Immersed boundary simulations of gravitational settling

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Outline

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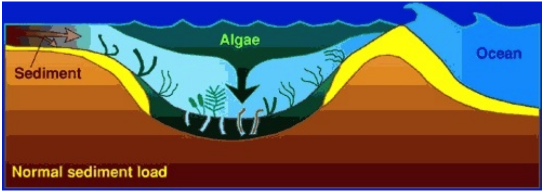
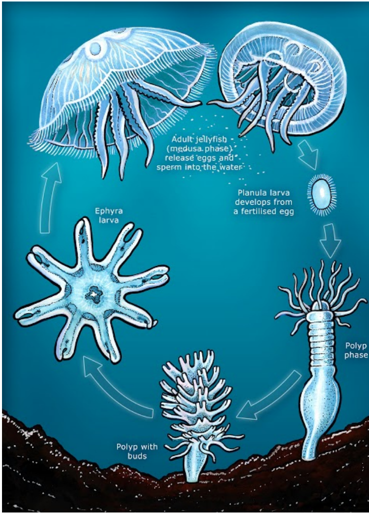
1. Gravitational settling
    - Analytical solutions
    - Experimental results
    - Numerical simulations
2. Immersed boundaries with mass
    - Mathematical formulation
    - Immersed boundary method
3. Simulations of settling cylinders
    - Single particle
    - Two particles and draft–kiss–tumble dynamics
4. Conclusions

Motivation: Sedimentation in applications

Sedimentation is the settling of particles under the influence of gravity:

- Biofilm dynamics.
- Marine organisms: algae, jellyfish.
- Industrial processes: wood pulp fibers, crystal precipitation, mine tailings.
- Natural phenomena: hailstorms, sediment transport in rivers and lakes.
- Tea leaves in a teacup.





Previous work on sedimentation

Gravitational settling of particle suspensions has been studied extensively in the literature using

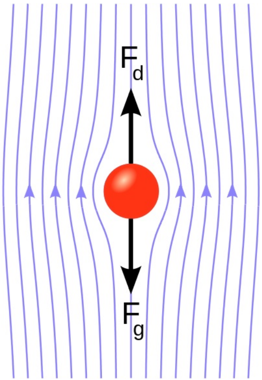
- mathematical analysis,
- experiments,
- numerical simulations.

Analytical solutions

- Stokes' law (1851): in a creeping flow of infinite extent, balancing gravity and drag forces yields settling velocity for a sphere in 3D:

$$V_s = \frac{gD^2(\rho_p - \rho_f)}{18\mu}$$

  where D = diameter, μ = viscosity.
- Analogous result can be derived for a 2D circular particle (infinite cylinder) $\Longrightarrow$ a nonlinear equation in V_s.
- An overview of more recent analytical results can be found in Guazzelli & Morris (2012).



Source: Wikipedia.

Experimental results

- An enormous experimental literature exists owing to the importance of sedimentation in industrial and other applications. [Davis & Acrivos, 1985]
- Of particular interest to us are estimates of wall-corrected settling velocity for a particle in a channel of width W:

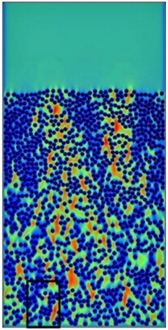
$$\widetilde{V}_s = \frac{V_s}{\lambda(k)} \qquad \text{where} \quad k = \frac{D}{W}$$

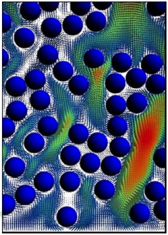
  and $\lambda(k)$ is a fitted correction factor.
- For example, Faxén's (1946) experiments yield

$$\lambda(k) \approx \frac{-4\pi}{0.9157 + \ln(k) - 1.724k^2 + 1.730k^4 - 2.406k^6 + 4.591k^8}$$

Numerical simulations

- Many authors have simulated 2D and 3D suspension flows numerically using:
  - finite element method,
  - lattice-Boltzmann method,
  - boundary element method,
  - . . .
- IB method has been applied to gravitational settling of
  - rigid fibers [Wang & Layton, 2009]
  - suspensions of swimming algal cells [Hopkins & Fauci, 2002]
- Direct-forcing IB approach has also been applied to sedimentation [Uhlmann, 2005] [Wang, Fan & Luo, 2008] [Breugem, 2012]
- However, there has not yet been an extensive validation of the IB method for particulate flows with settling.





