cycles

Global bifurcation – involves large regions of the phase plane, not just the neighbourhood of a fixed point.

Common mechanisms for the creation and destruction of limit cycles in 2-d:

① Hopf bifurcation – a local bifurcation (destabilization of a fixed point)

amplitude $\sim (|\mu-\mu_c|)^{1/2}$

period $\sim O(1)$

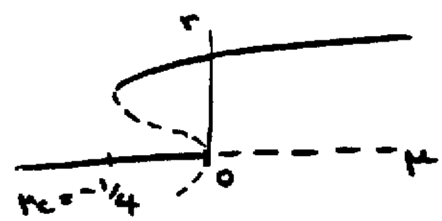
amplitude $\to 0$ as $\mu \to \mu_c$ – birth of small-amplitude limit cycles

② Saddle-Node Bifurcation of Cycles (Fold)

(eg in polar coordinates, saddle-node bifurcation of fixed points in radial equation)

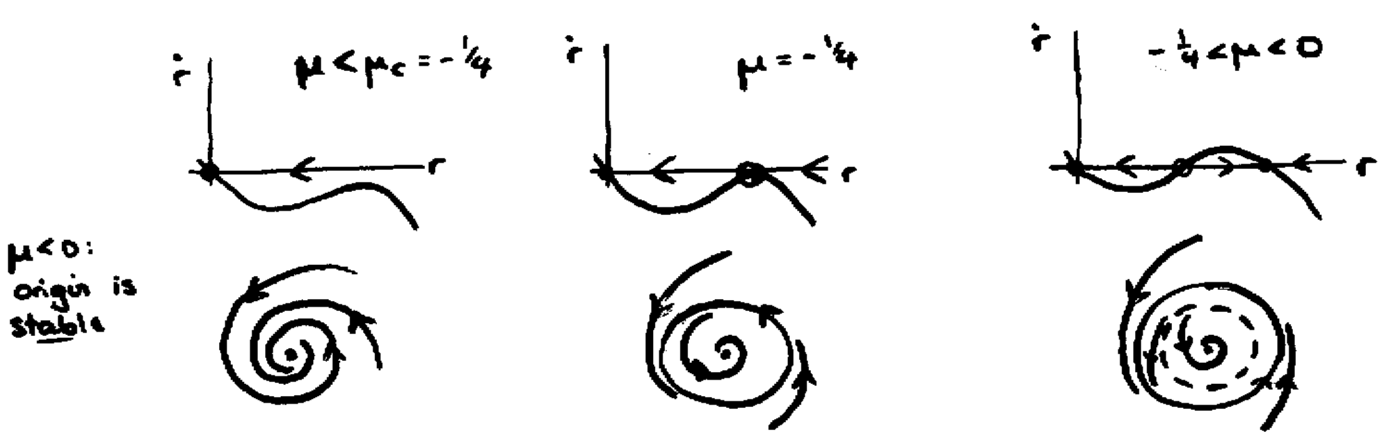
eg
$$\left.\begin{aligned} \dot r &= \mu r + r^3 - r^5 \\ \dot\theta &= \omega + b r^2 \end{aligned}\right\}$$

$\boxed{b \geq 0}$



$$\Rightarrow \dot r = r\left[\mu + \tfrac14 - (r^2 - \tfrac12)^2\right]$$

At $\mu = \mu_c = -\frac14$, saddle-node bifurcation of fixed points in r
$\Rightarrow$ saddle-node bifurcation of limit cycles in 2-d



Note: At bifurcation: amplitude $\sim O(1)$
period $\sim O(1)$

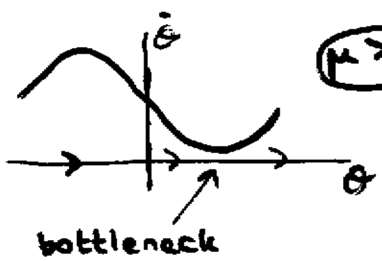
③ Infinite-Period Bifurcation

eg $\left.\begin{aligned}\dot r &= r(1-r^2)\\ \underbrace{\dot\theta = \mu - \sin\theta}\end{aligned}\right\}$ $\mu \geq 0$

