Immersed boundary simulations of gravitational settling

John Stockie

Department of Mathematics
Simon Fraser University
Burnaby, British Columbia, Canada

http://www.math.sfu.ca/~stockie

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Outline

1. Gravitational settling
   - Analytical solutions
   - Experimental results
   - Numerical simulations

2. Immersed boundaries with mass
   - Mathematical formulation
   - Immersed boundary method

3. Simulations of settling cylinders
   - Single particle
   - Two particles and draft–kiss–tumble dynamics

4. Conclusions
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Motivation: Sedimentation in applications

Sedimentation is the settling of particles under the influence of gravity:

- Biofilm dynamics.
- Marine organisms: algae, jellyfish.
- Industrial processes: wood pulp fibers, crystal precipitation, mine tailings.
- Natural phenomena: hailstorms, sediment transport in rivers and lakes.
- Tea leaves in a teacup.
Previous work on sedimentation

Gravitational settling of particle suspensions has been studied extensively in the literature using
- mathematical analysis,
- experiments,
- numerical simulations.
Analytical solutions

- **Stokes’ law (1851):** in a creeping flow of infinite extent, balancing gravity and drag forces yields settling velocity for a sphere in 3D:

\[ V_s = \frac{gD^2(\rho_p - \rho_f)}{18\mu} \]

where \( D \) = diameter, \( \mu \) = viscosity.

- Analogous result can be derived for a 2D circular particle (infinite cylinder) \( \implies \) a nonlinear equation in \( V_s \).

- An overview of more recent analytical results can be found in Guazzelli & Morris (2012).
Experimental results

- An enormous experimental literature exists owing to the importance of sedimentation in industrial and other applications. [Davis & Acrivos, 1985]

- Of particular interest to us are estimates of wall-corrected settling velocity for a particle in a channel of width $W$:

$$\tilde{V}_s = \frac{V_s}{\lambda(k)}$$

where $k = \frac{D}{W}$

and $\lambda(k)$ is a fitted correction factor.

- For example, Faxén’s (1946) experiments yield

$$\lambda(k) \approx \frac{-4\pi}{0.9157 + \ln(k) - 1.724k^2 + 1.730k^4 - 2.406k^6 + 4.591k^8}$$
Numerical simulations

- Many authors have simulated 2D and 3D suspension flows numerically using:
  - finite element method,
  - lattice-Boltzmann method,
  - boundary element method,
  - ...

- IB method has been applied to gravitational settling of
  - rigid fibers [Wang & Layton, 2009]
  - suspensions of swimming algal cells [Hopkins & Fauci, 2002]

- Direct-forcing IB approach has also been applied to sedimentation [Uhlmann, 2005] [Wang, Fan & Luo, 2008] [Breugem, 2012]

- However, there has not yet been an extensive validation of the IB method for particulate flows with settling.
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- rigid fibers  [Wang & Layton, 2009]
- suspensions of swimming algal cells
  [Hopkins & Fauci, 2002] ← basis for our approach!

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However, there has not yet been an extensive validation of the IB method for particulate flows with settling.
Aims of this study

- Sedimentation is very well-studied for rigid particles such as spheres, ellipsoids, fibers, ... 
- For simplicity, we consider spherical particles that are only slightly heavier than the suspending fluid: \( \frac{\rho_p - \rho_f}{\rho_f} \ll 1 \).
- We develop a very general numerical approach and validate it using results for rigid particles.
- Our long-term aim is to simulate sedimentation of both rigid and deformable particles. Hence, the need for the IB method!
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**Problem geometry**

- \( \Gamma^p \): Particle, diameter \( D \)
- \( \Gamma^w \): Walls, separated by \( W \)
- \( \Omega \): Fluid domain, size \( L_x \times L_y \)

Periodic boundary conditions
Governing equations

Variables: \( u(x, t) = \text{velocity}, \ p(x, t) = \text{pressure}, \ X(s, t) = \text{IB position} \)

Parameters: \( \rho_f = \text{fluid density}, \ \rho_p = \text{particle density}, \ \mu = \text{viscosity} \)

Incompressible Navier-Stokes equations: (Boussinesq approximation, \( \rho_p \gtrsim \rho_f \))

\[
\rho_f \frac{\partial u}{\partial t} + \rho_f u \cdot \nabla u = \mu \nabla^2 u - \nabla p + f_{IB} + f_G
\]
\[
\nabla \cdot u = 0
\]

