Mathematical Modelling of Sap Flow

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Introduction

In the plant hydraulic literature, biologists have proposed three types of mathematical models to describe the sap flow through trees: an RC model based on Ohm's law, a hydraulic model, and a porous media model based on Darcy's law. These models are described and implemented, and we show that there are certain mathematical equivalencies between these three models.

RC Model and Hydraulic Model

The RC model is based on an analogy between a tree's hydraulic infrastructure and an RC circuit. The variables that are treated as analogous are as follows. (We let 'pool' be a placeholder for the compartment of the tree; e.g., the crown, stem, ect.)

- Water content W (kg) and charge Q (c).
- Flow rate $f(pool)$ (kg/s) and current I $(A = c/s)$.
- Water potential ψ (Pa) and voltage V $(v = \frac{J}{c^2})$.
- Resistance in xylem/storage $R^{x/s}$ $(\frac{1}{m \cdot s})$ and resistance R $(\Omega = \frac{J \cdot s}{c^2})$.
- Capacitance $C(pool)$ $(m \cdot s^2)$ and capacitance C $(F = c^2/s)$.

The governing equations for tree water flow are then given by analogies to RC laws:

$$(1)\ f(pool) = \frac{dW}{dt} \quad \backsim \quad I = \frac{dQ}{dt}$$

$$(2)\ f(pool) = \frac{\Delta\psi}{R^{x/s}} \quad \backsim \quad I = \frac{\Delta V}{R}$$

$$(3)\ C(pool) = \frac{dW/dt}{d\psi/dt} = \frac{dW}{dt} \quad \backsim \quad C = \frac{dQ/dt}{dV/dt}$$

Model in Steppe et al.

We consider the tree to be divided into four sections: crown xylem, crown storage, stem xylem, and stem storage. Let $F_{crown/stem}$ be the flow rate in the crown/stem xylem, $W_{crown/stem}$ be the water content in the crown/stem storage, $R^{x/s}$ be the resistance in the xylem/storage, ψ_{roots} be the water potential of the soil (assumed to be constant), $\psi^s_{crown/stem}$ be the water potential of the crown/stem storage, and $E(t)$ (kg/s) be the transpiration rate. Using the basic equations (1), (2) and conservation of water mass, we obtain:

$$(4)\ F_{crown}(t) = \frac{dW_{crown}}{dt}(t) + E(t)$$

$$(5)\ F_{stem}(t) = \frac{dW_{stem}}{dt}(t) + F_{crown}(t)$$

$$(6)\ \frac{dW_{crown}}{dt}(t) = \frac{R^x}{A}\psi^s_{stem} - \frac{R^x + R^s}{A}\psi^s_{crown} + \frac{R^s}{A}\psi_{roots} - \frac{(R^s)^2 + 2R^xR^s}{A}E(t)$$

$$(7)\ \frac{dW_{crown}}{dt}(t) = -\frac{2R^x + R^s}{A}\psi^s_{stem} + \frac{R^x}{A}\psi^s_{crown} + \frac{R^x + R^s}{A}\psi_{roots} - \frac{R^xR^s}{A}E(t)$$

by letting $A = (R^x)^2 + 3R^xR^s + (R^s)^2$.

The difference between the hydraulic and RC model consists in the function that we select for $\psi^s(t)$. For the hydraulic model, we select

$$(8)\ \psi^s_{pool}(t) = \frac{\psi^s_{min}(pool)}{1 + exp(\frac{W_{pool}(t) - K_1}{K_2})}$$

with constants $\psi^s_{min}(pool)$, K_1, and K_2. For the RC model, we select

$$(9)\ \psi^s_{pool}(t) = \text{constant} + \frac{W_{pool}(t)}{C(pool)}.$$

We implemented these models choosing parameters in accordance with Steppe (2004) to yield the following results.