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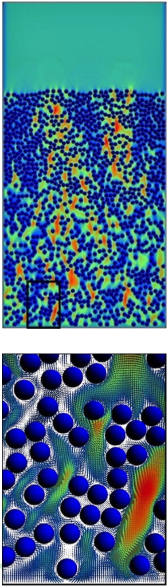

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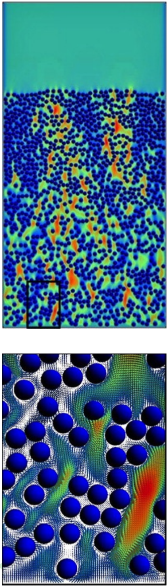

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  - suspensions of swimming algal cells [Hopkins & Fauci, 2002] ⟵ basis for our approach!
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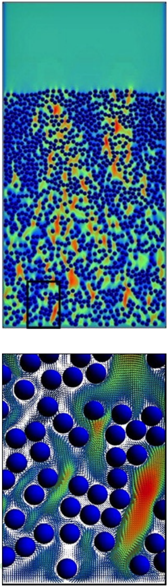

Aims of this study

- Sedimentation is very well-studied for rigid particles such as spheres, ellipsoids, fibers, . . .
- For simplicity, we consider spherical particles that are only slightly heavier than the suspending fluid: $\dfrac{\rho_p - \rho_f}{\rho_f} \ll 1$.
- We develop a very general numerical approach and validate it using results for rigid particles.
- Our long-term aim is to simulate sedimentation of both rigid and deformable particles. Hence, the need for the IB method!

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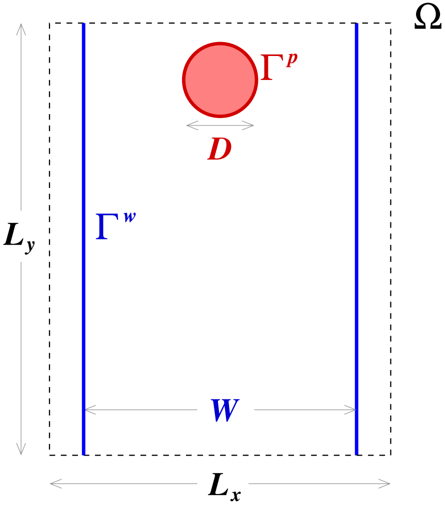
Problem geometry

Γ^p: Particle, diameter D
Γ^w: Walls, separated by W
Ω: Fluid domain, size $L_x \times L_y$
Periodic boundary conditions



Governing equations

Variables: $\mathbf{u}(\mathbf{x},t)$ = velocity, $p(\mathbf{x},t)$ = pressure, $\mathbf{X}(s,t)$ = IB position

Parameters: ρ_f = fluid density, ρ_p = particle density, μ = viscosity

Incompressible Navier-Stokes equations: (Boussinesq approximation, $\rho_p \gtrapprox \rho_f$)

$$\rho_f \frac{\partial \mathbf{u}}{\partial t} + \rho_f \mathbf{u} \cdot \nabla \mathbf{u} = \mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}_{IB} + \mathbf{f}_G$$

$$\nabla \cdot \mathbf{u} = 0$$

IB evolution equation:

$$\frac{\partial \mathbf{X}}{\partial t} = \int_\Omega \mathbf{u}(\mathbf{x},t)\,\delta(\mathbf{x} - \mathbf{X}(s,t))\,d\mathbf{x}$$

IB elastic force:

$$\mathbf{f}_{IB}(\mathbf{x},t) = \int_{\Gamma^{w,p}} \mathbf{F}_{IB}(s,t)\,\delta(\mathbf{x} - \mathbf{X}(s,t))\,ds$$

(specify discrete $\mathbf{F}_{IB}$ later)

Gravitational settling term:

$$\mathbf{f}_G(\mathbf{x},t) = -\begin{bmatrix} 0 \\ g \end{bmatrix} \int_{\Gamma^p} (\rho_p - \rho_f)\,\delta(\mathbf{x} - \mathbf{X}(s,t))\,ds$$

Immersed boundary method

We apply a straightforward discretization of the IB problem using:

- centered finite differences in space,
- cosine approximation for delta function,
- ADI for diffusion and advection terms,
- explicit treatment of IB force and settling terms,
- split-step projection scheme, with an FFT solve for the pressure Poisson equation.

