Parametric Resonance in Immersed Elastic Boundaries

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Outline

1. Review of
   • “normal” resonance $\Rightarrow$ external forcing
   • parametric resonance $\Rightarrow$ internal forcing

2. Floquet stability analysis for parametrically-excited springs

3. Immersed boundary model

4. Parametric resonance in immersed boundaries

5. Numerical simulations

6. Summary and future work
For a damped, mass-spring system:

\[ m\ddot{x} + \mu \dot{x} + \sigma x = f(t) \]
1. Review of Resonance

For a damped, mass-spring system:

\[ m\ddot{x} + \mu \dot{x} + \sigma x = f(t) \]

I. Unforced or Natural Oscillations, \( f(t) \equiv 0 \):

(a) Undamped system, \( \mu = 0 \):

sustained oscillations with natural frequency

\[ \omega_o = \sqrt{\frac{\sigma}{m}} \]

(b) Damped system, \( \mu > 0 \):

decaying oscillations with natural frequency

\[ \omega_o = \sqrt{\frac{\sigma}{m} - \frac{\mu^2}{4}} \]
II. Externally-forced Oscillations:

The system is subjected to an external periodic force (modeled as a separate term in the DE)

\[ m\ddot{x} + \mu\dot{x} + \sigma x = F_0 \cos \omega t \]

**Resonance** occurs when forcing is close to the natural frequency
\[ \implies \text{amplitude of resulting oscillations grows when undamped (} \mu = 0 \) \]

(a) Undamped, non-resonant \((\mu = 0, \omega \neq \omega_o)\):

persistent, mixed-frequency oscillations

(b) Undamped, resonant \((\mu = 0, \omega = \omega_o)\):

unstable, unbounded oscillations
(c) With damping ($\mu > 0$) — “real” systems:
- oscillations are **bounded**, transient dies out, and forcing persists
- **resonance** appears as a peak in max. amplitude, only if $\mu^2 < \frac{1}{2}\sigma^2$

Compare to the frequency–response curve for the **undamped system**:
III. Internal forcing:

Oscillations can also be excited via periodic variation in a system parameter:

\[ m\ddot{x} + \mu \dot{x} + \sigma(t)x = 0 \]

\( \sigma(t) \) periodic: \( \sigma(t) = \sigma(t + T) \) \implies \text{Hill’s equation (1886)}

special case: \( \sigma(t) = \sigma_0(1 + 2\varepsilon \cos \omega t) \) \implies \text{Mathieu equation (1868)}

Compare:

- Solution can become unstable whether or not \( \mu = 0 \)!
  \implies \text{called parametric resonance}

- The system responds at frequency \( \frac{1}{2}\omega \) [Why?]

- Internal forcing can also stabilize systems that are otherwise unstable

  e.g., inverted pendulum with gravitational modulation
A Heuristic Look at Parametric Resonance

Take \( m = 1, \mu = 0 \), and treat the periodic term as a forcing term:

\[
\ddot{x} + \omega_0^2 x = -2\varepsilon \omega_0^2 (\cos \omega t) x
\]

where \( \omega_0^2 = \sigma_0 \).

First solve the homogeneous problem:

\[
\ddot{x} + \omega_0^2 x = 0 \implies x(t) = A \cos(\omega_o t - \varphi)
\]

Substitute into the right hand side:

\[
\ddot{x} + \omega_0^2 x = -2A\varepsilon \omega_0^2 \cos \omega t \cos(\omega_o t - \varphi)
\]

\[
= -A\varepsilon \omega_0^2 \left\{ \cos \left[ (\omega + \omega_o) t - \varphi \right] + \cos \left[ (\omega - \omega_o) t - \varphi \right] \right\}
\]

Resonance ensues if \( \omega - \omega_o = \omega_o \) or \( \omega_o = \frac{1}{2} \omega \)

response frequency \( = \frac{1}{2} \) (forcing frequency)
Examples of Parametric Resonance

- Springs or pendula with moving supports
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- RLC circuits with periodically-varying inductance or capacitance
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- Surface waves in fluids and granular materials $\Rightarrow$ Faraday instability [Faraday, 1831; Benjamin & Ursell, 1954]
Examples of Parametric Resonance