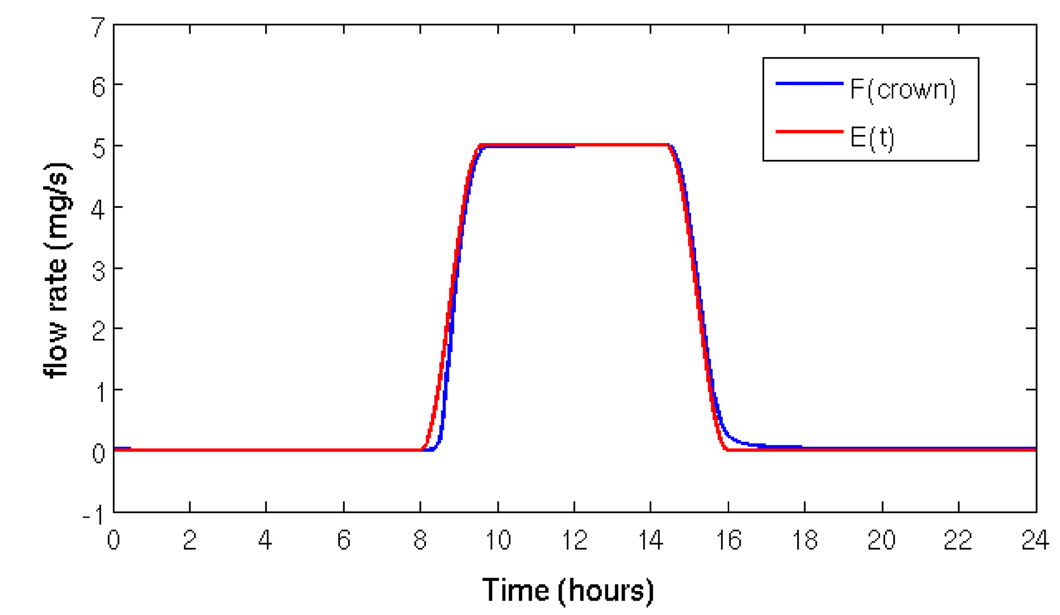

Figure 1: Flow rate in the crown xylem and transpiration rate.

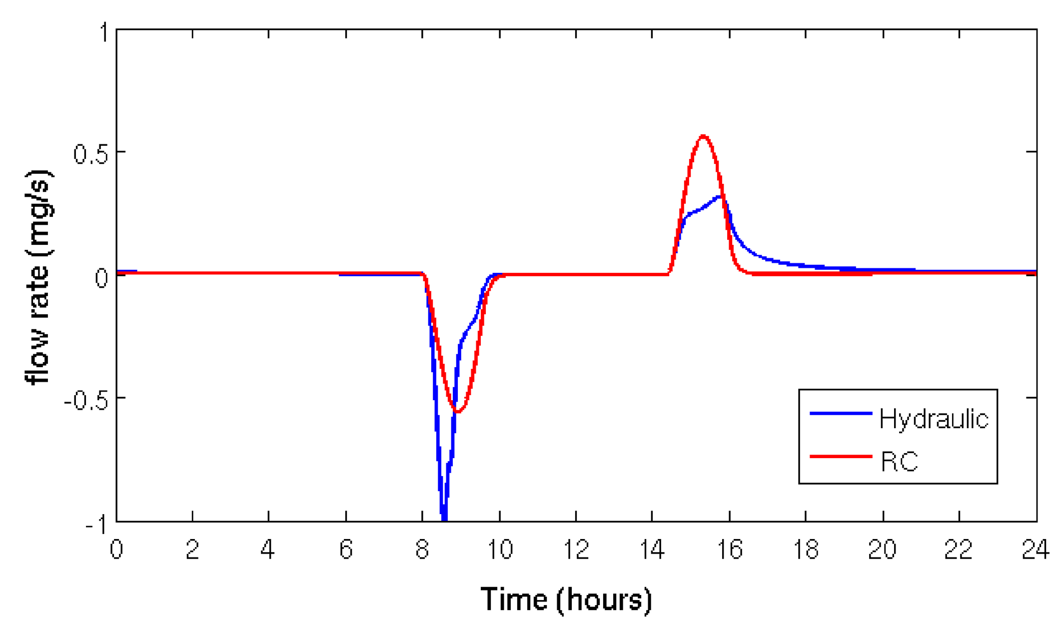

Figure 2: Flow rate in the crown storage.

