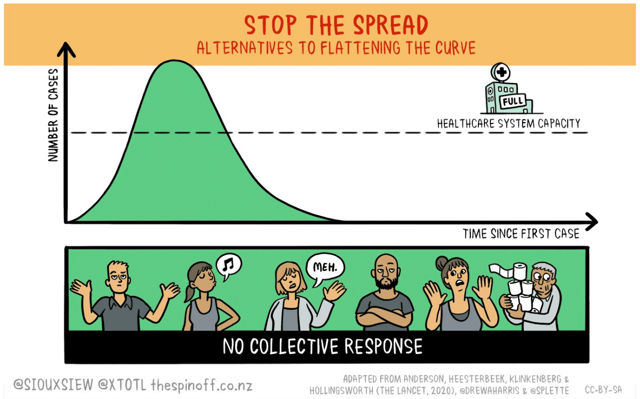




# Section 5x. ODEs and COVID-19

**John Stockie**
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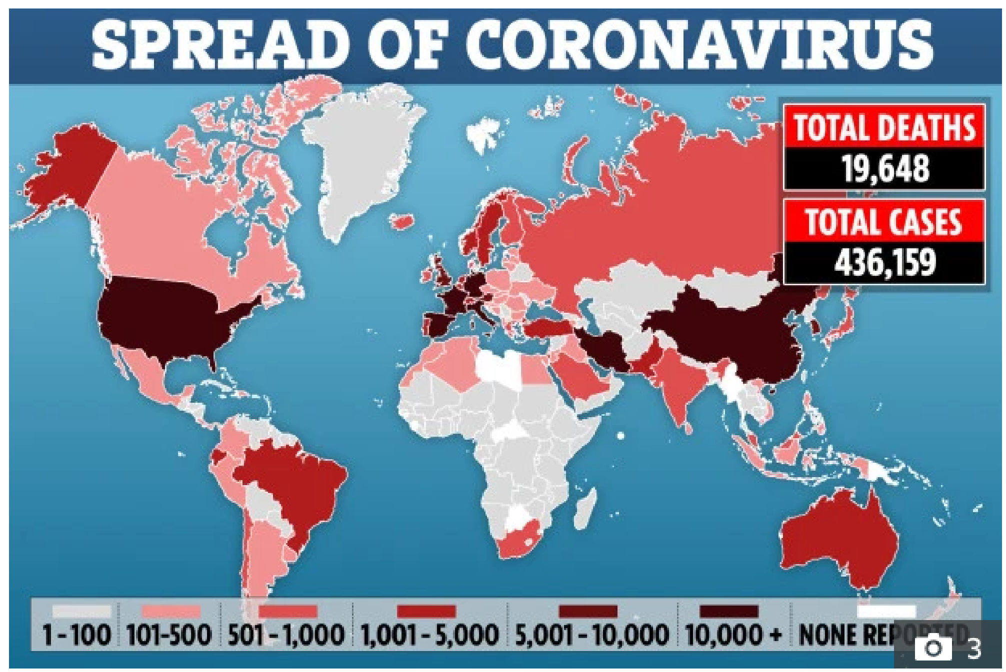
**MACM 316: Numerical Analysis**

# Section 5x. ODEs and COVID-19

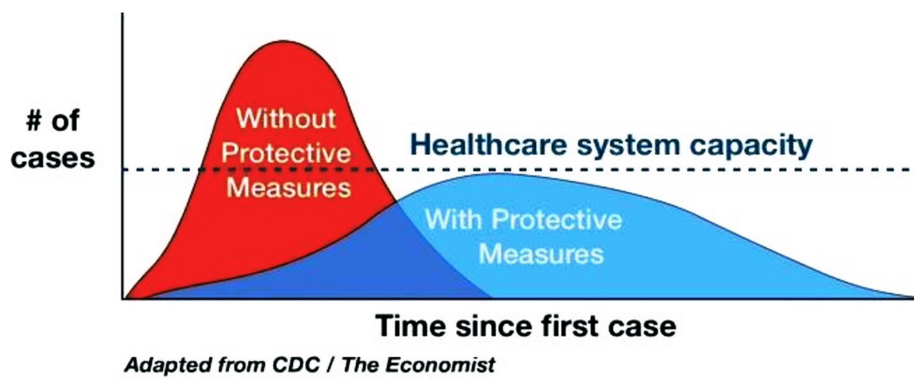
**What do (ordinary) differential equations (ODEs) and COVID-19 have in common?**

- **Like many other physical and biological systems, the spread of an infectious disease can be described mathematically**

**. . . as functions of time**

- **Rates of change – rates of contact, infection, spread, recovery, etc. – can be described using time derivatives $\Longrightarrow$ leads naturally to differential equations**
- **Models of disease transmission are used to study, treat and manage epidemics:**
  - ∗ **by health agencies to minimize infections and optimize healthcare delivery**
  - ∗ **by drug companies to develop new drugs or treatment strategies**
  - ∗ **by news agencies to explain these issues to the public**

**. . . and we've all heard lots about the importance of "flattening the curve" by "social distancing"!**



# Warning

- **What I'll describe is only a "toy model" for epidemics that has**
  - **many hidden assumptions**
  - **many approximations**
  - **several unknown parameters**
- **It can reproduce some (not all) realistic behaviours qualitatively**
- **BUT to make accurate predictions or reliable health-related decisions requires a real disease expert who is also intimately familiar with the mathematical models . . . and that's definitely not me!!**

# Ordinary Differential Equations (ODEs)

Text ch.5

- **ODE's are equations that involve:**
  - * a continuous function $u(t)$ of a single variable $t$, and
  - * any of its derivatives $\frac{du}{dt}, \frac{d^2u}{dt^2}, \ldots$
- **We will focus on first-order, ordinary differential equations having the form**

$$\frac{du}{dt} = f(t, u)$$

  - * **first-order** because only first derivatives are involved, AND
  - * **ordinary** because $u(t)$ depends on $t$ only, so $u'$ is an "ordinary" derivative.

**Aim:** To solve the **initial value problem (IVP)** consisting of an ODE and an initial condition:

$$\frac{du}{dt} = f(t, u) \quad \text{for } 0 \leqslant t \leqslant t_{max}, \quad \text{subject to } u(0) = u_0$$

---

**Examples:**

| $\frac{du}{dt} = au$ | $\frac{du}{dt} = au(1-u)$ | $\frac{du}{dt} = \cos(t) + \frac{u^2}{1+u^2}$ |
|---|---|---|
| exponential growth/decay | logistic equation | ... and anything else you can think of |

(you have likely seen ODEs already in 152/155/158 or MATH 310 or ...)