- Springs or pendula with moving supports
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- Solar surface heating generating atmospheric convection cells or patterned ground formation  [McKay, 1998]
- Surface waves in fluids and granular materials  $\rightarrow$ Faraday instability  [Faraday, 1831; Benjamin & Ursell, 1954]
- Two-fluid interfaces  [Kumar & Tuckerman, 1994]
- Bubble oscillations and sonoluminescence  [Brenner, Lohse & Dupont, 1995]
- Many problems in aerodynamics involving wing stability and flutter (but wing oscillations are small and have no influence on the flow)

There are many examples in fluids, but almost no analysis for fluid-structure interaction problems.
Faraday (1831) observed that:

- regular, symmetric patterns arise on the surface
- waves oscillate with **twice the period** of the parametric forcing

Source: Dept. of Physics, University of Toronto
http://mobydick.physics.utoronto.ca/faraday.html
Some Results from Floquet Theory

For homogeneous, periodic, linear systems:

\[ \dot{x} = A(t) \cdot x, \quad \text{where } A(t) \in \mathbb{R}^{n \times n}, \quad x(t) \in \mathbb{R}^n \quad (\ast) \]

and

\[ A(t + T) = A(t), \quad \text{for some } T > 0. \]

**Floquet Theorem** *(Bloch Theorem in solid state physics):*

Every fundamental matrix solution \( X(t) \) of \((\ast)\) has the form:

\[ X(t) = e^{Bt} \cdot P(t) \]

where \( P(t + T) = P(t) \) and \( B = \text{constant} \).

**Corollary:** Any solution of \((\ast)\) is a linear combination of functions of the form:

\[ e^{\gamma t} \cdot p(t) \]

"Floquet" or characteristic exponent, \( \gamma \in \mathbb{C} \)

a polynomial with \( T \)-periodic coefficients
2. Floquet Analysis of the Mathieu equation

\[ \ddot{x} + \mu \dot{x} + \sigma_o (1 + 2\varepsilon \cos \omega t) x = 0 \]

- Cannot have a solution \( x(t) = A \cos \omega t + B \sin \omega t \)
- Instead, the \( \omega \)-periodic term excites all Fourier modes with frequencies \( n\omega, \ n = 1, 2, \ldots \)

Longrightarrow  Look for solutions of the form (details in [Nayfeh & Mook, 1979])

\[
x(t) = e^{\gamma t} \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \text{ where } \gamma = \alpha + i\beta
\]

\( \alpha \) governs stability:
- \( \alpha > 0 \) \( \rightarrow \) unstable
- \( \alpha < 0 \) \( \rightarrow \) stable
- \( \alpha = 0 \) \( \rightarrow \) stability boundary

\( \beta \) is the frequency (consider only \( 0 \leq \beta \leq \frac{1}{2} \omega \)):
- \( \beta = 0 \) \( \rightarrow \) harmonic
- \( \beta = \frac{1}{2} \omega \) \( \rightarrow \) subharmonic
- \( 0 < \beta < \frac{1}{2} \omega \) \( \rightarrow \) decaying solutions (ignore these)
Substitute

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{(\gamma+in\omega)t} \]

into

\[ \ddot{x} + \mu \dot{x} + \sigma_o (1 + 2\varepsilon \cos \omega t)x = 0 \]

Interested only in the stability boundary where \( \alpha = 0 \) (i.e., \( \gamma = 0 + i\beta \)):

\[ \sum_{n=-\infty}^{\infty} c_n e^{i(\beta+n\omega)t} \left\{ - (\beta + n\omega)^2 + i\mu(\beta + n\omega) + \sigma_o \left[ 1 + \varepsilon \left( e^{i\omega t} + e^{-i\omega t} \right) \right] \right\} = 0 \]

shifts \( n \) by \( \pm 1 \)
Substitute

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{(\gamma + in\omega) t} \]

into

\[ \ddot{x} + \mu \dot{x} + \sigma_o (1 + 2\varepsilon \cos \omega t) x = 0 \]

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\[ \sum_{n=-\infty}^{\infty} c_n e^{i(\beta + n\omega)t} \left\{ -(\beta + n\omega)^2 + i\mu(\beta + n\omega) + \sigma_o \left[ 1 + \varepsilon \left( e^{i\omega t} + e^{-i\omega t} \right) \right] \right\} = 0 \]

\[ \Rightarrow [\sigma_o - (\beta + n\omega)^2 + i\mu(\beta + n\omega)]c_n + \varepsilon \sigma_o (c_{n-1} + c_{n+1}) = 0 \quad (***) \]

