Engaging undergraduate students in a modeling course on the mathematics of (mostly Olympic) sport

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Abstract

A course in mathematical modeling has a central role in most undergraduate mathematics curricula. The range of mathematical techniques that can be applied to solve real world problems is remarkably broad and so the content of such a course can likewise cover an extensive and sometimes intimidating array of mathematical topics. In order to bring some cohesion to such a broad-based modeling course, and to spark student interest and engagement in mathematics, we have initiated a third-year undergraduate mathematics course that is organized around the theme of Olympic sport. The mathematical topics range across the undergraduate math and statistics curriculum, including graphical data analysis, linear regression, differential equations, optimization, probability, and game theory … sprinkled with a solid dose of Newtonian physics. The course is targeted toward students from all faculties who are either taking a minor specialization in mathematics or are likely to consider taking such a minor. The purpose of this talk is to describe the course structure and to motivate through a specific example how mathematics can be exploited to obtain useful insights into sport. Suggestions are also provided on how to incorporate projects and technology into such a modeling class with an aim to improving student engagement with the material.

1. Introduction

The idea for a course on mathematical modeling in sport sprang from a departmental initiative to create a series of third-year topics courses that would appeal to students from many disciplines and attract new minors or even majors to mathematics. Offerings in this series over the past 6 years have included permutation puzzles, Ramsey theory, uncertainty quantification, symmetry and geometry, and set theory. As an applied mathematician, I was motivated to develop a course on mathematical modeling, for which there are copious examples of courses having a well-defined disciplinary focus with titles such as *"Mathematical modeling in X"* where *X* = biology, economics, finance, agriculture, environment, etc. My aim was to design a non-standard course that would be suitable for students in all departments, and so I sought a cross-cutting theme with universal appeal.

What better topic to capture students' interest and imagination than sport? Sport not only dominates the popular media but also consumes a significant proportion of the average citizen's leisure time. Inspired by the upcoming 2010 Olympic Winter Games hosted in my home city of Vancouver, I was therefore motivated to create a modeling course with sport as the unifying theme. The first offering of *Math 302: The mathematics of (mostly Olympic) sport* was in Fall semester of 2009 fortuitously stumbling across a beautiful book by Cohen and de Mestre (2007). The course has since been offered twice more and has proven very popular, attracting between 40 to 65 students.

2. Course structure and evaluation

The range of problems encountered in sport is enormous, and so the course naturally covers a broad range of techniques spanning statistics, probability, calculus, differential equations and game theory, among others. It was therefore logical to organize the material around "modules", each requiring from 1 to 8 one-hour lectures to complete and focused on a particular sport or common underlying aspect of several related sports. Consequently, there was no well-defined syllabus and I structured the course instead around a selection of leading questions, with a partial list below:

Topical questions	**Mathematical techniques**
1. Who really won the 2008 Summer (or 2010 Winter) Olympics?	Descriptive statistics (graphing and data analysis)
2. What are the limits of human performance? And will women ever outperform men?	Curve fitting (regression and least squares)
3. Is there an optimal technique for throwing a shot or taking a long jump? And what connects these two events?	Differential and integral calculus (projectile motion)
4. What determines a runner's top speed or endurance? And who is the fastest (wo)man on the planet?	Differential equations, control theory, dimensional analysis
5. What are the chances of winning a tennis (or badminton or table tennis) match?	Probability, game theory
6. Is there really a "home ice advantage" in a best-of-7 Stanley Cup playoff series?	Probability (tournament design)

I will next address several details related to evaluating student performance, use of computing technology, and interactive or web-based resources.

2.1 Student evaluation: Assignments, midterm and group projects

Students were assigned regular bi-weekly homework sets timed roughly to coincide with the completion of each module, with 5-6 assignments in total. Most questions were somewhat routine in nature, focusing on mathematical manipulations and extensions of the ideas from lecture to slightly modified scenarios or related sports. I placed particular emphasis in the marking criteria on written discussions (in complete sentences) and interpretation in the context of the sport under study.