IB evolution equation:

\[
\frac{\partial X}{\partial t} = \int_\Omega u(x, t) \delta(x - X(s, t)) \, dx
\]

IB elastic force:

\[
f_{IB}(x, t) = \int_{\Gamma_{w,p}} F_{IB}(s, t) \delta(x - X(s, t)) \, ds
\]

(specify discrete \( F_{IB} \) later)

Gravitational settling term:

\[
f_G(x, t) = - \left[ \frac{0}{g} \right] \int_{\Gamma_p} (\rho_p - \rho_f) \delta(x - X(s, t)) \, ds
\]
We apply a straightforward discretization of the IB problem using:

- centered finite differences in space,
- cosine approximation for delta function,
- ADI for diffusion and advection terms,
- explicit treatment of IB force and settling terms,
- split-step projection scheme, with an FFT solve for the pressure Poisson equation.

Details are in Ghosh & JS [arxiv:1304.0804, 2013].
Discrete IB force for the walls

- The stationary walls are divided into \( N_w \) equally-spaced tether points with fixed locations

\[
Y^w_\ell = \left[ \left( L_x \pm W \right)/2, \ell L_y/N_w \right] \quad \text{for } \ell = 1, 2, \ldots, N_w
\]

- Each wall IB point \( X_\ell(t) \) is connected to the corresponding tether point by a stiff spring with force density

\[
F^w_\ell(t) = \sigma_w(Y^w_\ell - X_\ell(t))
\]

- The force integral approximation involves a length scaling factor:

\[
f_{i,j} = \sum_{\ell=1}^{N_w} F^w_\ell \delta_h(x_{i,j} - X_\ell) \frac{L_y}{N_w}
\]
Discrete IB force for the particle

- “Uniform” triangulation of particle with nodes $\mathbf{X}_\ell(t)$ for $\ell = 1, 2, \ldots, N_p$.

- Following Alpkvist & Klapper (2007), edges generate spring forces with

  $$
  \mathbf{F}_\ell^p = \sigma_p \sum_{m=1}^{N_p} \mathbb{I}_{\ell,m} \frac{d_{\ell,m}}{d_{\ell,m}} (d_{\ell,m}(0) - d_{\ell,m})
  $$

  $$
  d_{\ell,m}(t) = \mathbf{X}_\ell(t) - \mathbf{X}_m(t)
  $$

  $$
  d_{\ell,m} = |d_{\ell,m}|
  $$

  $$
  \mathbb{I}_{\ell,m} = [ \text{0/1 incidence matrix} ]
  $$

- Force integral is scaled by an area factor:

  $$
  f_{i,j} = \sum_{\ell=1}^{N_p} \mathbf{F}_\ell^p \delta_h(\mathbf{x}_{i,j} - \mathbf{X}_\ell) \frac{\pi D^2}{4 N_p} \text{area}
  $$

[Hopkins & Fauci, 2002]
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For small $\Delta \rho = \rho_p - \rho_f$, the settling velocity $V_s$ approaches Faxén’s (1946) result as the channel length $L_y$ increases:
Wall-corrected $\tilde{V}_s$ formulas are only valid for small $k = W/D$.

Our simulations demonstrate physically reasonable behaviour as $k \to 1$. 
At Reynolds number $Re = 4.9$, a single particle released off-center migrates toward the centerline.

Hydrodynamic forces between the particle and the walls are in balance.
Simulations of two particles

Consider two initial configurations, centered and off-center, with particles separated by a distance $2D$:
Two particles at low Re: Drafting and kissing

At low Reynolds number (Re = 3), the particles approach each other (draft) and nearly touch (kiss):
Two particles at $Re = 80$: DKT behaviour

At higher Reynolds number ($Re = 80$), the particles undergo a **tumbling** motion after drafting and kissing:
Two particles at Re = 80: DKT behaviour (cont’d)

Results match qualitatively with FEM simulations of Feng, Hu & Joseph (1994).

[Video]
Two particles at $Re = 47$, off-center

More interesting behaviour arises at an intermediate Reynolds number ($Re = 47$) for two particles released off-center:

[Video]
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We developed a 2D immersed boundary method that handles gravitational settling in the presence of walls.

- Computed settling velocities match with experiments.
Current and future work

- Study settling of deformable, non-spherical particles.
- Investigate applications to:
  - biofilm floc deformation,
  - flexible fiber suspensions,
  - jellyfish swimming dynamics.
- Simulate large numbers of particles using Wiens’ parallel IB algorithm.
Thank-you!

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