$r=0$ unstable, $r=1$ stable

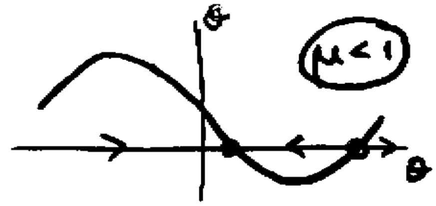
$r(0)>0 \Rightarrow r(t)\to 1$ as $t\to\infty$

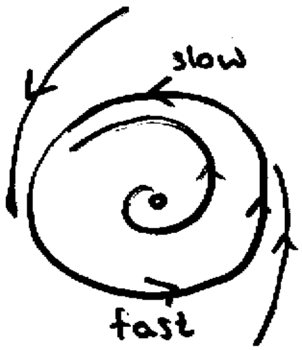
Saddle-node bifurcation at $\mu=1$, $\theta^* = \pi/2$



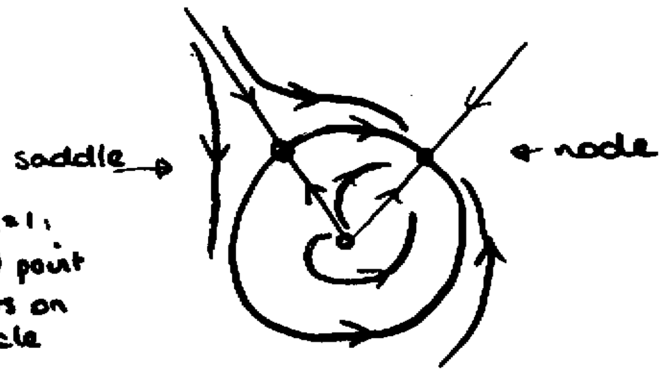
$\dot\theta > 0$: limit cycle at $r=1$, counter-clockwise

$\sin\theta^* = \mu$: invariant rays





$\mu = \mu_c = 1$: fixed point appears on circle



period $T \sim (|\mu-\mu_c|)^{-1/2}$ ← oscillation period becomes infinite at bifurcation

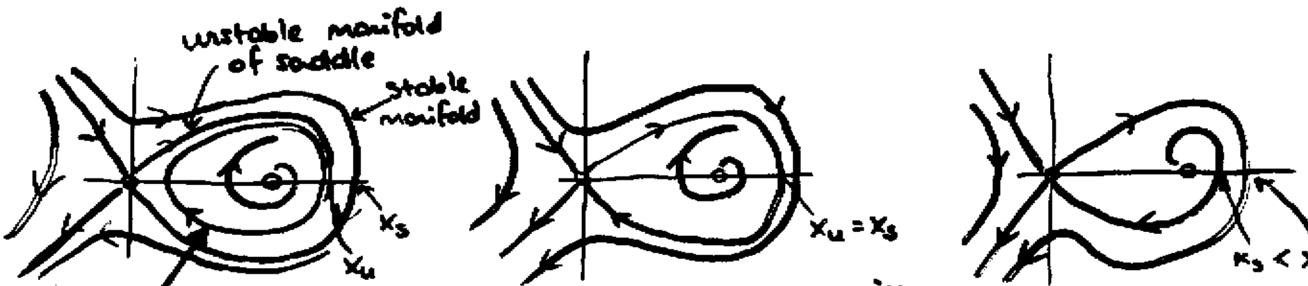
amplitude $\sim O(1)$

④ Homoclinic Bifurcation (Saddle-Loop)

eg $\left.\begin{aligned}\dot x &= y\\ \dot y &= \mu y + x - x^2 + xy\end{aligned}\right\}$

Fixed points
$(0,0)$ saddle
$(1,0)$ unstable spiral for $-1<\mu<1$

Bifurcation at $\mu = \mu_c \approx -0.8645$

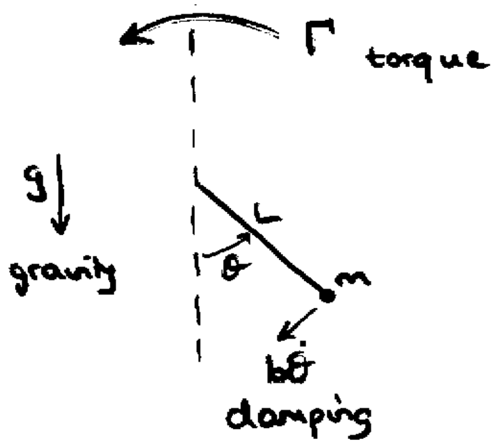


stable limit cycle | limit cycle collides with saddle: homoclinic loop | no cycle