# Example 1: A Simple ODE IVP

**Consider the first-order ODE initial value problem:**

$$u' = tu - \frac{t^3}{3} \qquad \text{for } 0 \leqslant t \leqslant \overset{[t_{max}]}{2.25} \qquad \text{with } u(0) = \overset{[u_0]}{0.5}$$

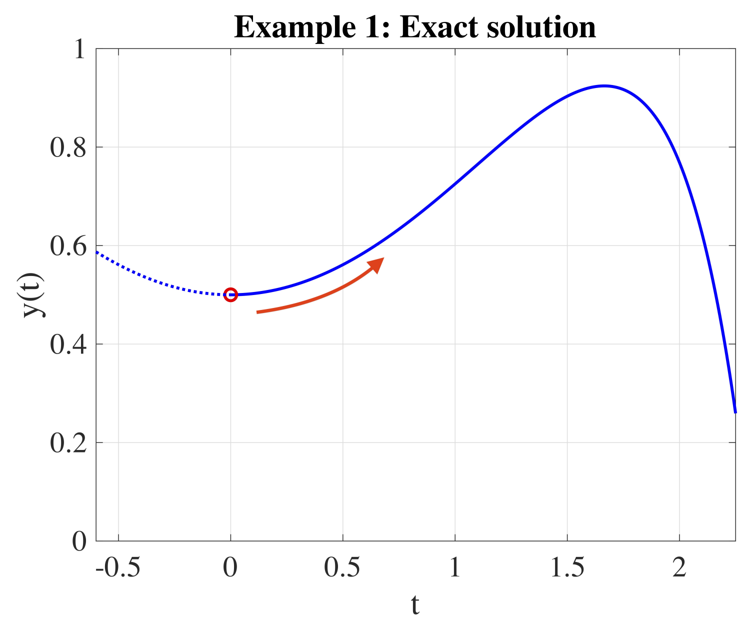
- **This problem has exact solution** $u(t) = \dfrac{4 + 2t^2 - e^{t^2/2}}{6}$
- **Verify by substituting:**

$$\text{LHS} = u' = \frac{4t - te^{t^2/2}}{6}$$

$$\text{RHS} = t\left(\frac{4 + 2t^2 - e^{t^2/2}}{6}\right) - \frac{t^3}{3} = \frac{4t - te^{t^2/2}}{6}$$

$$\text{IC} = u(0) = \frac{4 + 2(0)^2 - e^0}{6} = 0.5$$

**IVPs:** Think of $t$ as time so that only $t \geqslant 0$ makes sense



# Euler's Method

Text §5.2

- Partition time $0 \leqslant t \leqslant t_{max}$ into $n$ intervals of equal length $h = \frac{t_{max}}{n}$ with

$$t_j = jh \quad \text{for } j = 1, 2, \dots, n-1 \quad \Longrightarrow \quad t_0 = 0 \text{ and } t_n = t_{max}$$

- $h$ is usually called the **time step**, even if $t$ is not time
- Evaluate the ODE $u' = f(t, u)$ at $t = t_0$, replacing $u'$ with a forward difference approximation:

$$\frac{u(t_1) - u(t_0)}{h} \approx f(t_0, u(t_0)) \quad \Longrightarrow \quad u(t_1) = u_0 + h\, f(t_0, u(t_0))$$

- Simplify notation: $u(t_j) \to u_j$ for $j = 0, 1, 2, \dots, n$:

$$u_1 = u_0 + h\, f(t_0, u_0)$$

$$u_2 = u_1 + h\, f(t_1, u_1)$$

$$\vdots$$

$$\boxed{u_{j+1} = u_j + h\, f(t_j, u_j)}$$

[ a step-by-step algorithm! ]

$\Longrightarrow$ called the "Forward Euler Method" just or **"Euler's Method"**

# Example 1b: Apply Euler's Method

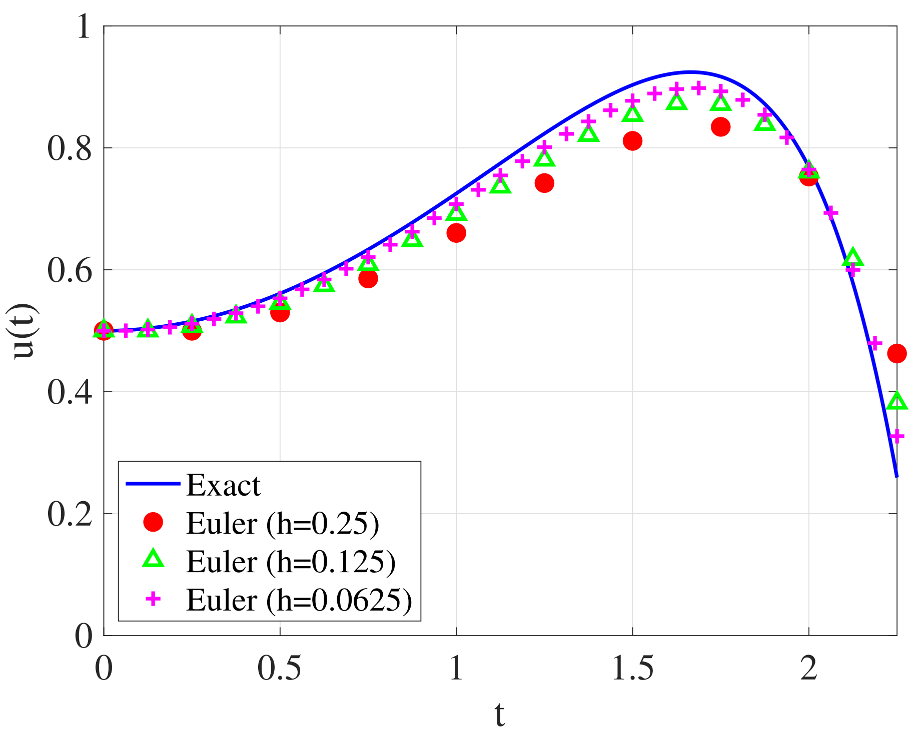
**Return to Example 1:**

$$u' = \underbrace{tu - \frac{t^3}{3}}_{f(t,u)} \qquad \text{with } 0 \leqslant t \leqslant 2.25$$

$$u(0) = 0.5$$

**Take $h = 0.25$ and organize calculations in tabular form . . .**

| $j$ | $t_j$ | $u_j$ | $f(t_j, u_j)$ | exact $u(t_j)$ | abs. error $\lvert u_j - u(t_j)\rvert$ |
|---|---|---|---|---|---|
| 0 | 0.00 | 0.500000 | 0.000000 | 0.500000 | — |
| 1 | 0.25 | 0.500000 | 0.119792 | 0.515543 | 0.015543 |
| 2 | 0.50 | 0.529948 | 0.223307 | 0.561142 | 0.031194 |
| 3 | 0.75 | 0.585775 | 0.298706 | 0.633369 | 0.047594 |
| 4 | 1.00 | 0.660451 | 0.327118 | 0.725213 | 0.064762 |
| 5 | 1.25 | 0.742231 | 0.276747 | 0.823467 | 0.081236 |
| 6 | 1.50 | 0.811417 | 0.092126 | 0.903297 | 0.091880 |
| 7 | 1.75 | 0.834449 | -0.326173 | 0.916841 | 0.082392 |
| 8 | 2.00 | 0.752906 | -1.160855 | 0.768491 | 0.015585 |
| 9 | 2.25 | 0.462692 | -2.755818 | 0.259299 | 0.203393 |