Because of the breadth of mathematical techniques represented in the topics above, and accounting for the wide range in students' mathematical backgrounds and abilities, it was evident from the outset that setting a comprehensive end-of-semester final exam covering such a wide range of topics was a recipe for disaster. However, there was a point near mid-semester – lying roughly between topics 3 and 4 above, and covering material for which all students had some previous familiarity – that was a natural juncture at which to set a mid-term test covering basic concepts in data analysis, curve fitting, and calculus. This test served as a fair measure of students' ability to apply techniques they were largely already familiar with, without requiring them to be proficient in a test situation with the full range of mathematical topics covered.

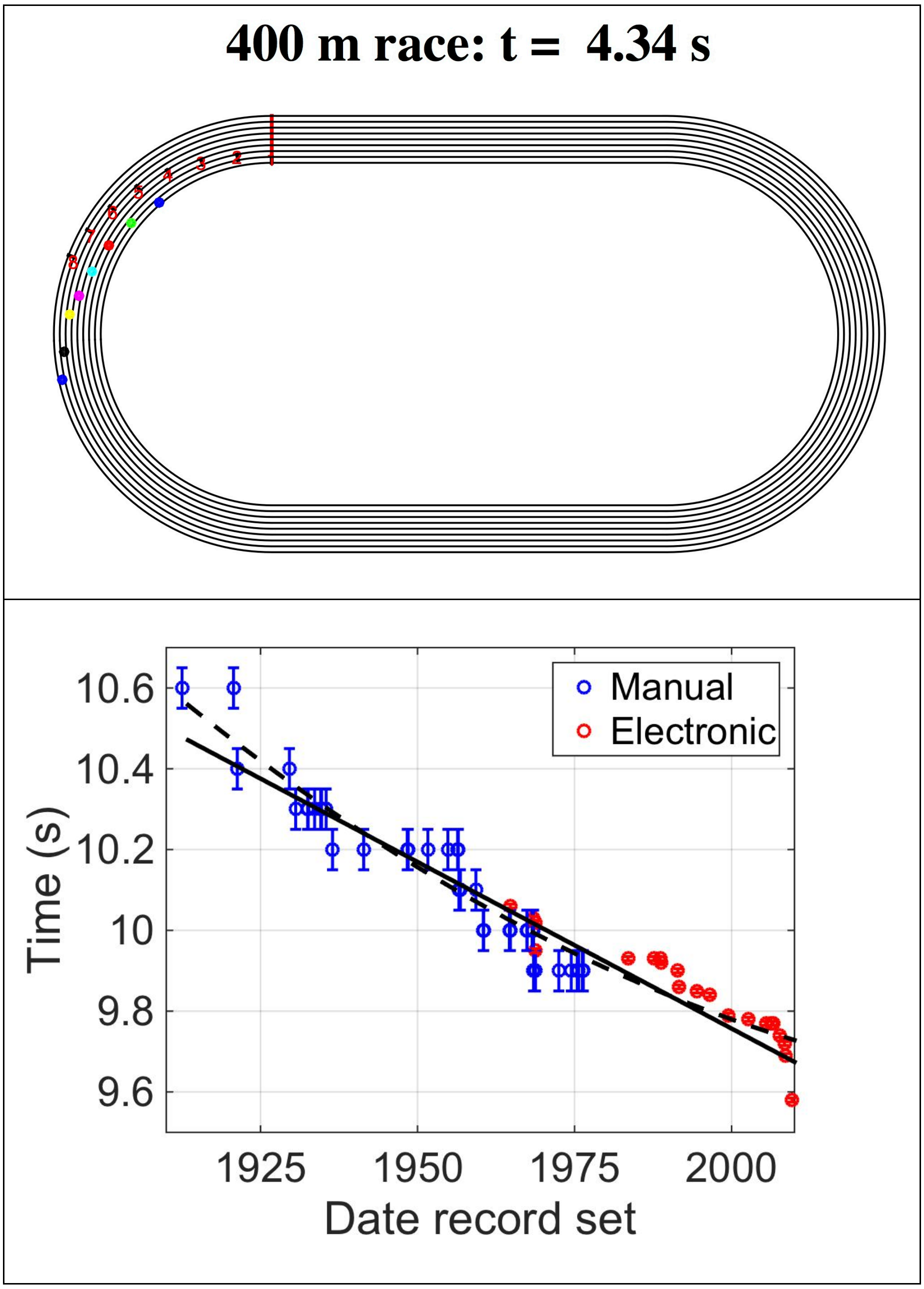
Instead of a final examination, the balance of the course grade was determined by a group project performed in groups of 3 to 5 students. Clear project milestones were laid out, beginning at the start of semester to ensure not only that students started early on, but also to help them identify problems of an appropriate difficulty. Below are the project requirements and rough deadlines (based on a 13-week semester):

