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Bounds on Rayleigh–Bénard convection with an imposed heat flux

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We formulate a bounding principle for the heat transport in Rayleigh–Bénard convection with fixed heat flux through the boundaries. The heat transport, as measured by a conventional Nusselt number, is inversely proportional to the temperature drop across the layer and is bounded above according to $Nu \leqslant c\hat{R}^{1/3}$, where $c < 0.42$ is an absolute constant and $\hat{R} = \alpha g \beta h^4/(\nu\kappa)$ is the 'effective' Rayleigh number, the non-dimensional forcing scale set by the imposed heat flux $\kappa\beta$. The relation among the parameter $\hat{R}$, the Nusselt number, and the conventional Rayleigh number defined in terms of the temperature drop across the layer, is $Nu\, Ra = \hat{R}$, yielding the bound $Nu \leqslant c^{3/2} Ra^{1/2}$.

1. Introduction

A quantity of fundamental interest in Rayleigh–Bénard convection is the total heat transport through the layer. This is often expressed in terms of the Nusselt number, Nu, measuring the enhancement of heat flux beyond pure conduction. The total flux depends on the temperature drop across the layer, the layer geometry and various material parameters which together determine a control parameter usually taken as the Rayleigh number, Ra. A major goal of both theory and experiment is to elucidate the relation between Nu and Ra.

In high Rayleigh number flows this relation may take the scaling form $Nu \sim Ra^p$, and much effort has gone into determining the value of the exponent p, both in experimental settings and from theoretical considerations. Dating back to the 1960s, Kraichnan (1962), Howard (1963), Busse (1969), and Spiegel (1971) all suggested – either from physical arguments or as estimates from the Boussinesq equations under some statistical assumptions – the value $p = \frac{1}{2}$ (with logarithmic corrections in Kraichnan's theory). In the 1990s an idea of Hopf (1941), relying on the decomposition of the dependent variables into 'background' and 'fluctuating' components, was revived and used to derive rigorous bounds for bulk-flow quantities in several fundamental fluid flows, with no *a priori* assumptions. When applied to Rayleigh–Bénard convection with fixed temperature boundary conditions, the resulting exponent in the bound was again $p = \frac{1}{2}$ (Doering & Constantin 1996).

Upper-bound analyses such as those referred to above have produced rather sharp results by comparison with experimental data or theoretical expectations (perhaps within logarithms) in a number of fluid flows including shear flow, infinite

Prandtl number convection, and porous-medium convection; see Doering & Constantin (1992, 1998) and Constantin & Doering (1999). In the case of finite Prandtl number convection, however, the upper bounds lie well above the experimental data. Indeed, typical scaling exponents seen in experiments typically fall in the range $0.27 \leqslant p \leqslant 0.31$ (Heslot, Castaing & Libchaber 1987; Niemela *et al.* 2000; Xu, Bajaj & Ahlers 2000), although there is some debate as to whether higher exponents have been measured (Chavanne *et al.* 1997). For a recent review see Kadanoff (2001).

It is natural to wonder whether these unresolved discrepancies could be due to some fundamental incompatibility between the model from which the theoretical bounds are deduced and the physical experiments they are intended to describe. In particular, previous analyses have assumed that the temperature of the boundaries in a convection experiment remains fixed, which requires that the bordering plates must be far better heat conductors than the fluid they contain. This is the case for many experiments when the conductivity of the plates is many orders of magnitude greater than that of the fluid and the Nusselt numbers are relatively modest. However, some convection experiments, for instance those using liquid mercury for the fluid and copper for the plates, provide an exception. The relative conductivities of mercury and copper differ only by a factor of 50 ($8.3\,\mathrm{J\,m^{-1}\,K^{-1}}$ for Hg, $401\,\mathrm{J\,m^{-1}\,K^{-1}}$ for Cu), so when the Nusselt number exceeds 50 (the experiment in Glazier *et al.* (1999) reaches Nusselt values of about 100), the fluid is effectively able to conduct heat at least as well as the plates. For such a flow the boundaries (i.e. the plates) cannot be maintained at fixed temperature. This problem presents itself for any convective flow whose Nusselt number exceeds the ratio of conductivities between plates and fluid, and should properly be taken into account in theoretical studies seeking uniform bounds in the asymptotic limit of large Rayleigh number.

