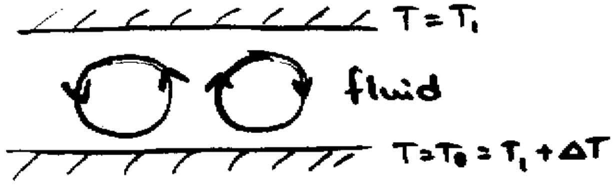
# Lorenz Equations

- model for waterwheel
- simplified model for convection rolls




$$\left.\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - x\underline{\underline{z}} \\ \dot{z} &= \underline{\underline{x}}y - bz \end{aligned}\right\}$$

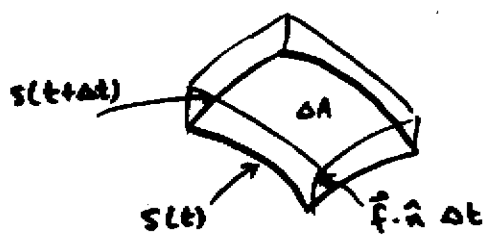
3-d dynamical system
(two nonlinear terms)
$r, \sigma, b > 0$

$\sigma$ ~ Prandtl number
$r$ ~ Rayleigh number
$b$ ~ aspect ratio

**Symmetry:** $(x, y, z) \to (-x, -y, z)$

**Dissipativity** - volume contraction

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$




Consider small volume $V(t)$, surface $S(t)$, normal $\hat{n}$.
- normal velocity $\vec{f}\cdot\hat{n}$

Small area element $\Delta A$, time $\Delta t$:
surface moves a distance $\vec{f}\cdot\hat{n}\,\Delta t$

$$\Rightarrow \Delta V = (\vec{f}\cdot\hat{n}\,\Delta t)\,\Delta A$$

Integrate: total change in volume

$$V(t+\Delta t) - V(t) = \left(\oint_S \vec{f}\cdot\hat{n}\,dA\right)\Delta t$$

$$\Rightarrow \frac{dV}{dt} = \oint_S \vec{f}\cdot\hat{n}\,dA \underset{\uparrow}{=} \int_V (\nabla\cdot\vec{f})\,dV$$

divergence theorem

Lorenz System:

$$\nabla\cdot\vec{f} = \frac{\partial\dot{x}}{\partial x} + \frac{\partial\dot{y}}{\partial y} + \frac{\partial\dot{z}}{\partial z} = \underbrace{-\sigma - 1 - b}_{\text{constant}} < 0$$

$$\Rightarrow \frac{dV}{dt} = -(\sigma+1+b)V \quad \Rightarrow V(t) = V(0)\,e^{-(\sigma+1+b)t}$$

$$\left.\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}\right\}$$

Volume contraction:

$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

- volumes in phase space shrink exponentially fast

$\Rightarrow$ no repelling fixed points or closed orbits "sources of volume"

no invariant tori or quasiperiodic solutions 

Fixed points: $\dot{x} = 0 \Rightarrow y = x$

$\dot{y} = 0 \Rightarrow x(r-1-z) = 0 \Rightarrow x = 0$ or $z = r-1$

$\dot{z} = 0 \Rightarrow xy = x^2 = bz$

$O$ : $(0,0,0)$ origin "conduction state"

$C^{\pm}$ : $x^* = y^* = \pm\sqrt{b(r-1)}$ $\quad r>1$

$z^* = r-1$ "convection rolls"

Stability: $$Df\big|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$\lambda_3 = -b$ : exponential decay in $z$-direction (z-axis $x=y=0$ is invariant, always stable) $z(t) \sim e^{-bt}$

$\lambda_{1,2}$ : eigenvalues of $\begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$ : $\tau = -(\sigma+1) < 0$

$\Delta = \sigma(1-r)$

$r<1$ : origin is linearly stable - in fact: nonlinear stability:

Liapunov function: $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2 > 0 \quad (\vec{x} \neq \vec{0})$

$$\frac{1}{2}\dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\dot{z} = x(y-x) + y(rx-y-xz) + z(xy-bz)$$
$$= -x^2 + (r+1)xy - y^2 - bz^2 = -\left(x + \frac{r+1}{2}y\right)^2 - \underbrace{\left(1 - \left(\frac{r+1}{2}\right)^2\right)}_{>0 \text{ for } r<1} y^2 - bz^2$$

$r<1 \Leftrightarrow \left(\frac{r+1}{2}\right)^2 < 1 \Rightarrow \dot{V} < 0$ for $\vec{x} \neq \vec{0}$

$\Rightarrow$ origin is globally stable for $r<1$: all trajectories $\to \vec{0}$.