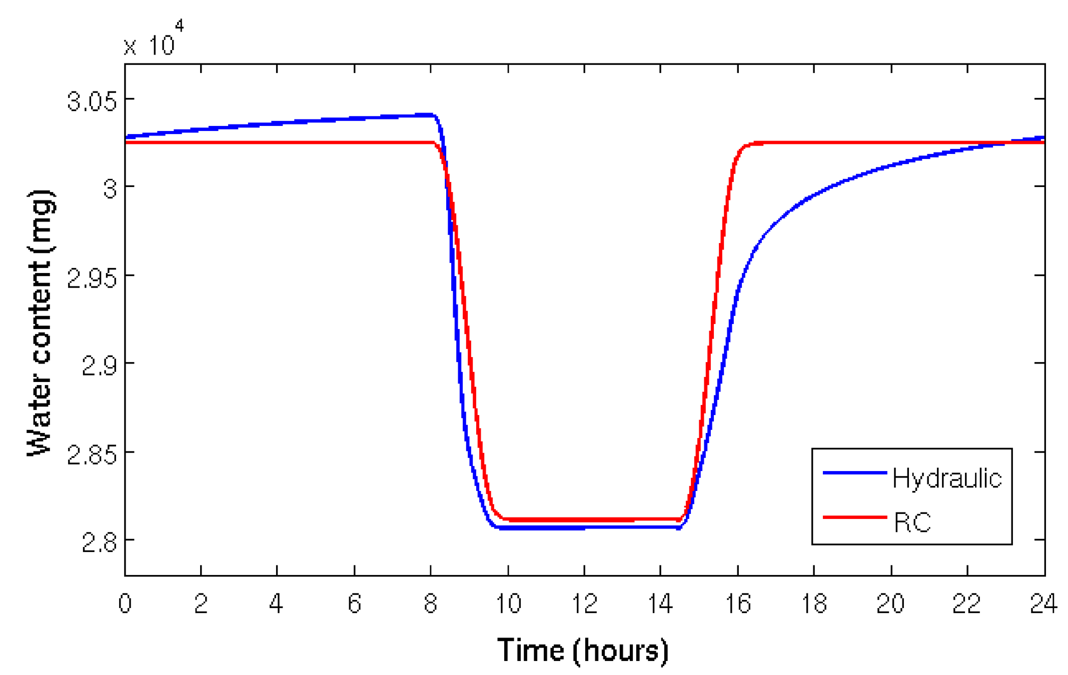

Figure 3: Water content in the crown storage.