Based on these considerations, we have begun to focus on how different temperature boundary conditions, taking the thermal properties of the plates into account, might affect the transport of the flow. One of the simplest variants of the boundary conditions to consider – apart from fixed temperature boundaries – is that of fixed heat flux. Such boundary conditions describe scenarios where the plates conduct heat poorly compared with the fluid. These boundary conditions have profound effects on the transition from conduction to convection: when the flux, rather than the boundary temperatures, is fixed, the bifurcation becomes a long-wave instability (see Hurle, Jakeman & Pike 1967; Chapman & Proctor 1980). In addition, the weakly nonlinear regime favours square-pattern convection, in contrast to the two-dimensional rolls favoured in the fixed temperature problem (Busse & Riahi 1980).

In this brief paper, we introduce the effect of varying temperature boundary conditions by presenting the fundamental uniform *a priori* bound for the fixed flux problem, following the methods of the 'background field' approach (see Doering & Constantin 1996). In subsequent work we will present the results of more comprehensive investigations, incorporating numerical and analytical studies of the fixed flux case, as well as of the experimentally more realistic situation in which the fluid is bounded by plates of finite thickness and conductivity.

The structure of this paper is as follows: In the next section we present the equations of motion and derive expressions for the fundamental flow quantities. In § 3 we formulate a bounding principle for the heat transport. A rigorous upper bound on Nu as a function of the (non-dimensional) imposed heat flux and the conventional Rayleigh number is derived in the final § 4.

2. Statement of the problem

The equations of motion based on the Boussinesq approximation are

$$\boldsymbol{u}^*_t + \boldsymbol{u}^* \cdot \nabla \boldsymbol{u}^* + \frac{1}{\rho}\nabla P^* = \nu \nabla^2 \boldsymbol{u}^* + \alpha g (T^* - T_0)\boldsymbol{e}_z, \quad \nabla \cdot \boldsymbol{u}^* = 0, \tag{2.1}$$

$$T^*_t + \boldsymbol{u}^* \cdot \nabla T^* = \kappa \nabla^2 T^*, \tag{2.2}$$

$$\left.\frac{\partial T^*}{\partial z}\right|_{z=0,h} = -\beta \quad \text{and} \quad \boldsymbol{u}^*|_{z=0,h} = 0, \tag{2.3}$$

where ν and κ are the diffusivity constants for momentum and temperature, α is the thermal expansion coefficient, g is the acceleration due to gravity, ρ the density at some reference temperature T_0, $\kappa\beta$ the constant heat flux at the boundaries and h the height of the layer. Variables with an asterisk are dimensional and we take periodic boundary conditions in the horizontal directions. A non-dimensional set of variables can be obtained by taking h, h^2/κ and $h\beta$ as the relevant space, time and temperature scales. The equations for the non-dimensional velocity $\boldsymbol{u} = (u, v, w)$ and temperature T are

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \sigma \nabla^2 \boldsymbol{u} + \sigma \hat{R}\, T \boldsymbol{e}_z, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{2.4}$$

$$T_t + \boldsymbol{u} \cdot \nabla T = \nabla^2 T, \tag{2.5}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=0,1} = -1 \quad \text{and} \quad \boldsymbol{u}|_{z=0,1} = 0, \tag{2.6}$$

where $\hat{R} = \alpha g \beta h^4/(\nu\kappa)$ and $\sigma = \nu/\kappa$ is the usual Prandtl number. The control parameter $\hat{R}$ is not generally the same as the usual Rayleigh number Ra, but is related to it and the familiar Nusselt number Nu by

$$\hat{R} = Ra\, Nu. \tag{2.7}$$

To see how this follows from the equations of motion, we first introduce some notation: for functions $f(x, y, z)$, $g(t)$ we define the horizontal and time averages by