**MATLAB: `odeex1b.m` on Canvas**

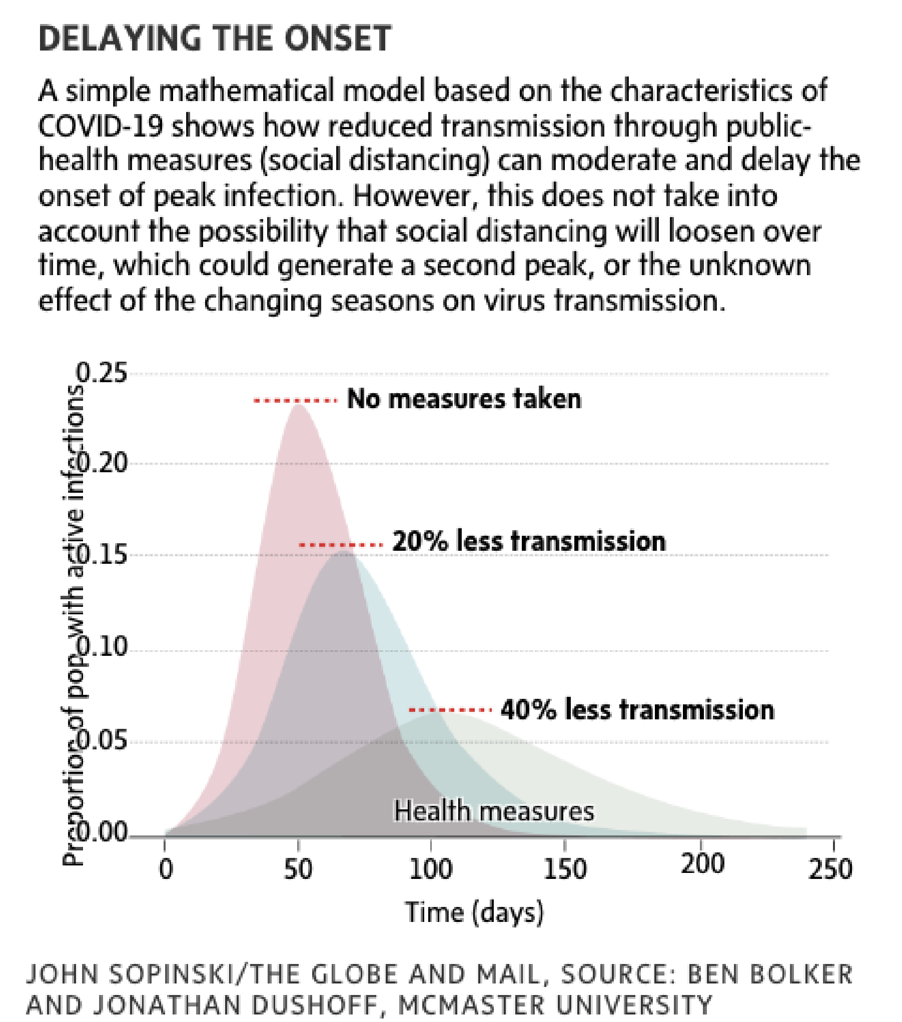
# MATLAB Code for Euler's Method

**This is a condensed version of the MATLAB code `odeex1b.m`:**

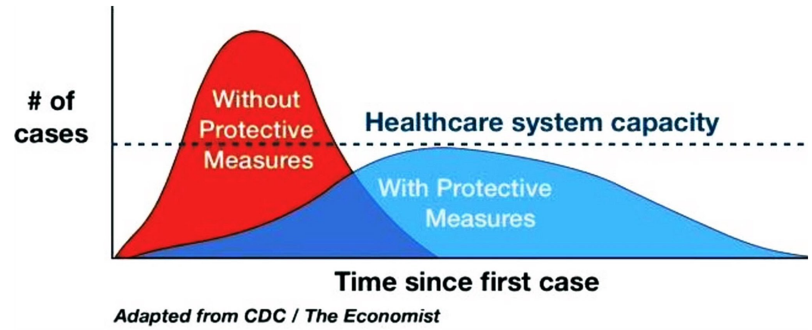
```
f = @(t,u) t.*u - t.^3/3;  % RHS function for u'=f(t,u)

tmax= 2.25;      % end time
nt  = 9;         % # steps
h   = tmax / nt;% time-step
t   = [0 : h : tmax];
u   = zeros(1,nt+1);
u(1)= 0.5;       % initial value u(0)

% Forward Euler time-stepping loop
for j = 1 : nt,
   u(j+1) = u(j) + h * f(t(j), u(j));
end

% Plot the computed solution
plot(t, u, 'ro')
```

# COVID-19 in the News

**Plots like the ones below are ubiquitous in the news right now:**




SOURCE: Globe & Mail, 16 March 2020
`https://tinyurl.com/vr4dy2g`




The Late Show with Stephen Colbert
17 March 2020

**Where do they come from?**

# Modelling Disease Transmission with ODEs

- A common mathematical model for disease transmission is the **SEIR model**
- Population is divided into "compartments" (fractions of population) that vary with time

$$S(t) = \text{Susceptible}$$
$$E(t) = \text{Exposed}$$
$$I(t) = \text{Infectious}$$
$$R(t) = \text{Recovered}$$

Transfer between compartments:

$$S \xrightarrow{\beta I} E \xrightarrow{\alpha} I \xrightarrow{\gamma} R$$

with $\boxed{S + E + I + R = 1}$ $\Longrightarrow$ total population remains constant!