**Aim:** To find real values of \( \varepsilon \):

- split (***) into real and imaginary parts,
- let \( c_n = c_n^r + ic_n^i \), and . . .
define \( \vec{c} = [\ldots, c_{n-1}^r, c_{n-1}^i, c_n^r, c_n^i, \ldots]^T \) so (***) can be written in matrix form:

\[
\begin{bmatrix}
\cdots & \cdots & & \cdots \\
0 & 0 & A_n & -B_n & 0 & 0 \\
0 & 0 & B_n & A_n & 0 & 0 \\
& & \cdots & & \cdots
\end{bmatrix}
\vec{c} + \varepsilon \sigma_o
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
& & \cdots & & \cdots
\end{bmatrix}
\vec{c} = 0
\]
define $\vec{c} = [\ldots, c_{n-1}^r, c_{n-1}^i, c_n^r, c_n^i, \ldots]^T$ so (**) can be written in matrix form:

$$
\begin{bmatrix}
0 & 0 & A_n & -B_n & 0 & 0 \\
0 & 0 & B_n & A_n & 0 & 0 \\
0 & 0 & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{bmatrix}
\begin{bmatrix}
\vec{c}
\end{bmatrix}
+ \varepsilon \sigma_o
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{bmatrix}
\begin{bmatrix}
\vec{c}
\end{bmatrix}
= 0
$$

where $A_n = \sigma_o - (\beta + n\omega)^2$ and $B_n = \mu(\beta + n\omega)$.

More compactly:

$$
(-\sigma_o D^{-1} E) \vec{c} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \vec{c},
$$

An infinite-dimensional eigenvalue problem with eigenvalues $\frac{1}{\varepsilon}$, and eigenvectors $\vec{c}$.
Details:

- Cut off at a finite $N$, i.e., $\sum_{-\infty}^{\infty} \rightarrow \sum_{-N}^{N}$ (since $c_n \rightarrow 0$ as $|n| \rightarrow \infty$)

- Apply “reality conditions”:
  - $c_{-n} = c_{n}^*$, if $\beta = 0$ (harmonic)
  - $c_{-n} = c_{n-1}^*$, if $\beta = \frac{1}{2}\omega$ (subharmonic)

  to eliminate all $c_n$ with $n < 0$ $\Rightarrow$ a $(2N + 2) \times (2N + 2)$ linear system

- Parameter values:
  - $\mu, \omega$: given
  - $\alpha = 0$: for stability boundaries
  - $\beta = 0, \frac{1}{2}\omega$: for harmonic/subharmonic cases
  - $N = 15$: chosen so that $|c_n| \ll 1$ for all $|n| > N$

  - $\varepsilon$: comes from solving the eigenvalue problem
  - $\sigma_o$: is a free parameter

**Basic Idea:** for each $\sigma_o$, we obtain $2N + 2$ values of $\varepsilon$

... pick the positive, real ones.
Stability of the Mathieu equation

\[ \ddot{x} + \mu \dot{x} + \sigma_0 (1 + 2\epsilon \cos \omega t) x = 0 \]

Ince-Strutt Diagram \((\omega = 2, \mu = 0)\):

- \[ \cdots \] = harmonic \((\beta = 0)\)
- \[ \cdots \] = subharmonic \((\beta = \frac{1}{2} \omega)\)

- Eigenvalues divide the plane into stable and unstable regions
  \[ \Rightarrow \text{tongues of instability} \]
- \(\epsilon \leq \frac{1}{2}\) are “physical” eigenvalues
- \(\epsilon = 0\) gives natural modes, \(\sigma_o = n^2\)
Stability of the Mathieu equation
\[ \ddot{x} + \mu \dot{x} + \sigma_o (1 + 2\varepsilon \cos \omega t)x = 0 \]

Increase damping further to \( \mu = 0.6 \) ➔ no more \textbf{physical} instabilities
Stability of the Mathieu equation
\[ \ddot{x} + \mu \dot{x} + \sigma_o (1 + 2\varepsilon \cos \omega t) x = 0 \]

Note:

- even small perturbations \((\varepsilon \approx 0)\) can lead to instability if \(\mu\) is small enough
- damping has a **stabilizing** influence
  \[\implies\] the \(\omega = 2\) problem is stable if \(\mu \gtrsim 0.6\)

Increase damping to \(\mu = 2.0\)

[ movie ]
3. Immersed Boundaries

Move on to fluid-structure interaction problems ...