$$\bar{f}(z) = \frac{1}{A}\int f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y, \tag{2.8}$$

and

$$\langle g \rangle = \limsup_{T\to\infty} \frac{1}{T}\int_0^T g(t)\, \mathrm{d}t, \tag{2.9}$$

where A is the (non-dimensional) area of the plates. In addition, $\int f$ denotes a volume integral over the entire fluid layer. We define the temperature difference as

$$\Delta T^* = \langle \overline{T}^*(0) - \overline{T}^*(h) \rangle, \tag{2.10}$$

$$\Delta T = \langle \overline{T}(0) - \overline{T}(1) \rangle \quad \text{(non-dimensional)}. \tag{2.11}$$

Recalling that $Ra = \alpha g h^3 \Delta T^*/(\nu\kappa)$ and $\Delta T^* = h\beta\Delta T$, we now see that $\hat{R}$ and Ra are related by

$$Ra = \hat{R}\, \Delta T. \tag{2.12}$$

To define the Nusselt number, Nu, we first write the temperature equation (2.5) as $T_t + \nabla \cdot \boldsymbol{J} = 0$, where $\boldsymbol{J} = \boldsymbol{u}T + \boldsymbol{J}_c$ is the heat current, and $\boldsymbol{J}_c = -\nabla T$ is the conductive part of $\boldsymbol{J}$. The Nusselt number is defined to be the ratio of the (average) total convective and conductive heat transport in the vertical direction to the purely conductive heat transport:

$$Nu = \frac{\frac{1}{A}\left\langle \int \boldsymbol{J} \cdot \boldsymbol{e}_z \right\rangle}{\frac{1}{A}\left\langle \int \boldsymbol{J}_c \cdot \boldsymbol{e}_z \right\rangle}. \tag{2.13}$$

Since

$$\frac{1}{A}\left\langle \int \boldsymbol{J}_c \cdot \boldsymbol{e}_z \right\rangle = \frac{1}{A}\left\langle \int -T_z \right\rangle \tag{2.14}$$

$$= \frac{1}{A}\left\langle \int \left(\int_0^1 -T_z dz \right) \mathrm{d}x\,\mathrm{d}y \right\rangle = \Delta T, \tag{2.15}$$

we have

$$Nu = 1 + \frac{\frac{1}{A}\left\langle \int wT \right\rangle}{\Delta T}. \tag{2.16}$$

Expression (2.16) may be further simplified by relating the quantities ΔT and $\langle \int wT \rangle$. To this end, we take the horizontal average of the temperature equation (2.5), multiply by z and integrate over z to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z\overline{T}\,\mathrm{d}z \right) + \int_0^1 z(\overline{wT} - \overline{T}_z)_z\,\mathrm{d}z = 0. \tag{2.17}$$

Integrating by parts the second term, we find

$$\int_0^1 z(\overline{wT} - \overline{T}_z)_z\,\mathrm{d}z = z(\overline{wT} - \overline{T}_z)\big|_0^1 - \int_0^1 (\overline{wT} - \overline{T}_z)\,\mathrm{d}z \tag{2.18}$$

$$= 1 - \frac{1}{A}\int wT + \overline{T}(1) - \overline{T}(0). \tag{2.19}$$

Using this last expression in (2.17) and taking the time average gives

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 z\overline{T}\,\mathrm{d}z \right) \right\rangle + 1 - \frac{1}{A}\left\langle \int wT \right\rangle - \Delta T = 0. \tag{2.20}$$

The first term in (2.20) vanishes because $\int T^2$ is uniformly bounded in time, a fact that can be deduced from the analysis in this paper (see the analogous discussion in Doering & Constantin 1992). Thus we have the identity

$$\frac{1}{A}\left\langle \int wT \right\rangle + \Delta T = 1. \tag{2.21}$$

This expression may now be used in (2.16) to deduce

$$Nu = \frac{1}{\Delta T}, \tag{2.22}$$

which, together with (2.12), implies

$$\hat{R} = Nu\,Ra. \tag{2.23}$$

We may also relate Nu to the viscous energy dissipation rate in the flow. Taking the inner product of the momentum equation (2.4) with $\boldsymbol{u}$, integrating over the layer and taking the time average, and then using (2.21) gives

$$\Delta T = 1 - \frac{1}{A\hat{R}} \left\langle \int |\nabla \boldsymbol{u}|^2 \right\rangle . \tag{2.24}$$