Details are in Ghosh & JS [arxiv:1304.0804, 2013].

Discrete IB force for the walls

- The stationary walls are divided into N_w equally-spaced tether points with fixed locations

$$\mathbf{Y}_\ell^w = \Big[(L_x \pm W)/2,\ \ell L_y/N_w\Big] \qquad \text{for } \ell = 1, 2, \ldots, N_w$$

- Each wall IB point $\mathbf{X}_\ell(t)$ is connected to the corresponding tether point by a stiff spring with force density

$$\mathbf{F}_\ell^w(t) = \sigma_w(\mathbf{Y}_\ell^w - \mathbf{X}_\ell(t))$$

- The force integral approximation involves a length scaling factor:

$$\mathbf{f}_{i,j} = \sum_{\ell=1}^{N_w} \mathbf{F}_\ell^w\, \delta_h(\mathbf{x}_{i,j} - \mathbf{X}_\ell)\, \frac{L_y}{N_w}$$

Discrete IB force for the particle

- "Uniform" triangulation of particle with nodes $\mathbf{X}_\ell(t)$ for $\ell = 1, 2, \ldots, N_p$.
- Following Alpkvist & Klapper (2007), edges generate spring forces with

$$\mathbf{F}^p_\ell = \sigma_p \sum_{\substack{m=1 \\ \mathbb{I}_{\ell,m}\neq 0}}^{N_p} \mathbb{I}_{\ell,m} \frac{\mathbf{d}_{\ell,m}}{d_{\ell,m}} (d_{\ell,m}(0) - d_{\ell,m})$$

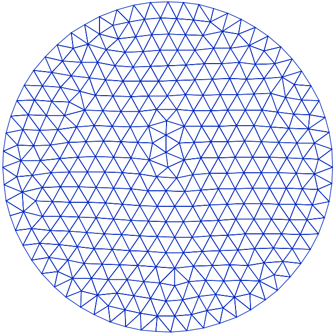
$$\mathbf{d}_{\ell,m}(t) = \mathbf{X}_\ell(t) - \mathbf{X}_m(t)$$

$$d_{\ell,m} = |\mathbf{d}_{\ell,m}|$$

$$\mathbb{I}_{\ell,m} = [\text{ 0/1 incidence matrix }]$$

- Force integral is scaled by an area factor:

$$\mathbf{f}_{i,j} = \sum_{\ell=1}^{N_p} \mathbf{F}^p_\ell \, \delta_h(\mathbf{x}_{i,j} - \mathbf{X}_\ell) \underbrace{\frac{\pi D^2}{4N_p}}_{\triangle \text{ area}}$$



Particle triangulation
with $N_p = 2015$ nodes.

[Hopkins & Fauci, 2002]

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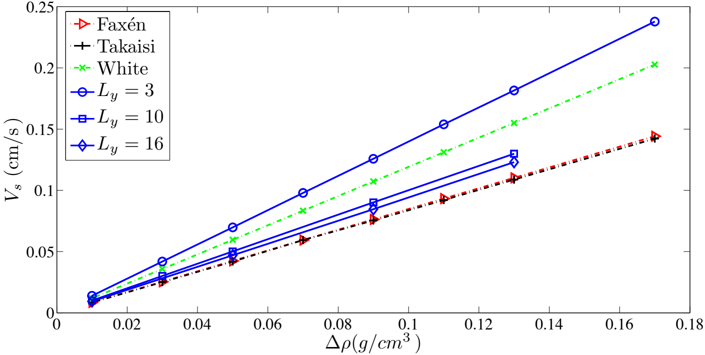
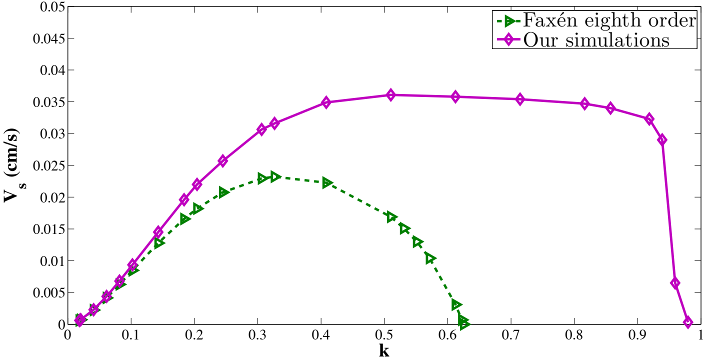
Single particle: Comparison to wall-corrected $\widetilde{V}_s$