“Life is . . . fiber-reinforced fluid.”

– C. S. Peskin (1999)
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Biological fibers, and surfaces constructed of fibers, immersed in fluid:

• heart and blood vessels
• worms and leeches
• flagellae and cilia
• plant cells (esp. wood)
• microtubules and actin filaments
• suspensions of proteins, DNA, polymers, etc.
Peskin and McQueen (2000): Simulations of the beating heart

Source: [link](http://www.psc.edu/science/Peskin/Peskin.html)

[ movie0 ] [ movie1 ] [ movie2 ] [ movie3 ]
Fiber Architecture of the Heart Muscle Wall

[Dissected pig heart: Carolyn Thomas, 1957]
The Immersed Boundary Model

Simplifying assumptions:

- 2D fluid with a single, impermeable, elastic membrane (or fiber)
- fiber has zero volume and mass, and is neutrally buoyant
- fluid lies both inside and outside the fiber
- domain is periodic
- in 3D, surfaces are built from an interwoven mesh of fibers
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\[ \Omega = \text{fluid domain} \]
\[ p(\vec{x}, t) = \text{fluid pressure} \]
\[ \vec{u}(\vec{x}, t) = \text{fluid velocity} \]

\[ \Gamma = \text{fiber} \]
\[ s = \text{arclength parameter} \]
\[ \vec{X}(s, t) = \text{fiber position} \]
\[ \vec{U}(s, t) = \text{fiber velocity} \]
\[ \vec{F}(s, t) = \text{fiber force density} = \sigma \partial_s^2 \vec{X} \text{ ("springs") } \]
Dynamic interaction between fluid and fiber:

- the fiber exerts a **singular force** on adjacent fluid particles:

\[
\rho \left( \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} \right) = \mu \Delta \vec{u} - \nabla p + \int_{\Gamma} \vec{F}(s, t) \delta(\vec{x} - \vec{X}(s, t)) \, ds
\]

\[
\nabla \cdot \vec{u} = 0
\]

(Navier-Stokes equations with a singular force)
Dynamic interaction between fluid and fiber:

- the fiber exerts a **singular force** on adjacent fluid particles:

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\nabla \cdot \vec{u} = 0 \]

(Navier-Stokes equations with a singular force)

- fiber moves at the fluid velocity – **the no slip condition**:

\[
\partial_t \vec{X} = \vec{u}(\vec{X}(s,t), t) = \int_{\Omega} \vec{u}(\vec{x}, t) \delta(\vec{x} - \vec{X}(s,t)) \, d\vec{x} 
\]

\[\Rightarrow\] interactions are mediated by **delta functions**!

An alternate formulation eliminates delta functions in favour of **jump conditions** \[\Rightarrow\] immersed interface method [Leveque & Li, 1994]
Previous Work

1. A linear stability analysis for unforced fibers, initially deformed and then released [JS & Wetton, 1995]:
   - unconditionally stable ($\alpha > 0$)
   - dependence of solution on parameters ($Re, \sigma_o$) is non-trivial
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Question:
What happens when a fiber is pulsed periodically (like a heart muscle fiber)?
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   **Question:**
   What happens when a fiber is pulsed periodically (like a heart muscle fiber)?

3. A related problem [Wang, 2003]:
   - Floquet-type analysis for buckling instabilities in a headbox, assuming small deformations
   - flow oscillations drive the structure BUT the structure has no effect on the fluid
4. Stability Analysis for Immersed Boundaries

Investigate the stability of a perturbed circular fiber with periodically-varying stiffness

\[ \sigma(t) = \sigma_0 (1 + 2\varepsilon \cos \omega t) \]
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Investigate the stability of a perturbed circular fiber with periodically-varying stiffness

\[ \sigma(t) = \sigma_o (1 + 2\varepsilon \cos \omega t) \]

**Simplifications:**

- Assume *small* perturbations about \( r = 1 \)
- Linearize the Navier–Stokes equations \( \Rightarrow \) Stokes’ equations
- Convert to stream function & vorticity: \( u, v \longrightarrow \psi, \xi \)
- Integrate the Navier-Stokes equations across the fiber \( \Rightarrow \) eliminates \( \delta \)-functions in favour of jump conditions
Inner \((r < 1)\) and outer \((r > 1)\) solutions both obey:

\[
\Delta \psi = -\xi
\]

\[
\partial_t \xi = \frac{\mu}{\rho} \Delta \xi
\]