We are now ready to discuss how one might establish a bound for the heat transport in such a flow. The key is to bound the temperature difference, ΔT, from below in terms of $\hat{R}$, thereby bounding $Nu = 1/\Delta T$ from above and $Ra = \hat{R}\,\Delta T$ from below. These estimates can then be combined into a single inequality producing an upper bound for Nu in terms of Ra.

3. Formulation of the bound

We now decompose the temperature field into a background $\tau(z)$ – which carries the boundary conditions of the flow – and a fully space- and time-dependent component θ:

$$T = \tau(z) + \theta, \tag{3.1}$$

where we impose $\tau'(0) = \tau'(1) = -1$, so the boundary conditions on $\theta = \theta(x, y, z, t)$ are $\theta_z\big|_{z=0,1} = 0$.

Inserting (3.1) into the temperature equation produces an evolution equation for θ:

$$\theta_t + \boldsymbol{u} \cdot \nabla\theta = \nabla^2\theta + \tau'' - w\tau'. \tag{3.2}$$

Multiplying by θ and taking the space–time average of this last expression gives the constraint

$$\Delta T = \frac{1}{A} \left\langle \int |\nabla\theta|^2 + \int \theta_z\tau' + \int \theta w\tau' \right\rangle + \Delta\tau, \tag{3.3}$$

where $\Delta\tau = \tau(0) - \tau(1)$. In (3.3) the term $\theta_z\tau'$ appears and turns out to be inconvenient when formulating a variational principle for the bound we seek. To eliminate it we multiply the θ-equation (3.2) by τ and integrate. After some integration by parts, time averaging and the use of incompressibility, we deduce that

$$\frac{1}{A} \left\langle \int \theta_z\tau' \right\rangle = \frac{1}{A} \left\langle \int \theta w\tau' \right\rangle - \int_0^1 \tau'^2 \, \mathrm{d}z + \Delta\tau. \tag{3.4}$$

Using this expression we may rewrite (3.3) as

$$\Delta T = \frac{1}{A} \left\langle \int |\nabla\theta|^2 + 2\int \theta w\tau' \right\rangle + 2\Delta\tau - \int_0^1 \tau'^2 \, \mathrm{d}z. \tag{3.5}$$

Taking a weighted average of (2.24) and (3.5), we have

$$\Delta T = b\Delta T + (1-b)\Delta T \tag{3.6}$$

$$= 2b\Delta\tau - b\int_0^1 \tau'^2 \, \mathrm{d}z + (1-b) + \frac{b}{A} \left\langle \int \left(|\nabla\theta|^2 + \frac{b-1}{b\hat{R}} |\nabla \boldsymbol{u}|^2 + 2\theta w\tau' \right) \right\rangle . \tag{3.7}$$

Hence

$$\Delta T = B_b(\tau) + \frac{b}{A} \mathscr{Q}_{\tau,b}(\theta, w), \tag{3.8}$$

where

$$B_b(\tau) = 2b\Delta\tau - b\int_0^1 \tau'^2 \, \mathrm{d}z + (1-b), \tag{3.9}$$