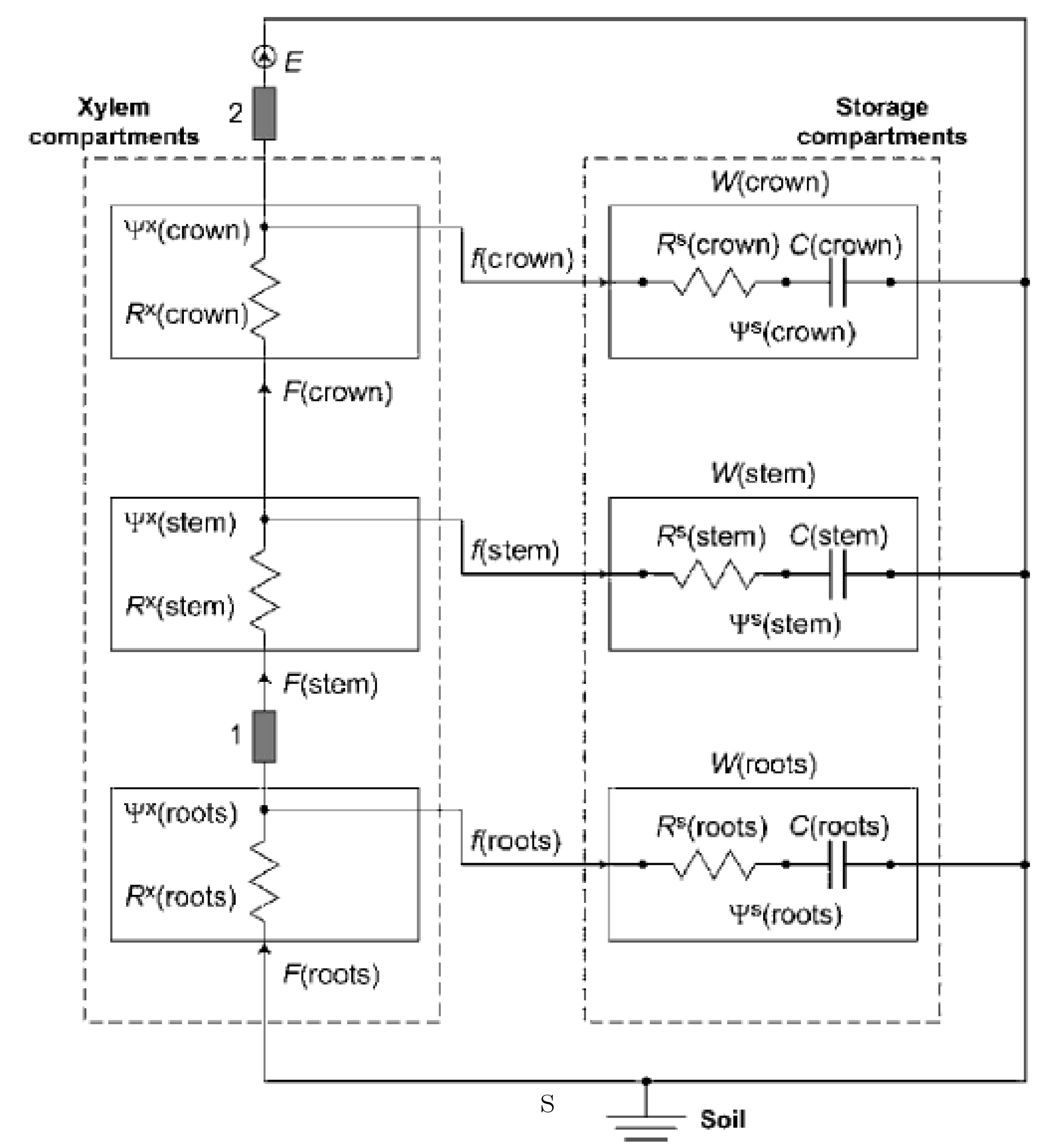

Figure 4: Steppe's electric circuit.

Equivalence of these Models

The hydraulic model and the RC model are in fact the same model. They have the same basic equations (1) and (2), and (9) can be derived from (8). For specified parameters K_1, K_2, $\psi^s_{min}(pool)$, we Taylor expand (8) around W_0 chosen so that the linear coefficient of the Taylor expansion is equal to $1/C(pool)$. We then truncate the Taylor series to its linear portion and obtain equation (9).

A Porous Media Model

In Holtta et al. (2006), a sap flow model based on Darcy's law is described. In this model, a tree is divided up into four radial compartments and discretized in the vertical direction into N compartments. The equation that governs flow rate between adjacent compartments is a discretized version of Darcy's law, given by:

$$(10)\ f = \frac{Ak\rho}{\eta l} \cdot \Delta\psi \quad (kg/s)$$

where k (m^2) is axial water permeability, η (Ns/m^2) is the dynamic viscosity of water, l (m) is the compartment length, A (m^2) is the cross-sectional area, ρ is the density of water, and $\Delta\psi$ is the difference in water potential.

The equivalence between this discretized porous media model and the RC model can be shown by relating equation (10) with equation (2), which have the same mathematical form. Furthermore, by equating these, we obtain:

$$(11)\quad R^{x/s} = \frac{\eta l}{Ak\rho}.$$

This gives us an explicit equation for the resistances used in the RC model. In Steppe (2004, 2005), the following explicit formula is given for the resistance of a tree's storage compartment:

$$(12)\quad R^s = \frac{1}{A\rho L},$$

where $L(\frac{m}{Pa \cdot s})$ is the radial hydraulic conductivity of the membrane separating the storage and the xylem. This suggests the following explicit formula for this radial hydraulic conductivity:

$$(13)\quad L = \frac{k}{\eta l}.$$

Conclusion

The three models that we examined are mathematically the same. The RC model and the hydraulic model have the same basic equations (1) and (2), and the RC model's equation for $\psi^s(pool)$ (equation 9) is a linear Taylor approximation to the hydraulic model's equation for $\psi^s(pool)$ (equation 8). The porous media model is the same as the RC model because the discretized version of Darcy's law for flow rate (equation 10) has the same mathematical form as the RC model's equations for flow rate based on Ohm's law (equation 2). This equivalence suggests an explicit formula for the resistance (equation 11) and radial hydraulic conductivity (equation 13) terms used in RC models.

References

[1] T. Holtta et al., Trees **20**: 67-78 (2006).

[2] K. Steppe et al., Tree Physiology **26**: 257-273 (2005).

[3] Steppe, Kathy *Diurnal Dynamics of Water Flow Through Trees: Design and Validation of a Mathematical Flow and Storage Model.* PhD diss., Ghent University, (2004).

Acknowledgements

This project was supported by an NSERC undergraduate research award and Simon Fraser University.