Immersed boundary equation:

\[
\partial_t \vec{X} = (\partial_\theta \psi, -\partial_r \psi)|_{r=1}
\]

Jump conditions, with \([\cdot] = (\cdot)|_{r=1^+} - (\cdot)|_{r=1^-}:

\[
[[\psi]] = 0, \quad [[\xi]] = \frac{p^2 \sigma_o}{i \omega \mu} (\psi(1) - \partial_r \psi(1))
\]

\[
[[\partial_r \psi]] = 0, \quad [[\partial_r \xi]] = \frac{p^2 (p^2 - 1) \sigma_o}{i \omega \mu} \psi(1)
\]

Boundary conditions:

\(\psi\) and \(\xi\) bounded as \(r \to \infty\)

periodic matching at \(\theta = 0\) and \(2\pi\)
Floquet–type solutions:

For $\xi$ (and $\psi$):

$$\xi(r, \theta, t) = e^{ip\theta} \sum_{n=-\infty}^{\infty} \xi_n(r) e^{(\gamma + in\omega)t}$$

same as before

For $X^r$ (and $X^\theta$):

$$X^r(\theta, t) = e^{ip\theta} \sum_{n=-\infty}^{\infty} X_n^r e^{(\gamma + in\omega)t}$$

Substitute and obtain a Bessel equation for $\xi_n$:

$$z^2 \xi''_n(z) + z \xi'_n(z) + (z^2 - p^2) \xi_n(z) = 0$$

where $z = - (\Omega_n r)^2$ and $\Omega_n \equiv \sqrt{\frac{\rho(\gamma + in\omega)}{\mu}}$
Solution: can be written in terms of Bessel functions, $J_p$ and $H_p$:

$$
\xi_n(r) = \begin{cases} 
    a_n H_p(i\Omega_n r), & \text{if } r > 1 \text{ (outer)} \\
    b_n J_p(i\Omega_n r), & \text{if } r < 1 \text{ (inner)}
\end{cases}
$$

and

$$
\psi_n(r) = \text{combinations of } J_p, H_p, r^{-p} \text{ and } r^p
$$

Applying jump conditions yields the two equations:

$$
0 = i \left\{ \frac{\mu^2}{\sigma_o} \Omega_n^3 \left[ \frac{H_p(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_p(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] + ip \right\} X_r^n
$$

$$
+ \left\{ \frac{\mu^2}{\sigma_o} \Omega_n^3 \left[ \frac{H_p(i\Omega_n)}{H_{p-1}(i\Omega_n)} + \frac{J_p(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] - ip^2 \right\} X_\theta^n
$$

$$
+ i\varepsilon_p \left( X_{r,n-1}^r - X_{r,n+1}^r \right) - \varepsilon^2 \left( X_{\theta,n-1}^\theta - X_{\theta,n+1}^\theta \right)
$$

$$
0 = i \left\{ \frac{\mu^2}{\sigma_o} \Omega_n^4 \left[ 2 - \frac{H_{p+1}(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_{p-1}(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] + 2p(p^2 - 1) \right\} X_r^n
$$

$$
+ \frac{\mu^2}{\sigma_o} \Omega_n^4 \left[ \frac{H_{p+1}(i\Omega_n)}{H_{p-1}(i\Omega_n)} - \frac{J_{p-1}(i\Omega_n)}{J_{p+1}(i\Omega_n)} \right] X_\theta^n + 2\varepsilon p(p^2 - 1) \left( X_{r,n-1}^r - X_{r,n+1}^r \right)
$$

View as a linear system in the unknowns $[\text{Re}(X_r^n), \text{Im}(X_r^n), \text{Re}(X_\theta^n), \text{Im}(X_\theta^n)]$!
After some messy algebra, obtain another eigenvalue problem

\[ (-D^{-1}E) \vec{c} = \left( \frac{1}{\varepsilon} \right) \vec{c} \]

where \( D \) and \( E \) are infinite, real matrices consisting of \( 4 \times 4 \) blocks.