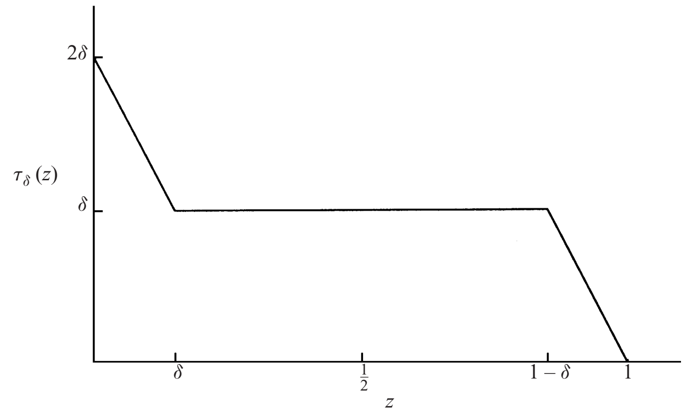

FIGURE 1. A graph of $\tau_\delta(z)$. Note that $\Delta\tau = \tau(0) - \tau(1) = 2\delta$.

and

$$\mathcal{Q}_{\tau,b}(\theta, w) = \left\langle \int \left(|\nabla\theta|^2 + \frac{b-1}{b\hat{R}} |\nabla \boldsymbol{u}|^2 + 2\theta w \tau' \right) \right\rangle. \tag{3.10}$$

The technical basis for the bounding procedure is as follows: if we can choose $\tau(z)$ and $b > 1$ so that the quadratic form $\mathcal{Q}_{\tau,b}(\theta, w) \geqslant 0$ for all relevant fields θ and w, then

$$\Delta T \geqslant B_b(\tau). \tag{3.11}$$

4. An explicit bound

We will now produce a background profile for which the quadratic form $\mathcal{Q}_{\tau,b}$ is positive definite. Consider the one-parameter family of profiles shown in figure 1; note that for this choice of the profile, $\Delta\tau = \int_0^1 \tau'^2 \mathrm{d}z = 2\delta$. We choose $b = 1 + c\delta$, with $c > 0$.

With these choices for τ and b it is easy to verify that

$$B = (2-c)\delta + 2c\delta^2 \geqslant (2-c)\delta, \tag{4.1}$$

$$\mathcal{Q} = \left\langle \int \left(|\nabla\theta|^2 + \frac{c\delta}{(1+c\delta)\hat{R}} |\nabla \boldsymbol{u}|^2 + 2\theta w \tau' \right) \right\rangle, \tag{4.2}$$

where we have dropped subscripts on B and $\mathcal{Q}$. To incorporate the incompressibility constraint – which is absolutely essential in this analysis – we will use the horizontal periodicity of the layer and recast the problem in Fourier variables:

$$w(\boldsymbol{x}, z) = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} w_{\boldsymbol{k}}(z), \tag{4.3}$$

where $\boldsymbol{x} = (x, y)$ and $\boldsymbol{k}$ is the horizontal wave vector; we will also use the notation $k^2 = |\boldsymbol{k}|^2$. The other variable θ may be expanded in the same way. Using incompressibility (and writing $\tilde{w}$ for the complex conjugate of w and $\mathrm{D} = \mathrm{d}/\mathrm{d}z$) we may express the quadratic form as

$$\mathcal{Q} \geqslant \sum_{\boldsymbol{k}} \mathcal{Q}_{\boldsymbol{k}}, \tag{4.4}$$

where

$$\mathcal{Q}_k = \left\langle \int_0^1 \left(|\mathrm{D}\theta_k|^2 + k^2|\theta_k|^2 + \frac{c\delta}{(1+c\delta)\hat{R}} \left(\frac{1}{k^2}|\mathrm{D}^2 w_k|^2 + 2|\mathrm{D}w_k|^2 + k^2|w_k|^2 \right) \right.\right.$$

$$\left.\left. + 2Re[\theta_k \tilde{w}_k]\tau' \right) \mathrm{d}z \right\rangle. \quad (4.5)$$

Note that equality in (4.4) holds for functions of x and z alone, and that the quadratic form $\mathcal{Q}$ is positive if for all $\boldsymbol{k}$, $\mathcal{Q}_k \geqslant 0$ as a quadratic form.