- Group selection (week 3): I left this up to students and intervened only when necessary.
- Topic choice (week 4): a single page summary of topic and mathematical approach, with references. I supplied detailed feedback at this stage, with suggestions on adjusting the project scope/difficulty or providing additional references to the literature.
- Progress report (week 8): a 3-5 page report with an expanded outline, details on methodology, preliminary results and expanded bibliography.
- Poster presentation (week 12): with the main results provided in a pleasing visual poster format, with each group also giving a short oral summary (of 5-10 minutes). Students were also encouraged to use the remainder of the poster session to view other students' work and ask questions.
- Final report (week 13): a 15-20 page written document containing all the mathematical details and results.

The choice of topics was not constrained to the Olympics but was open to the full range of amateur and professional sport. Students' enthusiasm regarding projects was infectious, and the quality of the posters was generally very high. With only a few exceptions, these group projects proved to be an unqualified success and were the high point for most students' Math 302 experience, as evidenced by uniformly positive written comments on course evaluations.

2.2 Computer-based demonstrations and assignment questions

Many aspects of sport are data-intensive, for example the study of Olympic medals, progression of records in timed/measured events, or estimation of winning probabilities from match/tournament outcomes. Fortunately, copious sources of data are readily available on web sites of the IOC, sporting federations, Wikipedia, etc. However, dealing with such large data sets by hand is impractical and it was necessary to employ computer software to illustrate results. It was also natural to have students explore the data with the computer on their homework assignments. Consequently, most assignments contained at least one "computational" question for which I provided code or a Microsoft Excel spreadsheet and required students to either modify the code and/or use it to plot or process the data. Two examples are provided below:



- On the top is a screen shot from a computer simulation of a 400m running race, incorporating the solution from a differential equation model of running presented by Townend (1983). The code constructs a regulation-size running track (whose precise geometry we study in class) and then animates the dynamics of 8 runners with randomly-selected peak speeds. The excitement generated in a classroom full of students who are cheering on their own favourite "coloured dot" to the finish line is simply amazing!
- On the bottom is a plot of the world record progression in the 100m sprint that illustrates the application of least squares curve fitting using both linear and quadratic polynomials.

The interested reader can download the Matlab code and supporting files for both examples from http://www.math.sfu.ca/~stockie/mathsport.

2.3 Sports trivia, videos and interactive demos

The study of sport is distinguished by the availability of an enormous diversity of on-line material with which to motivate students. I have already pointed out the rich sources of sport data, and a few other examples are listed below:

- *Sports trivia:* In most classes, I include at least one multi-choice sports trivia question on some aspect of the topic under study. In each case, I take an informal "show of hands" poll and then we discuss the outcome.
- *News flashes:* Sports pervade the media and so it is easy to provide regular news reports on some aspect of the Olympic Games or the specific sport being studied in class. I found it especially stimulating to discuss the frequent errors in mathematical or statistical reasoning that are made by reporters or subjects of media interviews (Paulos, 1997).
- *Videos and podcasts:* Every topic is illustrated with at least one video, podcast or other media source that explains the mechanics of the sport under study (useful to students who may be unfamiliar with the sport). Interviews with scientists or sports experts are also a great way to engage students' interest. YouTube (http://www.youtube.com) and the *Maths and Sport* site *(Millennium Mathematics Project,* http://sport.maths.org) are two excellent sources.
- *Interactive demos:* Several news outlets (mostly notably the *New York Times*) regularly publish beautiful high-resolution graphics or interactive on-line applets that permit easy exploration of sports data. My all-time favourite is the NY Times Olympic Medal Map (http://tinyurl.com/ob983ta).