**Basic Idea** (same as for the Mathieu equation):

- Cut off at finite \( N \)
- Look for stability boundaries, \( \gamma = 0 \) and \( \frac{1}{2} i \beta \)
- Parameter space is now the \( p-\varepsilon \) plane (\( \sigma_o \) is scaled out)
- Results are reported in terms of the nondimensional parameters

\[
\kappa = \frac{\sigma_o}{\rho \omega_o^2 R^2} \quad \text{and} \quad \nu = \frac{\mu}{\rho \omega_o R^2}
\]

**Question:**
Are there regions of instability for physically-reasonable parameter values?
Stability Diagrams for the Forced Problem

**Case I:**
\[ \kappa = 0.04, \ \nu = 0.00056 \]
\[ p = 4 \text{ mode is unstable} \]

**Case II:**
\[ \kappa = 0.4, \ \nu = 0.001 \]
\[ p = 2 \text{ mode is unstable} \]
Compile the minimum $\varepsilon$ over a range of $\kappa$ and $\nu$:

\[
\nu = 0.0389 \cdot \kappa^{0.626}
\]

(. . . interesting . . . can this be explained?)

Next, compare to both computations and previous analyses . . .
Natural modes – unforced problem

- consider a fiber with no forcing, \( \varepsilon = 0 \)
- take the \( n = 0 \) mode
- the eigenvalue equations reduce to a single dispersion relation

The decay rate and frequency can be compared to direct numerical simulations

This represents an exact (asymptotic) solution that can be used to validate 2D computations!!!
In the limit as $\nu \to 0$, we obtain

Decay rate: $\alpha = 0$

Frequency: $\beta = \omega_N \sqrt{\kappa}$ where $\omega_N(p) = p \sqrt{(p^2 - 1)/2}$.

$\Rightarrow$ matches the inviscid analysis of [Cortez & Varela, 1997]
Natural modes – small $\nu$ limit

In the limit as $\nu \to 0$, we obtain

Decay rate: $\alpha = 0$

Frequency: $\beta = \omega_N \sqrt{\kappa}$ where $\omega_N(p) = p \sqrt{(p^2 - 1)/2}$.

$\implies$ matches the inviscid analysis of [Cortez & Varela, 1997]

Asymptotic expansion of the dispersion relation in $\nu$ yields:

Decay rate: $\alpha \sim -\frac{p}{2\sqrt{2}} \omega_N^{1/2} \kappa^{1/4} \sqrt{\nu}$

Frequency: $\beta \sim \omega_N \sqrt{\kappa} - \frac{p}{2\sqrt{2}} \omega_N^{3/4} \kappa^{1/4} \sqrt{\nu}$

$\implies$ matches the viscous analysis of [JS & Wetton, 1995]
5. Numerical Simulations

“Usual” implementation of the immersed boundary method [Peskin, 1977]

- second order centered differences in space
- split step projection method
- delta function approximation reduces spatial accuracy to first order
- first order and explicit in time

**Notice:** small discrepancies between analysis and numerics owing to

- artificial dissipation from the numerical scheme
- numerical errors from the first order time–stepping

We have also run fully second-order simulations using the blob projection method [Cortez & Minion, 2000]
Unforced fiber simulations

Fiber is initially perturbed with a $p$–mode of magnitude 0.05

**Case I:** $\kappa = 0.04$, $\nu = 0.00056$, $p = 4$

- Initial
- Amplitude of $p = 4$ mode

[ movie ]
Fiber is initialized and forced with the same resonant $p$–mode

**Case I:** $\kappa = 0.04$, $\nu = 0.00056$, $p = 4$

Amplitude of $p = 4$ mode

Vary parameters

[ movie ]
Case II: $\kappa = 0.5$, $\nu = 0.001$, $p = 2$

Amplitude of $p = 2$ mode

Vary Parameters

[ movie ]
Energy Transfer from Resonant Modes

Because of numerical errors, the fiber motion is not a pure $p$-mode

- most of the energy remains in the resonant mode
- **Even** $p$: small perturbations appear in all $np$-modes
- **Odd** $p$: asymmetry leads to a small "drift" or $p = 1$ mode

$\implies$ all $p$-modes are excited! \[ \sin(p \pm q) = \sin p \cos q \pm \cos p \sin q \]