Scaling: amplitude $\sim O(1)$, period $T \sim O(-\ln|\mu-\mu_c|)$ ← time for trajectory to pass saddle

Damped, Driven Pendulum

Bifurcations & Hysteresis



Balance of torques

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

$$\Rightarrow \frac{L}{g}\ddot{\theta} + \frac{b}{mgL}\dot{\theta} + \sin\theta = \frac{\Gamma}{mgL}$$

Nondimensionalize (different choice of time scale from before)

$$\tau = t/t_0\,, \quad t_0 = \sqrt{L/g}\,, \quad \alpha = \frac{b}{mgL}\sqrt{\frac{g}{L}}\,, \quad \gamma = \frac{\Gamma}{\Gamma_c} = \frac{\Gamma}{mgL}$$

$$\Rightarrow \quad \boxed{\theta'' + \alpha\theta' + \sin\theta = \gamma}$$

$$\Rightarrow \left.\begin{aligned} \theta' &= v \\ v' &= \gamma - \sin\theta - \alpha v \end{aligned}\right\}$$

cylindrical phase space

$\alpha \geq 0$ - physical

$\gamma \geq 0$ - without loss of generality

Fixed points $\quad v^* = 0, \quad \sin\theta^* = \gamma$

$\gamma > 1$ no fixed points
$\gamma < 1$ two fixed points

saddle-node bifurcation at $\gamma = 1$

Jacobian $\begin{pmatrix} 0 & 1 \\ -\cos\theta^* & -\alpha \end{pmatrix}$ $\qquad \tau = -\alpha < 0$

$\Delta = \cos\theta^* = \pm\sqrt{1-\gamma^2}$

$\Delta = +\sqrt{1-\gamma^2} > 0$: sink $\qquad \Delta = -\sqrt{1-\gamma^2} < 0$: saddle

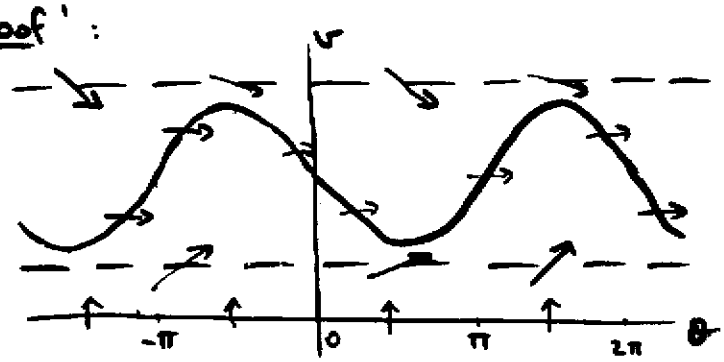
$\tau^2 - 4\Delta = \alpha^2 - 4\sqrt{1-\gamma^2}\begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix}$ - sink is a $\begin{Bmatrix} \text{node} \\ \text{spiral} \end{Bmatrix}$ ← γ near 1, or strong damping

$\gamma = 1$: node and saddle coalesce

$$\left.\begin{aligned} \theta' &= v \\ v' &= \gamma - \sin\theta - \alpha v \end{aligned}\right\} \qquad \gamma > 1: \text{ no fixed points}$$

- all trajectories attracted to <u>unique stable limit cycle</u>

'<u>Proof</u>':



$v = v_2 : v' < 0$

v-nullcline $v = \dfrac{\gamma - \sin\theta}{\alpha}$

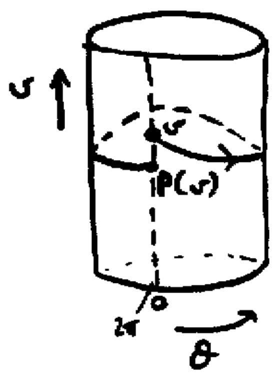
$v = v_1 : v' > 0$

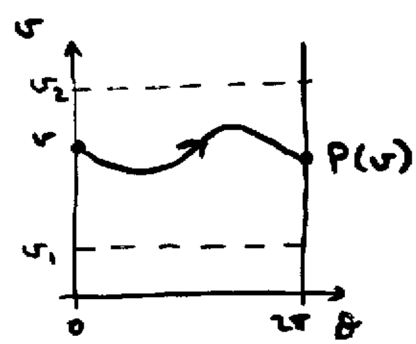
$v > 0 \Rightarrow \theta$ is increasing