- The rates of change of each fraction are determined by **rate / transfer laws**:
  - $*$ Interactions between people in two compartments $(x, y)$ occur with probability proportional to $x * y$
  - $*$ Suppose interactions cause a transfer from $x \longrightarrow y$. Then the rate of transfer $= \pm(\text{const}) * x * y$

$$\Longrightarrow \quad \frac{dx}{dt} = -\alpha xy \quad \text{and} \quad \frac{dy}{dt} = +\alpha xy \qquad [\text{ "conservation of people" }]$$

  - $*$ Simpler transfers $x \longrightarrow y$ with no interactions obey a simpler rate law

$$\Longrightarrow \quad \frac{dx}{dt} = -\alpha x \quad \text{and} \quad \frac{dy}{dt} = +\alpha x$$

# SEIR Transmission Steps

**Step 1:** Susceptibles become exposed (S $\longrightarrow$ E) due to S/I interactions at a contact rate $\beta$:

$$\frac{dS}{dt} = -\beta SI$$

**Step 2:** Susceptibles become exposed (S $\longrightarrow$ E) at same rate $\beta$ and exposeds become infectious (E $\longrightarrow$ I) at an infection rate $\alpha$:

$$\frac{dE}{dt} = +\beta SI - \alpha E$$

$\alpha = 1$ / (average incubation period)

**Step 3:** Exposeds become infectious (E $\longrightarrow$ I) and infectious people also recover (I $\longrightarrow$ R) at a recovery rate $\gamma$:

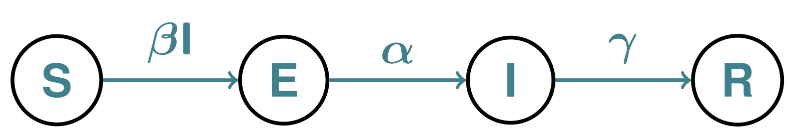
$$\frac{dI}{dt} = +\alpha E - \gamma I$$

$\gamma = 1$ / (average infectious period)

**Step 4:** Infectious become recovered (I $\longrightarrow$ R):

$$\frac{dR}{dt} = +\gamma I$$

- So far ... no-one dies from the virus
- That's unrealistic **BUT** the death rate is small



# Vector Form

- Define the solution as a vector function

$$\vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \end{bmatrix}$$

- Rewrite the system of four ODEs as in vector form $\frac{d\vec{u}}{dt} = \vec{f}(\vec{u})$:

$$\begin{bmatrix} \frac{dS}{dt} \\ \frac{dE}{dt} \\ \frac{dI}{dt} \\ \frac{dR}{dt} \end{bmatrix} = \begin{bmatrix} -\beta SI \\ \beta SI - \alpha E \\ \alpha E - \gamma I \\ \gamma I \end{bmatrix} \implies \frac{d\vec{u}}{dt} = \begin{bmatrix} -\beta u_1 u_3 \\ \beta u_1 u_3 - \alpha u_2 \\ \alpha u_2 - \gamma u_3 \\ \gamma u_3 \end{bmatrix} = \vec{f}(\vec{u})$$

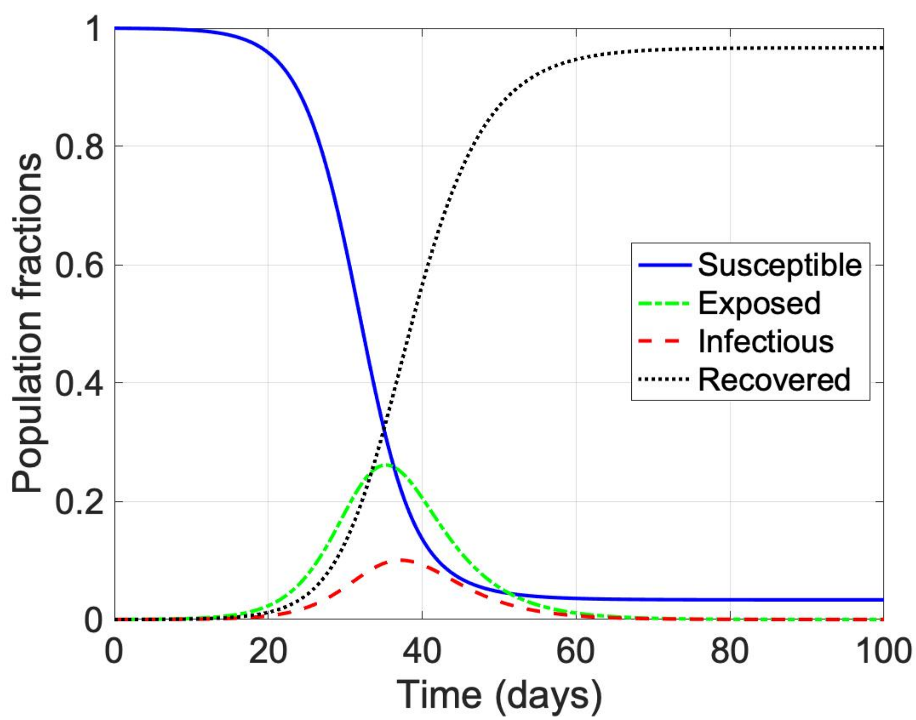
- Euler's method is applied component-wise to systems of ODEs

# Model Parameters for COVID-19

**We start with some very rough estimates of the parameters:**

- **Rate of incubation:** $\alpha = 0.2$ [ incubation period of 5 days ]
- **Average S/I contact rate:** $\beta = 1.75$
- **Rate of recovery:** $\gamma = 0.5$ [ infectious period is 2 days ]
- **Reproduction number:** $R_0 = \beta/\gamma = 3.5$
- **Total population:** $N_0 = 10{,}000$
- **Initial values:** $\vec{u}(0) = [1 - 1/N_0,\ 0,\ 1/N_0,\ 0]$ [ start with one infectious person ]

[ Based on literature cited in Christian Hubbs' blog post ]