Case I, initialized with $p = 3$ mode

Energy transfers to the resonant $p = 4$ mode

[ movie ]
# Application to Atrial Fibrillation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Human heart</th>
<th>3D numerics (Peskin-McQueen)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>normal</td>
<td>fibrillated</td>
</tr>
<tr>
<td>$\rho$</td>
<td>g/cm$^3$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu$</td>
<td>g/cm s</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>g/cm s$^2$</td>
<td>1000?</td>
<td>1000?</td>
</tr>
<tr>
<td>$\omega_o$</td>
<td>rad/s (bt/min)</td>
<td>6.3 (60)</td>
<td>18.3 (170)</td>
</tr>
<tr>
<td>$R$</td>
<td>cm</td>
<td>3.2</td>
<td>3.2</td>
</tr>
<tr>
<td>$\kappa = \frac{\sigma_0}{\rho \omega^2 R^2}$</td>
<td>–</td>
<td>2.47</td>
<td>0.29</td>
</tr>
<tr>
<td>$\nu = \frac{\mu}{\rho \omega R^2}$</td>
<td>–</td>
<td>0.00062</td>
<td>0.00021</td>
</tr>
</tbody>
</table>

The diagram shows contours of eigenvalues, $\varepsilon$, with labels for Stable and Unstable regions. The axes are labeled $\Lambda$, $\nu$, $\kappa$, $\sigma_0$, and $\rho$. The diagram includes points labeled Norm, Num, and Fib.
• Most studies of atrial fibrillation point to **electrophysiological** causes.

• Our analysis suggests a possible fluid–mechanical contribution to fibrillation through **feedback** into propagation of electrical signals (?)

However, we still don’t expect instabilities to develop in normal hearts!
Possible explanations:

• non-linearities tend to stabilize:
  $$R_o \neq 0$$
  - the fiber, introducing a rest length $R_o \neq 0$
  - the fluid, moving to higher Reynolds number

• 3D effects are missing:
  - added stability of an interwoven fiber mesh
  - thickness of the heart wall
  - a heart beat is actually a spiral wave, not a homogeneous pulse
Application to Cochlear Mechanics

The cochlea is a spiral-wound, fluid-filled tube, which propagates waves along the basilar membrane (BM):

Recent research aims to explain the amazing sensitivity of the BM to sound waves in the presence of large viscous damping:

- (experiments) outer hair cells change length in response to shearing
- modulates the stiffness of the BM
- introduces a mechanical feedback that amplifies BM motions
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Cochlea</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>g/cm$^3$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu$</td>
<td>g/cm s</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_o$</td>
<td>g/cm s$^2$</td>
<td>1000</td>
</tr>
<tr>
<td>$\omega$</td>
<td>rad/s</td>
<td>1000+</td>
</tr>
<tr>
<td>$R$</td>
<td>cm</td>
<td>0.2</td>
</tr>
<tr>
<td>$\kappa$</td>
<td></td>
<td>0.025</td>
</tr>
<tr>
<td>$\nu$</td>
<td></td>
<td>0.00025</td>
</tr>
</tbody>
</table>

\[ \kappa = \frac{\sigma_o}{\rho \omega^2 R^2} \]
\[ \nu = \frac{\mu}{\rho \omega R^2} \]

- Current models of cochlear mechanics focus on mechanical resonance [Martin et al., 2000], [Nobili et al., 1998]
- Our analysis suggests that fluid-mechanical feedback may also play a significant role
- **BUT** we’re still missing 3D effects, nonlinearities in BM stiffness, and coupling to mechanical effects
6. Summary

- performed a Floquet analysis of a parametrically-forced immersed boundary
- derived an analytical (leading-order, asymptotic) solution which can be used to validate 2D numerical simulations – the first such exact solution!
- matched results with previous analyses of the unforced problem in the small-viscosity limit
- identified parameter ranges in which forced fiber dynamics are unstable

**Warning:**  
(for people simulating active, biological interfaces)  
watch out for parametric resonances, which can easily be mistaken for numerical instabilities!

- suggested possible parametric resonance in biological systems such as the human heart and cochlea
Future Work

- a 3D numerical study of forced immersed boundaries
- spatially-varying stiffness, $\sigma(\theta, t)$ simulates spiral wave propagation
- extend to fiber spring force with non-zero resting length, $R_o$:

\[ \vec{f}(s, t) = \sigma \partial_s^2 \vec{X} \implies \vec{f}(s, t) = \sigma \partial_s \left[ \partial_s \vec{X} \left( 1 - \frac{R_o}{|\partial_s \vec{X}|} \right) \right] \]

- “step-function” forcing is better than

- optimal control of forced immersed boundaries, with application to pacemaker design

\[ \implies \text{external forcing } f(t) = A \cos \Omega t, \text{ with } \Omega \text{ a control parameter} \]
References