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At $r=1$ : supercritical pitchfork bifurcation : $C^{\pm}$ created

$r>1$ : origin is an unstable saddle $\begin{pmatrix} \text{2 stable directions} \\ \text{1 unstable direction} \end{pmatrix}$

$$\left.\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}\right\}$$

Fixed points $O = (0,0,0)$ : stable for $r<1$

$C^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$

Stability of $C^{\pm}$ : linearly stable for $1 < r < r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$

Subcritical Hopf bifurcation at $r = r_H$ :

For $\sigma=10, b=8/3$
$r_H \approx 24.74$

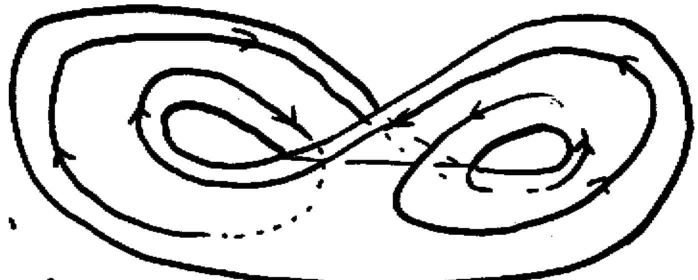
unstable limit cycles for $r<r_H$ :

saddle cycles : 2-d stable manifold
2-d unstable manifold

$r > r_H$ :


- trajectories are bounded
- no stable limit cycles — r slightly above $r_H$
- aperiodic long-term dynamics

Strange Attractor:



Trajectories approach an attracting set of zero volume:

they appear to approach a pair of surfaces, merging into one - but trajectories cannot intersect

More detailed analysis:

"infinite complex of surfaces" extremely close

"Fractal" dimension ~ 2.05

Sensitive Dependence on Initial Conditions: → if $\lambda > 0$
(SDIC)

Exponentially fast separation of neighbouring trajectories :


two trajectories starting close together rapidly diverge

$\|\vec{\delta}(t)\| \sim \|\vec{\delta}(0)\| e^{\lambda t}$ (small t)

$\lambda$: (largest) Liapunov exponent

- breakdown of prediction

- to $t_{horizon} \sim \frac{1}{\lambda} \ln \frac{a}{\|\vec{\delta}(0)\|}$ (a : tolerance)

**Chaos** : Aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions

Aperiodic - no attracting fixed points, periodic or quasiperiodic orbits

Deterministic - no noise

**Attractor** : a set A to which all neighbouring trajectories converge

- positively invariant - if $\vec{x}(0) \in A$, then $\vec{x}(t) \in A$ for $t \geq 0$
- basin of attraction : A attracts an open set U of initial conditions
  $\searrow$ U  $A \subset U$
- A is minimal

Attractors - eg
- stable fixed points
- stable periodic orbits
- strange attractors ← attractor with SDIC

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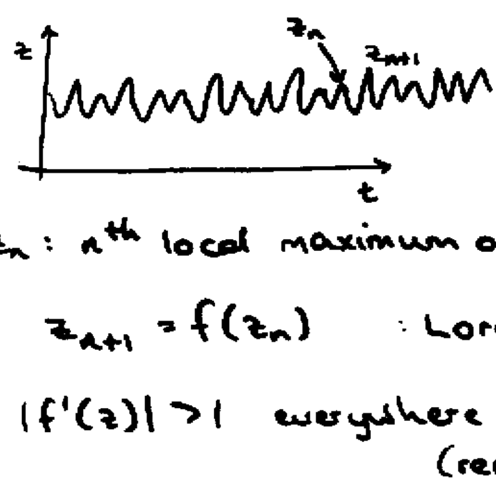
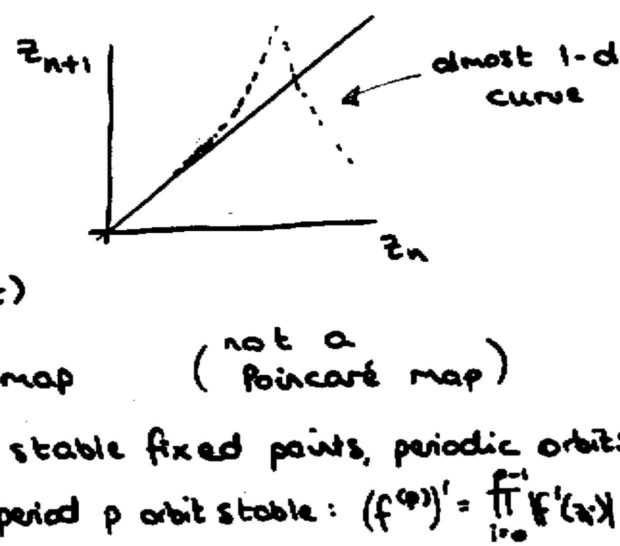
## Lorenz Map




$z_n$ : $n^{th}$ local maximum of $z(t)$

$z_{n+1} = f(z_n)$ : Lorenz map  (not a Poincaré map)

$|f'(z)| > 1$ everywhere — no stable fixed points, periodic orbits
(recall : period p orbit stable : $(f^{(p)})' = \prod_{i=0}^{p-1} |f'(x_i)| < 1$

## Bifurcations in Parameter Space

$\sigma = 10$
$b = 8/3$

$r = r_{hom} \approx 13.926$ homoclinic bifurcation
$r = r_s \approx 24.06$ strange attractor created
$r = r_H \approx 24.74$ Hopf bifurcation