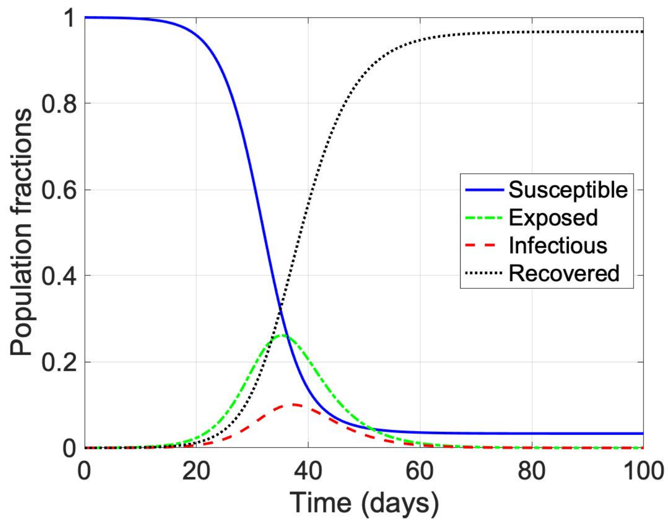
# Euler Code for SEIR Model

**MATLAB: `covid19.m` on Canvas**

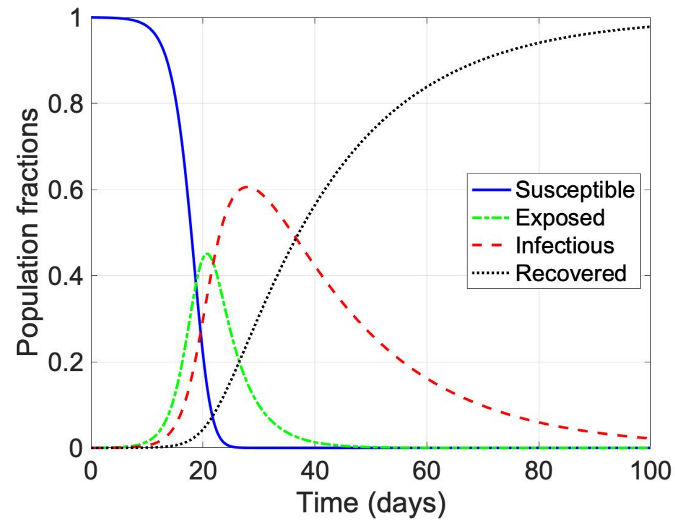
```
alpha = 0.2;       % incubation period (5 days)
beta  = 1.75;      % average contact rate
gamma = 0.5;       % 1/gamma is mean infectious period (2 days)
R0 = beta/gamma; % reproduction number

% Normalized population with S+E+I+R = 1.
N0 = 10000;        % initial population, one person infectious
u0 = [1-1/N0, 0, 1/N0, 0];  % initial values with sum(u0)=1

% RHS function for the SEIR ODE with u=[S,E,I,R]
f = @(u) [ -beta*u(1)*u(3),       beta*u(1)*u(3)-alpha*u(2), ...
           alpha*u(2)-gamma*u(3), gamma*u(3) ];

dt = 0.1;
t = [0 : dt : 100]';
nt = length(t)-1;
u = zeros(nt+1, 4);
u(1,:) = u0;  % initialize SEIR values

% Euler time-stepping loop
for k = 1 : nt,
  u(k+1,:) = u(k,:) + dt * f(u(k,:));
end
```

# Play Around!

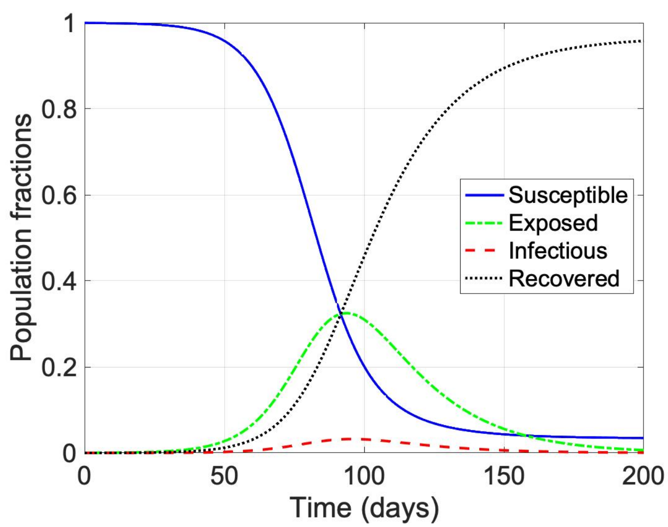
**Original parameters**
**(5-day incubation & 2-day infectious period)**