Because w_k vanishes at both plates for all $\boldsymbol{k}$ so does the product $\theta_k \tilde{w}_k$. Hence, for $z \leqslant \frac{1}{2}$, we have

$$|\theta_k \tilde{w}_k(z)| = \left| \int_0^z \mathrm{D}(\theta_k(\zeta)\tilde{w}_k(\zeta))\,\mathrm{d}\zeta \right| \leqslant \int_0^z |\theta_k \mathrm{D}\tilde{w}_k|\,\mathrm{d}\zeta + \int_0^z |\mathrm{D}\theta_k \tilde{w}_k|\,\mathrm{d}\zeta. \quad (4.6)$$

Furthermore, since w_k and $\mathrm{D}w_k$ both vanish at $z = 0$ (the latter by virtue of the incompressibility constraint and the 'no-slip' boundary condition for the velocity field), the Fundamental Theorem of Calculus and the Cauchy–Schwarz inequality imply

$$|w_k(z)| \leqslant \sqrt{z} \left(\int_0^{1/2} |\mathrm{D}w_k(\zeta)|^2\,\mathrm{d}\zeta \right)^{1/2} \equiv \sqrt{z}\, \|\mathrm{D}w_k\|_{[0,1/2]}, \quad (4.7)$$

$$|\mathrm{D}w_k(z)| \leqslant \sqrt{z} \left(\int_0^{1/2} |\mathrm{D}^2 w_k(\zeta)|^2\,\mathrm{d}\zeta \right)^{1/2} \equiv \sqrt{z}\, \|\mathrm{D}^2 w_k\|_{[0,1/2]}. \quad (4.8)$$

Using these estimates in (4.6), another application of the Cauchy–Schwarz inequality, and the fact that $AB \leqslant \frac{1}{2}(A^2/\alpha + \alpha B^2)$ for any $\alpha > 0$, we obtain

$$|\theta_k \tilde{w}_k(z)| \leqslant \left(\frac{\alpha}{k^2}\|\mathrm{D}^2 w_k\|^2_{[0,1/2]} + \gamma\|\mathrm{D}w_k\|^2_{[0,1/2]} + \frac{k^2}{\alpha}\|\theta_k\|^2_{[0,1/2]} + \frac{1}{\gamma}\|\mathrm{D}\theta_k\|^2_{[0,1/2]} \right) \frac{z}{2\sqrt{2}}, \quad (4.9)$$

where $\gamma > 0$ is also adjustable. A similar estimate holds for $z \geqslant \frac{1}{2}$, and using them together we have

$$\left| \int_0^1 \theta_k \tilde{w}_k \tau'\,\mathrm{d}\zeta \right| \leqslant \int_0^\delta |\theta_k \tilde{w}_k\,\mathrm{d}\zeta| + \int_{1-\delta}^1 |\theta_k \tilde{w}_k|\,\mathrm{d}\zeta \quad (4.10)$$

$$\leqslant \frac{\delta^2}{4\sqrt{2}} \left(\frac{\alpha}{k^2}\|\mathrm{D}^2 w_k\|^2_{[0,1]} + \gamma\|\mathrm{D}w_k\|^2_{[0,1]} + \frac{k^2}{\alpha}\|\theta_k\|^2_{[0,1]} + \frac{1}{\gamma}\|\mathrm{D}\theta_k\|^2_{[0,1]} \right). \quad (4.11)$$

Inserting this estimate into (4.5) and choosing $\alpha = \gamma = \delta^2/2\sqrt{2}$ and $\delta \sim (8c/\hat{R})^{1/3}$ ensures that $\mathcal{Q} \geqslant 0$. The resulting lower bound on ΔT is

$$\Delta T \geqslant (8c)^{1/3}(2-c)\hat{R}^{-1/3}. \quad (4.12)$$

The coefficient above has a maximum value of $3/2^{1/3}$ (when $c = \frac{1}{2}$), resulting in the bound

$$\Delta T \geqslant \frac{3}{2^{1/3}} \hat{R}^{-1/3}, \tag{4.13}$$

so that using (2.12) and (2.22),

$$Ra \geqslant \frac{3}{2^{1/3}} \hat{R}^{2/3} \quad \text{and} \quad Nu \leqslant \frac{2^{1/3}}{3} \hat{R}^{1/3}, \tag{4.14}$$

and we finally obtain the bound in terms of Nu and Ra:

$$Nu \leqslant \left(\frac{2^{1/3}}{3} \right)^{3/2} Ra^{1/2} < 0.28 \times Ra^{1/2}. \tag{4.15}$$

This is the main result of the paper; we see that while fixing the flux at the boundaries does not (apparently) lower the upper bound exponent from $p = \frac{1}{2}$, it certainly does not raise it either.† It remains to be seen if this exponent really is optimal, and just how far the prefactor will be decreased by more precise analyses.