The interactivity of the course could be further enhanced by incorporating a classroom response system with "clickers", whose pedagogical value in more traditional math courses has already been proven (Bode et al., 2009). Multiple-choice trivia questions lend themselves naturally to clickers, and I plan to extend my question bank to include more subject-based questions that test and explore students' understanding of mathematical concepts and their applications to sport.

3. A sample module: The mathematics of running

As a concrete illustration, I next give a brief description of the module "mathematics of running" (topical question 4 from the list in section 2). This is the longest of all modules in the course (requiring 8 one-hour lectures to cover) which provides a nice glimpse of the range of mathematical techniques that can be applied to a single sport, as well as the connections between them. This module consists of three main parts:

a) *Geometry of track design*, illustrating that regulation-size IAAF 400m tracks require a great deal of precision to properly measure lane boundaries and start/finish lines in the various races. This is a straightforward exercise in geometry and algebra that involves only high school mathematics.
b) *An ODE model for sprints*, which is based on the ordinary differential equation $\frac{dv}{dt} = F - kv$ for speed $v(t)$ with initial condition $v(0) = 0$. My discussion follows a lovely paper by Townend (1983) who treats the case of constant propulsion force F and obtains the solution $v(t) = \frac{F}{k}(1 - e^{-kt})$ by direct integration. Townend also provides a wonderful application of least squares fitting to obtain parameters F and k from measurements of split times in a 100m race. This model can be easily extended to include other effects such as acceleration out of

the starting block and runner fatigue by adding an explicit time-dependence to $F(t)$. I then conclude by explaining how Keller (1974) extends the sprint model to medium- and long-distance races by incorporating a second ODE governing energy expenditure and then reformulating the equations as an optimal control problem. I make no attempt to solve this much more difficult problem in class, but a discussion of Keller's solution nonetheless provides students with some key insights into race strategies.

c) *Estimating peak running speed using techniques from dimensional analysis,* focusing first on how to identify the essential physical parameters in a problem, and then applying the Buckingham Pi Theorem to obtain a functional dependence of peak speed on other parameters.

Several additional sub-topics are touched on throughout this module, including the effects of lane curvature on peak running speed (due to higher centripetal forces in the inner lanes) and drag force due to air resistance (leading into a discussion of the rules on the validity of wind-assisted record times).

It is worth noting that the Matlab track simulation pictured above is a vivid demonstration of how the equations describing running track geometry can be combined with the ODE model for a sprint race. Indeed, this parallels what often occurs in the mathematical modeling of many real world problems – namely, that the desired solution is seldom achieved in a single step, but rather requires the synthesis of solutions from several distinct sub-problems.

4. Closing remarks

Sport is an ideal source of problems to illustrate mathematical modeling concepts from the undergraduate curriculum. There is a wide range of books available to instructors who wish to teach such a course focused on general application of mathematics in sport (e.g., Cohen and de Mestre, 2007; Sadovskii and Sadovskii, 1993; Townend, 1983) or more specifically focused on a topic such as projectile motion (de Mestre, 1990). The web site *Maths and Sport* hosted at the University of Cambridge was developed leading up to the 2012 London Summer Olympics and contains a wealth of teaching resources suitable for all levels from elementary school to university. By capitalizing on students' interest and enthusiasm for sport outside the classroom, it is much easier to motivate the importance of mathematical and statistical techniques for understanding aspects of sport that range over biomechanics, performance, judging, tournament design and strategic games.

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