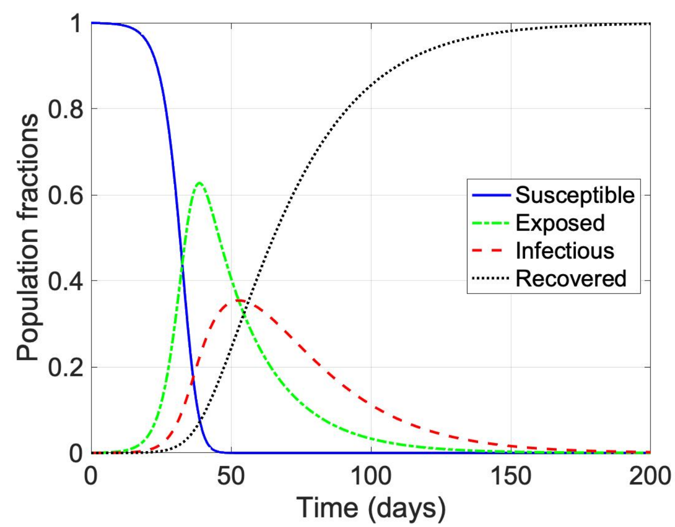

**Decrease $\gamma$ : 0.5 $\rightarrow$ 0.05**
**(20-day infectious period)**




**Decrease $\alpha$ : 0.2 $\rightarrow$ 0.05**
**(20-day incubation period)**



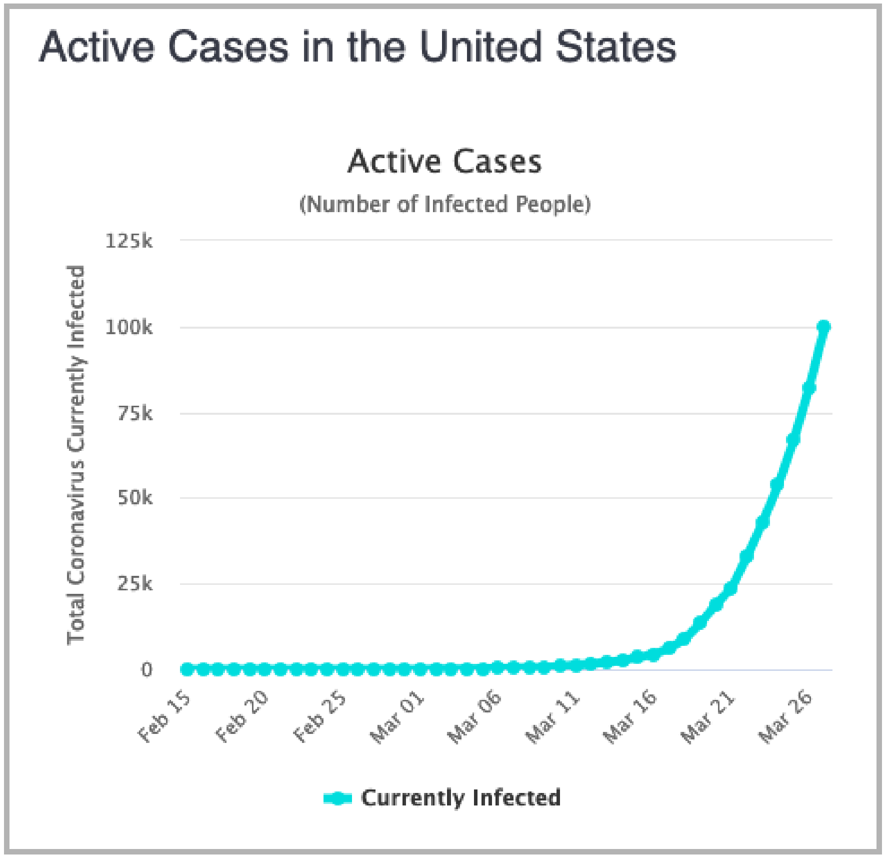

**BOTH: $\alpha = \gamma = 0.05$**



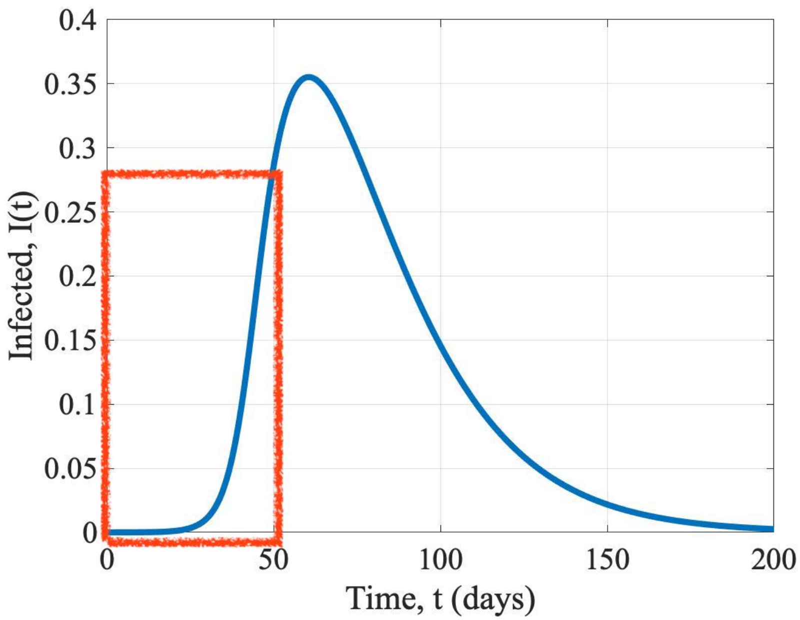
# Exponential Growth of Infections . . . Initially!

- **Early stages of an epidemic exhibit exponential growth**
- **Infections must eventually slow down (population is finite)**
- **BUT predicting / identifying the turn-around or inflection point is difficult**