All trajectories are attracted to the strip

$$v_1 \le v \le v_2 \qquad v_1 < \frac{\gamma - 1}{\alpha}, \quad v_2 > \frac{\gamma + 1}{\alpha}$$

<u>Cylindrical Phase Space</u> $S^1 \times \mathbb{R}$





<u>Poincaré map</u>

(first return map)

follow trajectory from $\theta = 0$ to $\theta = 2\pi = 0 \pmod{2\pi}$

- how does v change in one turn around cylinder?

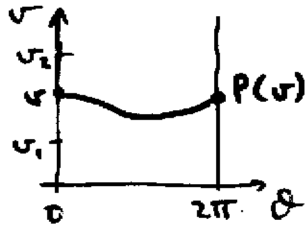
<u>Poincaré map</u>

- if trajectory has height v at $\theta = 0 \pmod{2\pi}$,
  $P(v)$ gives height when it first returns to $\theta = 0 \pmod{2\pi}$

<u>Properties</u> $P(v_1) > v_1$ $\qquad v' > 0$ on $v = v_1$: v increases

$P(v_2) < v_2$

Poincaré map $\quad v \longmapsto P(v)$



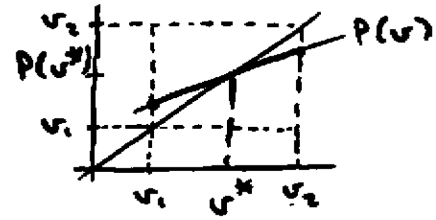
$P(v_1) > v_1$

$P(v_2) < v_2$

$P(v)$ is continuous — solutions of ODE depend continuously on initial conditions

$P(v)$ is monotonic (trajectories cannot cross)

$\Rightarrow P$ has a fixed point.

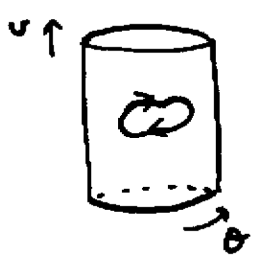


Fixed point of Poincaré map

$\Rightarrow$ periodic orbit

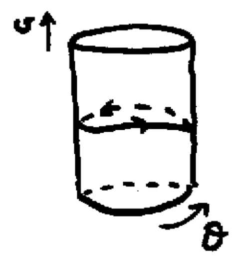
Isolated fixed point $\Rightarrow$ limit cycle

Topologically distinct limit cycles:



libration

- must enclose a fixed point (index theory)

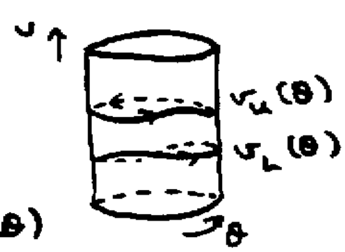


rotation

(whirling)

No fixed points for $\gamma > 1 \Rightarrow$ no librations

Uniqueness of limit cycle



derive a contradiction ...

Assume there are two limit cycles

Trajectories cannot cross $\Rightarrow v_u(\theta) > v_L(\theta)$ (all θ)

$$\frac{dv}{d\theta} = \frac{v'}{\theta'} = \frac{v'}{v}$$

Change in energy $E = \frac{1}{2}v^2 - \cos\theta$ after one cycle:

$$0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\theta}\, d\theta = \int_0^{2\pi} \left(v\frac{dv}{d\theta} + \sin\theta\right) d\theta = \int_0^{2\pi} \left(v\frac{v'}{v} + \sin\theta\right) d\theta$$

$$= \int_0^{2\pi} (\gamma - \cancel{\sin\theta} - \alpha v + \cancel{\sin\theta})\, d\theta = 2\pi\gamma - \alpha\int_0^{2\pi} v\, d\theta$$