For small $\Delta\rho = \rho_p - \rho_f$, the settling velocity V_s approaches Faxén's (1946) result as the channel length L_y increases:



Single particle: Varying particle size

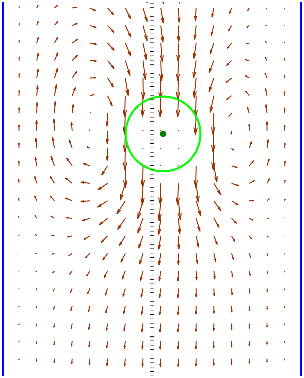
Wall-corrected $\widetilde{V}_s$ formulas are only valid for small $k = W/D$.

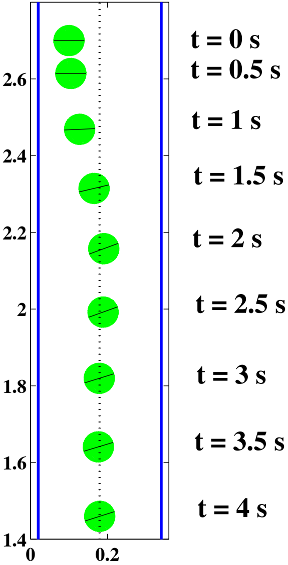


Our simulations demonstrate physically reasonable behaviour as $k \longrightarrow 1$.

Single particle: Released off-center

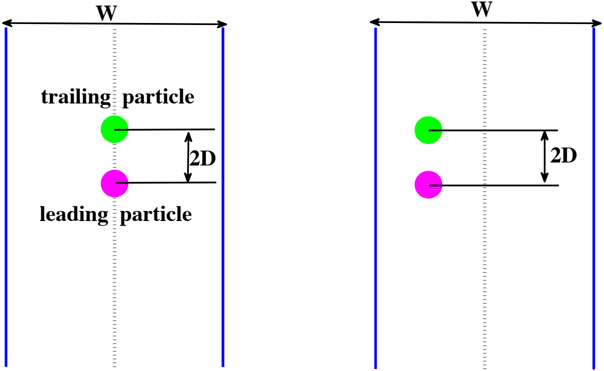
- At Reynolds number Re = 4.9, a single particle released off-center migrates toward the centerline.
- Hydrodynamic forces between the particle and the walls are in balance.





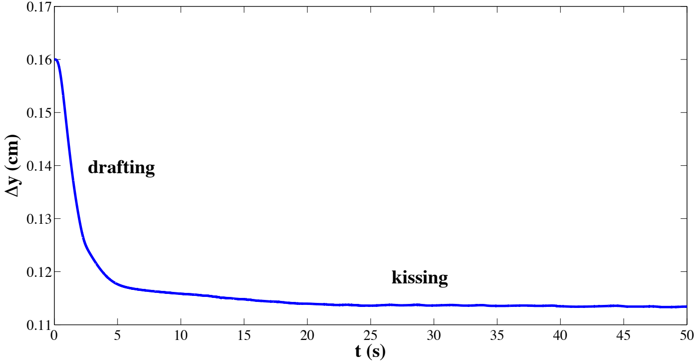
Simulations of two particles

Consider two initial configurations, centered and off-center, with particles separated by a distance $2D$:



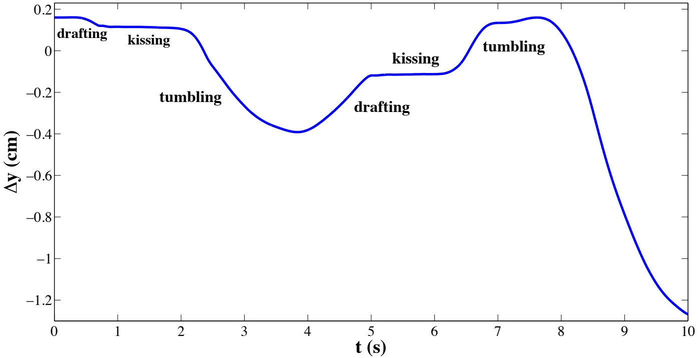
Two particles at low Re: Drafting and kissing