**Key Question: Where on the infection curve are we sitting NOW?**



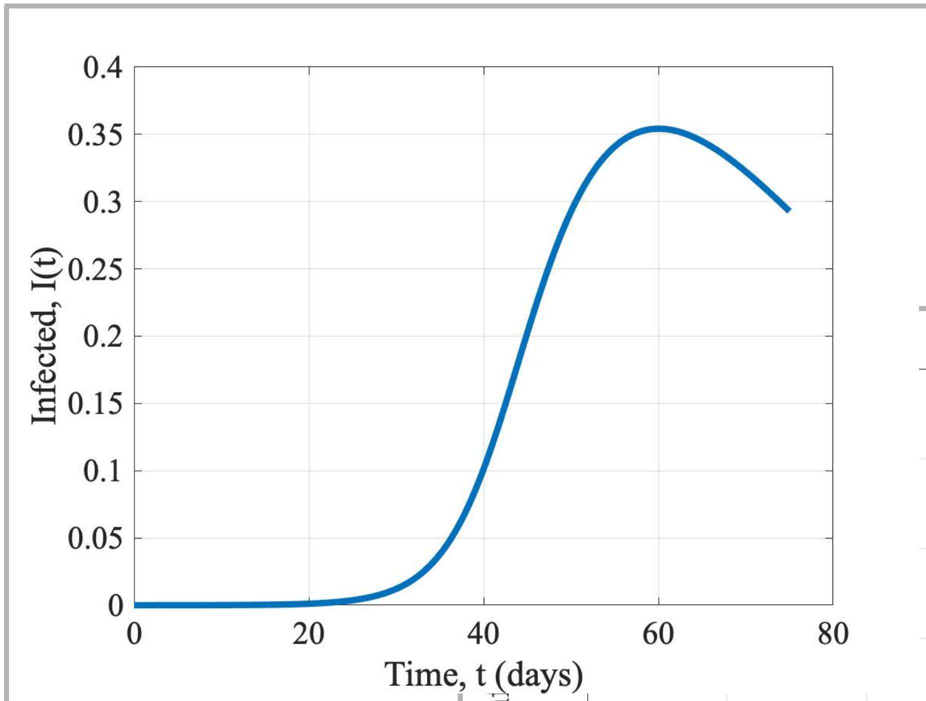

https://www.worldometers.info/coronavirus

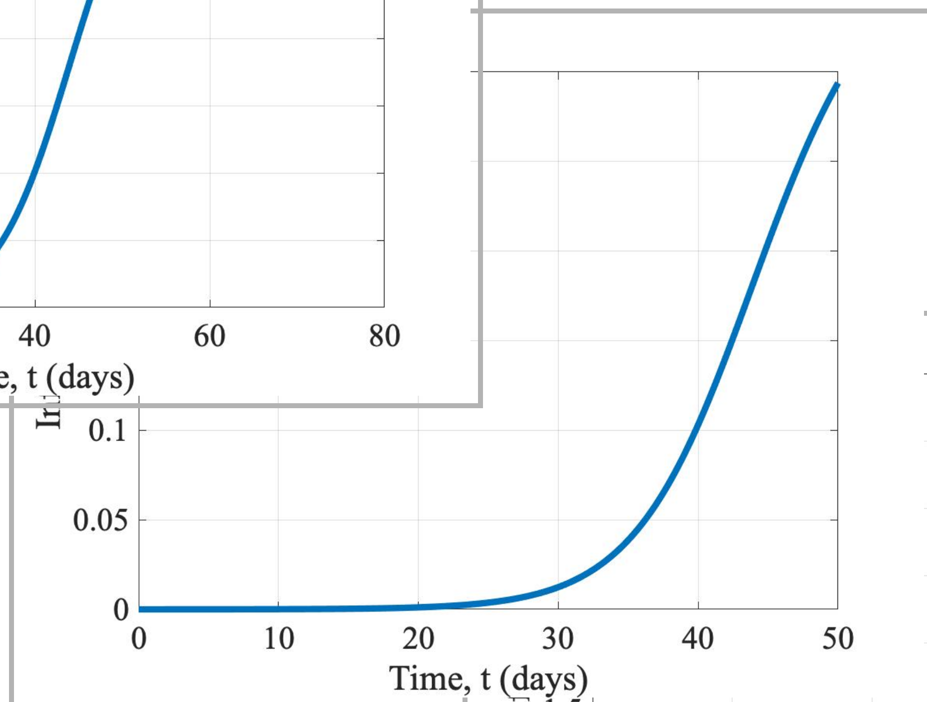


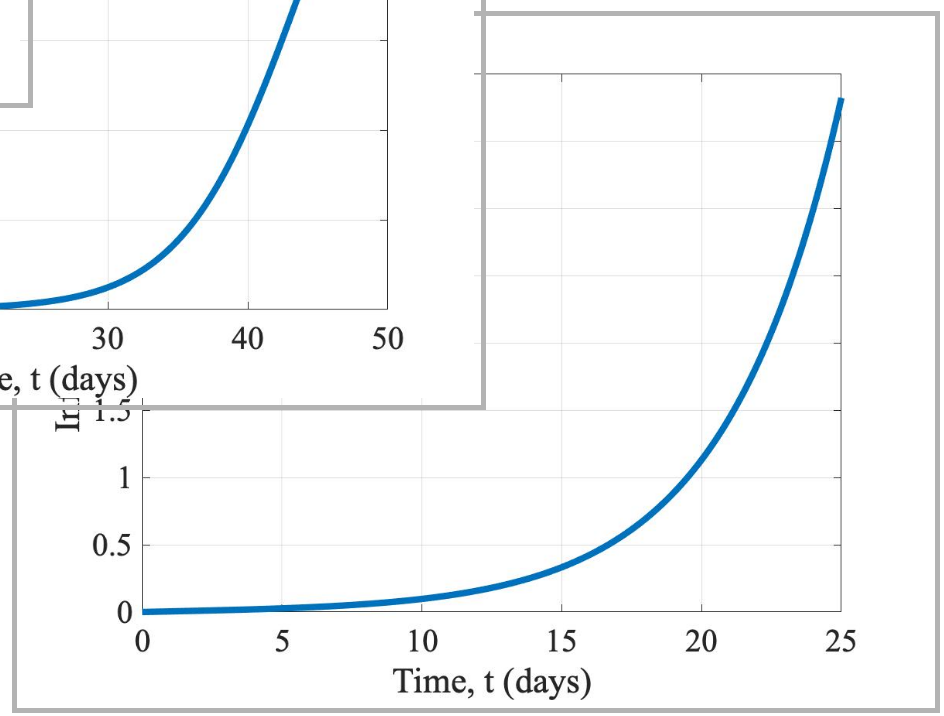

**Excellent video by minutephysics:**
**"How to tell if we're beating COVID-19"**
https://youtu.be/54XLXg4fYsc

# When is the Infection Slowing Down?

**It can be difficult to identify when the exponential growth is starting to turn around**



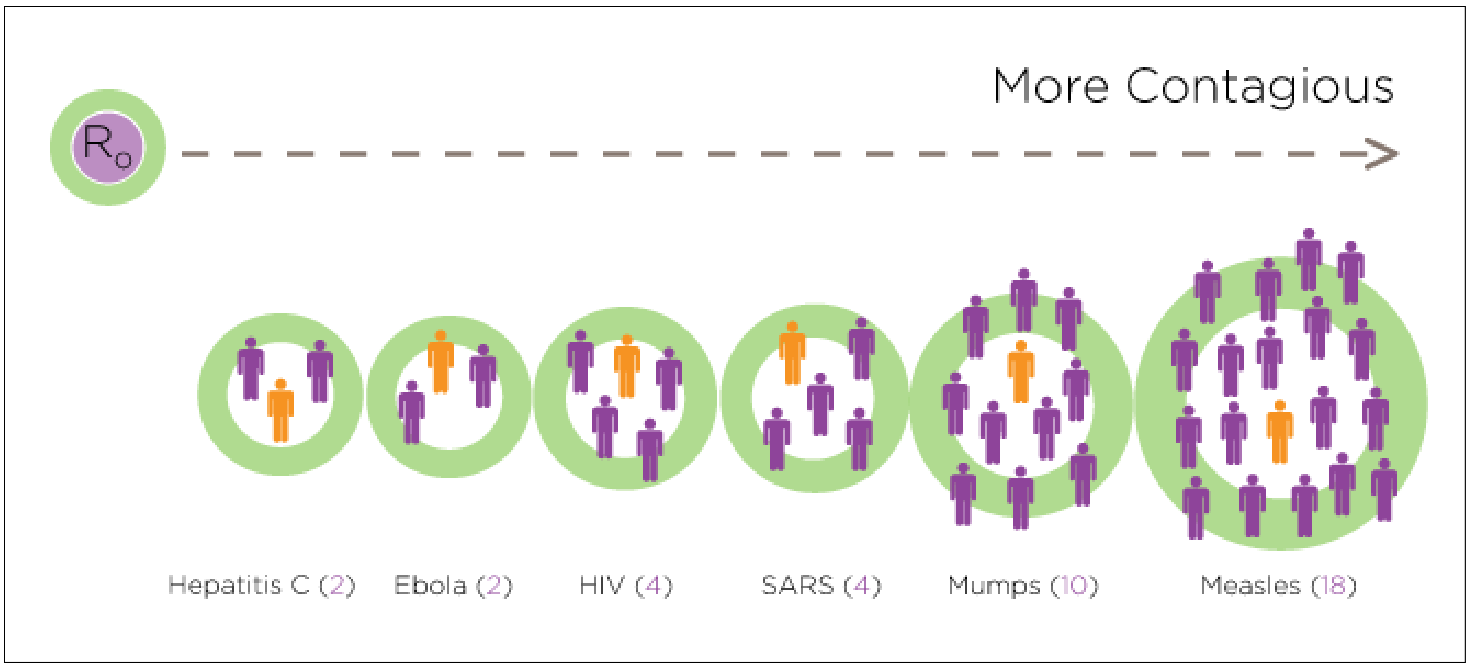



# Reproduction Number $R_0$

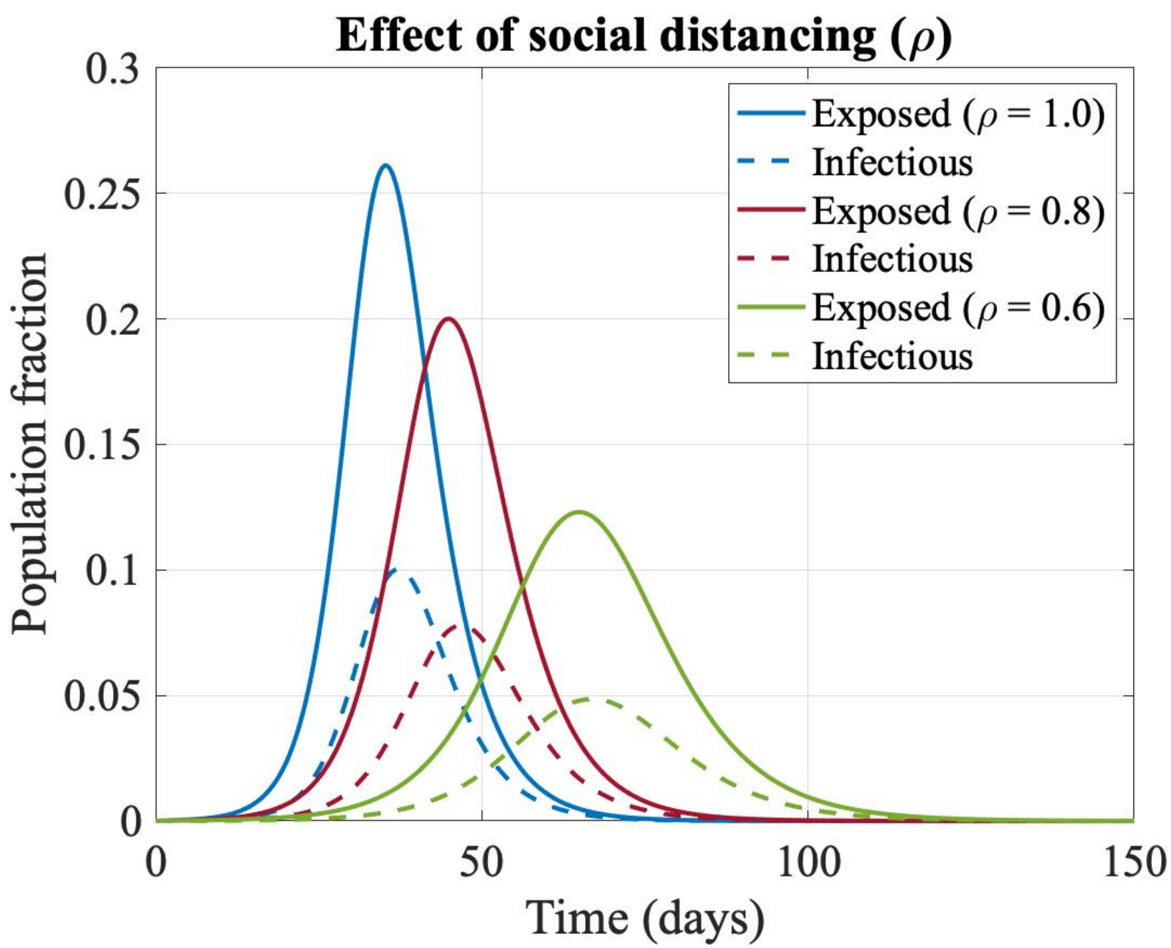
**A common measure of how infectious a disease is:**

- $R_0 = \beta/\gamma$ is the expected number of cases resulting from one infectious case in a population where everyone is susceptible to infection
- Basic theoretical results:
  - If $R_0 > 1$, the infection can spread in a population
  - If $R_0 < 1$, the infection dies out naturally
  - The larger $R_0$ is, the harder it is to control an epidemic
- $R_0$ can't be determined from data – it's estimated from SEIR model simulations
- Compare the estimate of $R_0 \approx 3.5$ for COVID-19 to past epidemics:



# Importance of Social Distancing

- **Social distancing is being promoted as an effective measure to control the COVID-19 virus by reducing the S/I contact rate $\beta$** [ same effect as reducing $R_0 = \beta/\gamma$ ]
- **Mimic social distancing by replacing $\beta \longrightarrow \rho\beta \implies$ simulate with $\rho = 1.0, 0.8, 0.6$**

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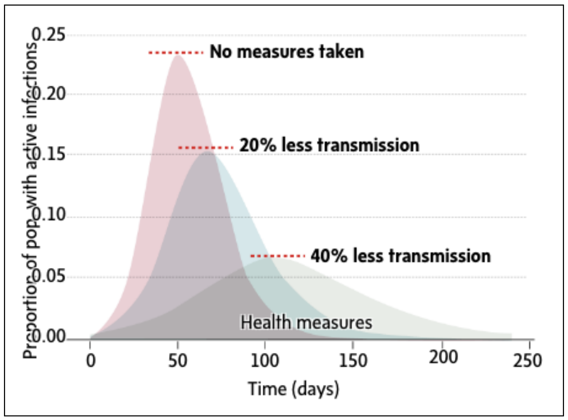