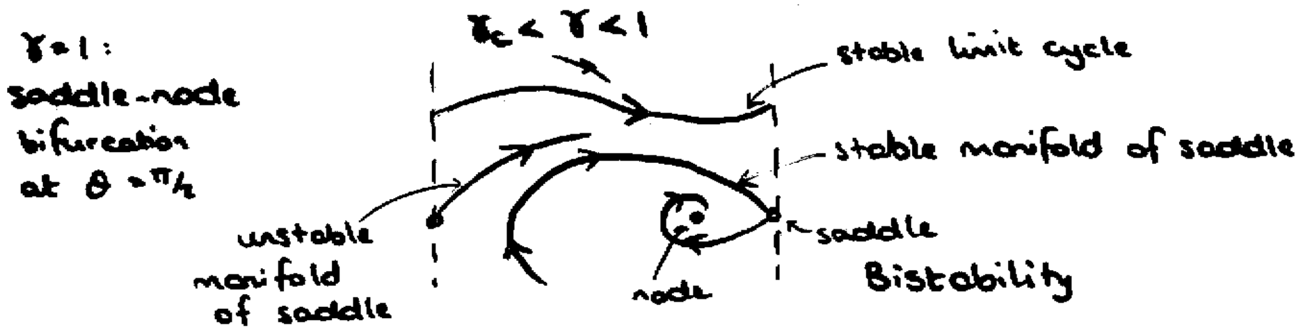
$$\Rightarrow \int_0^{2\pi} v\, d\theta = \frac{2\pi\gamma}{\alpha}$$, any rotation. But $\int_0^{2\pi} v_u\, d\theta > \int_0^{2\pi} v_L\, d\theta$!

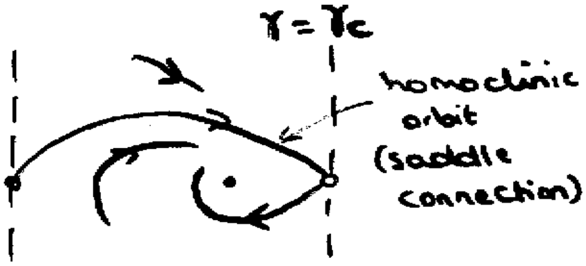
Unique stable limit cycle for $\gamma > 1$.

For small damping (small α):

As torque Γ is decreased, pendulum struggles to make it over the top. At some critical value $\gamma_c < 1$, torque is insufficient to overcome gravity and damping:

<u>Global bifurcation</u> at $\gamma = \gamma_c < 1$: (small α)

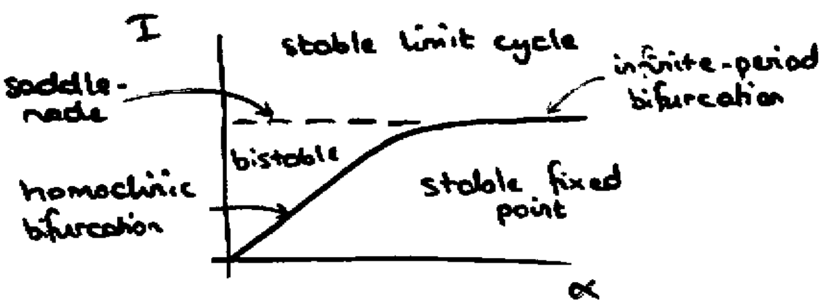


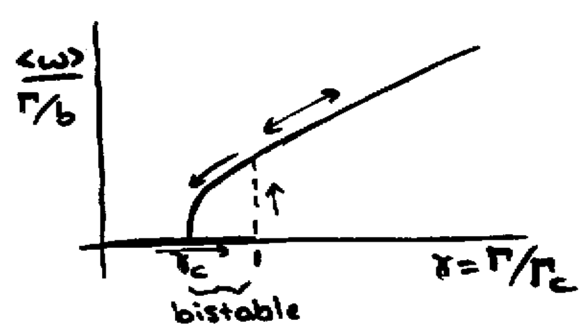


As γ decreases, the limit cycle moves down.