At low Reynolds number ($\mathrm{Re} = 3$), the particles approach each other (draft) and nearly touch (kiss):

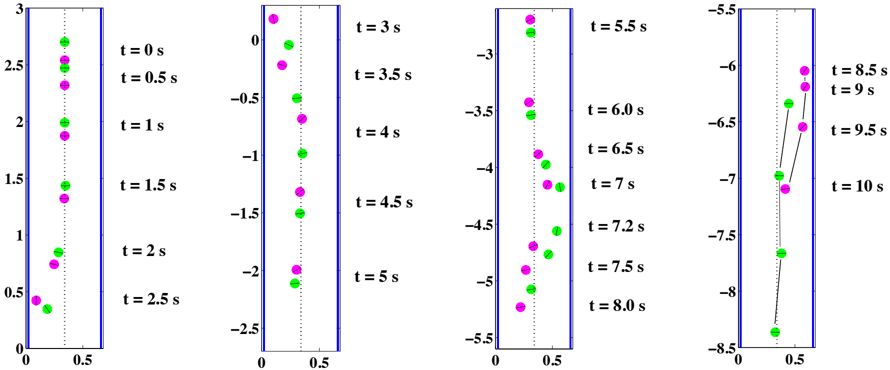


Two particles at Re = 80: DKT behaviour

At higher Reynolds number (Re = 80), the particles undergo a tumbling motion after drafting and kissing:



Two particles at Re = 80: DKT behaviour (cont'd)

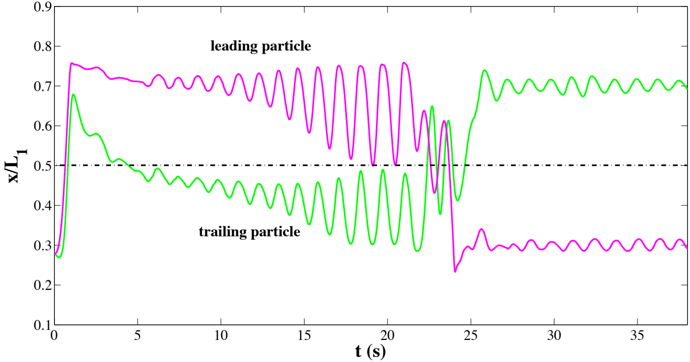


Results match qualitatively with FEM simulations of Feng, Hu & Joseph (1994).

[Video]

Two particles at Re $= 47$, off-center

More interesting behaviour arises at an intermediate Reynolds number (Re $= 47$) for two particles released off-center:



[Video]

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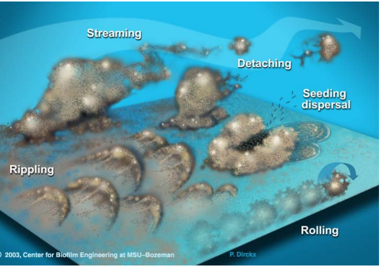
Closing remarks

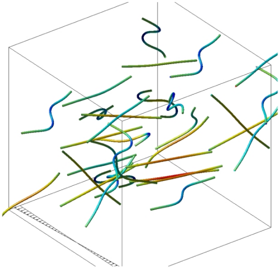
- We developed a 2D immersed boundary method that handles gravitational settling in the presence of walls.
- Computed settling velocities match with experiments.
- More complicated two-particle dynamics are consistent with simulations of Feng, Hu & Joseph (1994).

Current and future work

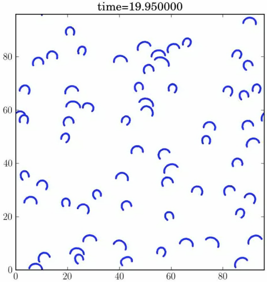
- Study settling of deformable, non-spherical particles.
- Investigate applications to:
    - biofilm floc deformation,
    - flexible fiber suspensions,
    - jellyfish swimming dynamics.
- Simulate large numbers of particles using Wiens' parallel IB algorithm.
  [Wiens & JS, submitted to *J. Comput. Phys.*, arXiv:1305.3976, 2013]



Biofilms
(MSU Bozeman)



Flexible fibers
(M. Shelley, NYU, 2004)



Jellyfish in 2D
(Jeff Wiens, 3:00 Wed)

Thank-you!

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