# Reality is Much More Complicated

**Many other effects can be added to such models that increase complexity:**

- **deaths and births** $\Longrightarrow$ $S + E + I + R \neq 1$
- **maternally inherited immunity at birth** $\Longrightarrow$ **MSEIR model**
- **age-structured population** $\Longrightarrow$ **different contact / infection / recovery / death rates**
- **time-varying transmission** $\Longrightarrow$ **parameters depend on** $t$
- **effects of hospitalization / treatment / vaccination**
- **differences in geography and national response** $\Longrightarrow$ **add spatial dependence**
- 

**political response to (and politicization of) results of disease models!!**

"Coronavirus modelers factor in new public health risk: Accusations their work is a hoax"
Washington Post, March 27, 2020
`https://tinyurl.com/sd439w5`

# References


- **Christian Hubbs, "Social distancing to slow the coronavirus",** ***Towards Data Science blog*** **(`https://towardsdatascience.com`), 12 March 2020.** [ light & easy read ]
- **Wikipedia, "Compartmental models in epidemiology".**
- **Herb Hethcote, "The mathematics of infectious diseases",** ***SIAM Review*** **42(4):599-653, 2000.** [ mathematical details ]
- **Nina Fefferman, "The role of applied math in real-time pandemic response: How basic disease models work", NIMBioS Webinar, 31 March 2020.** [ public lecture ]
  **On YouTube:** `https://youtu.be/cYiD7f066t4`
- **Daniel Coombs, "Real-time modelling of the 2020 coronavirus epidemic", UBC Physics and Astronomy Colloquium, 19 March 2020.** [ latest research, SEIR model $\approx$ 31:00 ]
  **On YouTube:** `https://youtu.be/dLVLEZMzIOk`