$\gamma = \gamma_c$: limit cycle merges with unstable manifold of saddle - <u>homoclinic bifurcation</u>

(for <u>large</u> α - overdamped limit - saddle-node bifurcation <u>on</u> the limit cycle : infinite-period bifurcation)





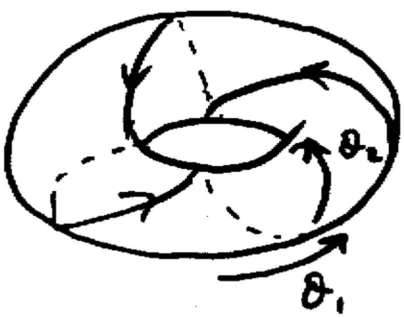
Quasiperiodicity

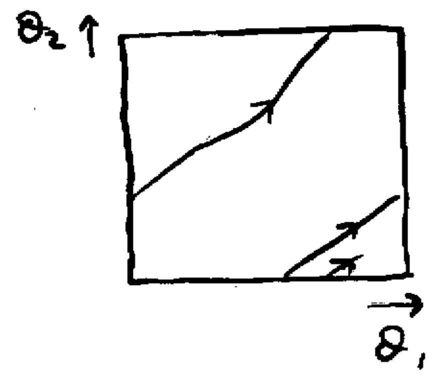
Dynamics on a torus

$$\left.\begin{aligned} \dot\theta_1 &= f_1(\theta_1, \theta_2) \\ \dot\theta_2 &= f_2(\theta_1, \theta_2) \end{aligned}\right\}$$

f_1, f_2 periodic in both θ_1 and θ_2

- natural phase space: torus $S^1 \times S^1$





periodic boundary conditions!

Uncoupled oscillators

$$\left.\begin{aligned} \dot\theta_1 &= \omega_1 \\ \dot\theta_2 &= \omega_2 \end{aligned}\right\}$$

Trajectories on square: slope $\dfrac{d\theta_2}{d\theta_1} = \dfrac{\omega_2}{\omega_1}$

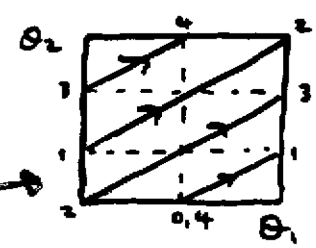
Two cases:

- $\dfrac{\omega_1}{\omega_2} = \dfrac{p}{q}$ rational p, q integers

  $\Rightarrow$ all trajectories are closed orbits

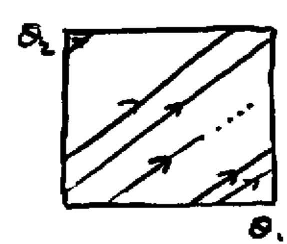
θ_1 completes p revolutions while θ_2 completes q revolutions

$p=3, q=2 \longrightarrow$



periodic

- $\dfrac{\omega_1}{\omega_2}$ irrational



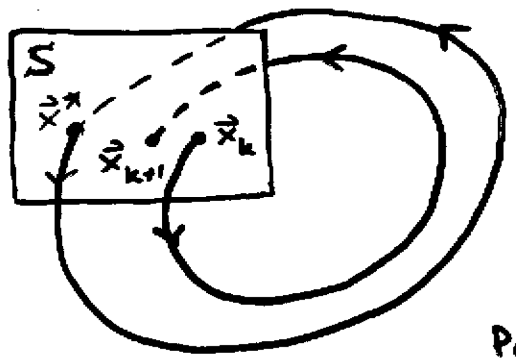
trajectories never close: quasiperiodic motion

(regular, not chaotic)

- densely covers the torus (trajectory comes arbitrarily close to any point)
- most complicated behaviour in 2-d systems

Poincaré Maps

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \qquad \vec{x} \in \mathbb{R}^n$$



S : surface of section
$(n-1)$-dimensional
transverse to flow
(vector field not parallel to S)

Poincaré Map $\quad P : S \to S$

$\vec{x}_{k+1} = P(\vec{x}_k)$: trajectory starting at $\vec{x}_k \in S$
$\vec{x}_{k+1}$ is next intersection
(first return)

Fixed point of Poincaré map: $\vec{x}^* = P(\vec{x}^*) \Rightarrow$ closed orbit of $\dot{\vec{x}} = \vec{f}(\vec{x})$

Stability of periodic orbits $\Leftrightarrow$ stability of fixed point $\vec{x}^*$ of P

$\vec{x}_k = \vec{x}^* + \vec{v}_k \leftarrow$ small perturbation

$$\vec{v}_{k+1} = \vec{x}_{k+1} - \vec{x}^* = P(\vec{x}_k) - \vec{x}^* = P(\vec{x}^* + \vec{v}_k) - \vec{x}^*$$
$$= \underbrace{P(\vec{x}^*)}_{=\vec{x}^*} + DP(\vec{x}^*)\,\vec{v}_k + \dots \quad - \vec{x}^*$$

$$\Rightarrow \quad \vec{v}_{k+1} = DP(\vec{x}^*)\,\vec{v}_k + O(\|\vec{v}_k\|^2)$$