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REFERENCES

Busse, F. 1969 On Howard's upper bound for heat transport by turbulent convection. *J. Fluid Mech.* **37**, 457–477.

Busse, F. H. & Riahi, N. 1980 Nonlinear convection in a layer with nearly insulating boundaries. *J. Fluid Mech.* **96**, 243–256.

Chapman, C. J. & Proctor, M. R. E. 1980 Nonlinear Rayleigh–Bénard convection between poorly conducting boundaries. *J. Fluid Mech.* **101**, 759–782.

Chavanne, X., Chilla, F., Castaing, B., Hebral, B., Chabaud, B. & Chaussy, J. 1997 Observation of the ultimate regime in Rayleigh–Bénard convection. *Phys. Rev. Lett.* **79**, 3648–3651.

Constantin, P. & Doering, C. R. 1999 Infinite Prandtl number convection. *J. Statist. Phys.* **94**, 159–172.

Doering, C. R. & Constantin, P. 1992 Energy dissipation in shear driven turbulence. *Phys. Rev. Lett.* **79**, 1648–1651.

Doering, C. R. & Constantin, P. 1996 Variational bounds on energy dissipation in incompressible flows. III. Convection. *Phys. Rev.* E **53**, 5957–5981.

Doering, C. R. & Constantin, P. 1998 Bounds for heat transport in a porous layer. *J. Fluid Mech.* **376**, 263–296.

Glazier, J., Segawa, T., Naert, A. & Sano, M. 1999 Evidence against ultrahard thermal turbulence at very high Rayleigh numbers. *Nature* **398**, 307–310.

Heslot, F., Castaing, B. & Libchaber, A. 1987 Transitions to turbulence in helium gas. *Phys. Rev.* A **36**, 5870–5873.

Hopf, E. 1941 Ein allgemeiner Endlichkeitssatz der Hydrodynamik. *Math. Ann.* **117**, 764–775.

Howard, L. N. 1963 Heat transport by turbulent convection. *J. Fluid Mech.* **17**, 405–432.

† Interestingly, this statement may not apply to the so-called 'single-wavenumber' bound where preliminary computational and analytical results from continuing investigations by the authors, E.G. Evstatiev and L.N. Howard suggest a bound of the form $Ra^{5/12}$ in contrast to the $Ra^{3/8}$ estimate for the fixed temperature problem (Howard 1963; Doering & Constantin 1996).

Hurle, D., Jakeman, E. & Pike, E. 1967 On the solution of the Bénard problem with boundaries of finite conductivity. *Proc. R. Soc. Lond.* A **296**, 469–475.

Kadanoff, L. 2001 Turbulent heat flow: Structures and scaling. *Physics Today* **54**, 34.

Kraichnan, R. H. 1962 Relationships among some deductive theories of turbulence. In *Mécanique de la Turbulence* (*Marseille, 1961*), pp. 99–106. Editions Centre Nat. Recherche Sci., Paris.

Niemela, J. L., Skrbek, L., Sreenivasan, K. & Donnell, R. 2000 Turbulent convection at very high Rayleigh numbers. *Nature* **404**, 837–840.

Spiegel, E. 1971 Convection in stars I. Basic Boussinesq convection. *Annu. Rev. Astron. Astron.* **9**, 323.

Xu, X., Bajaj, K. & Ahlers, G. 2000 Heat transport in turbulent Rayleigh–Bénard convection. *Phys. Rev. Lett.* **84**, 4357–4360.