$DP(\vec{x}^*)$: linearized Poincaré map at $\vec{x}^*$: $(n-1)\times(n-1)$ matrix

Let λ_j, $j = 1, \dots, n-1$ be the eigenvalues of $DP(\vec{x}^*)$, with eigenvectors $\vec{e}_j$

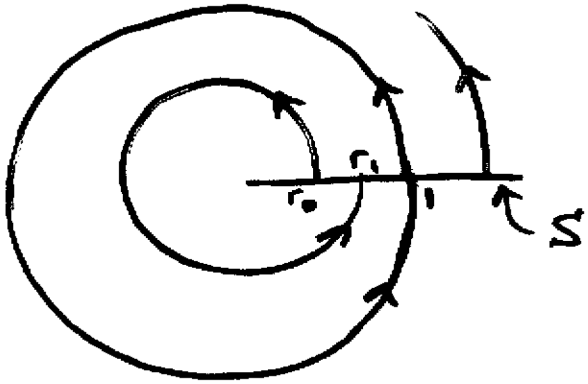
$$\vec{v}_k = \sum_{j=1}^{n-1} a_j^{(k)} \vec{e}_j \quad \Rightarrow \quad \vec{v}_{k+1} = DP(\vec{x}^*)\vec{v}_j + \dots = \sum_{j=1}^{n-1} a_j^{(k)} \lambda_j \vec{e}_j + \dots$$

So $\|\vec{v}_k\| \longrightarrow 0$ as $k \to \infty$ provided $|\lambda_j| < 1$, $j = 1 \dots n-1$

$|\lambda_j| > 1$, some j: perturbations in direction $\vec{e}_j$ grow

$|\lambda_j| = 1$: bifurcation of periodic orbits

λ_j, $j = 1 \dots n-1$: Characteristic (Floquet) multipliers (and $\lambda_0 = 1$: perturbations along orbit) eg $\left.\begin{aligned} \dot r &= r(1-r^2) \\ \dot\theta &= 1 \end{aligned}\right\}$ $\quad$ $r=0$: unstable fixed point; $r=1$: stable limit cycle



positive x-axis
ie $\theta \bmod 2\pi = 0$

$\dot\theta = 1 \Rightarrow$ time to return to S is $t = 2\pi$.

$r_1 = P(r_0)$: $\quad \dfrac{dr}{dt} = r(1-r^2) \Rightarrow \displaystyle\int \frac{dr}{r(1-r^2)} = \int dt$

$$\Rightarrow \int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi$$

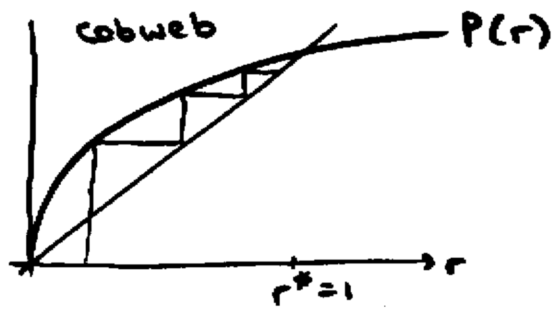
$$\Rightarrow \dots \Rightarrow r_1 = \left[1 + e^{-4\pi}(r_0^{-2} - 1)\right]^{-1/2} \equiv P(r_0)$$

<u>Stability</u>:

$P'(r^*) = P'(1) = e^{-4\pi} < 1$

$\Rightarrow$ stable limit cycle

Floquet multiplier



$\Big\lceil$ $r = 1+\eta \Rightarrow \dot r = r(1-r^2) = (1+\eta)(1-(1+\eta)^2)$

$\dot\eta = (1+\eta)(-2\eta - \eta^2) = -2\eta + O(\eta^2)$

$\Rightarrow \eta(t) \approx \eta_0 e^{-2t}$ for small η

After time 2π, $\quad \eta_1 \approx \underbrace{e^{-4\pi}}_{\text{multiplier}} \eta_0$ $\Big\rfloor$