The Natural Oscillation of Immersed Elastic Membranes: Theory and Experiment

William Ko, Darrell Tse and John M. Stockie

Department of Mathematics, Simon Fraser University

SFU

Introduction

The immersed boundary (IB) method is a mathematical framework to model and simulate the two-way fluid-structure interaction between an elastic solid and its surrounding fluid [1]. The IB method has been used in a wide variety of biological and industrial applications. However, little has been done experimentally to show how the IB method reflects the real world. The aim of this project is to further validate the IB method by performing experiments on a popular IB test problem: The natural oscillation of an immersed spherical water balloon.

Immersed Boundary Model

- Consider a spherical elastic membrane (e.g. rubber balloon) which is filled with and immersed in a fluid (e.g. water).
- The fluid is governed by the incompressible Navier-Stokes equations,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\Delta\mathbf{u} + \mathbf{f},$$

$$\nabla\cdot\mathbf{u} = 0,$$

  where $\mathbf{u}$ is the fluid velocity and p is the pressure. The density and viscosity of the fluid is denoted by ρ and μ, respectively.
- The membrane, represented by $\mathbf{X}(\xi,\eta,t)$, moves with the local fluid velocity,

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{u}(\mathbf{X},t).$$

- The force exerted on the fluid is due to deformations of the membrane

$$\mathbf{f}(\mathbf{x},t) = \int_0^{\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{X},t)\delta(\mathbf{x}-\mathbf{X})\sin(\eta)\mathrm{d}\xi\mathrm{d}\eta,$$

$$\mathbf{F}(\mathbf{x},t) = \frac{1}{2}\sigma\Delta\mathbf{X},$$

  where σ is the membrane stiffness parameter and $\delta(\mathbf{x})$ is the Dirac delta function.

Linear Stability Analysis

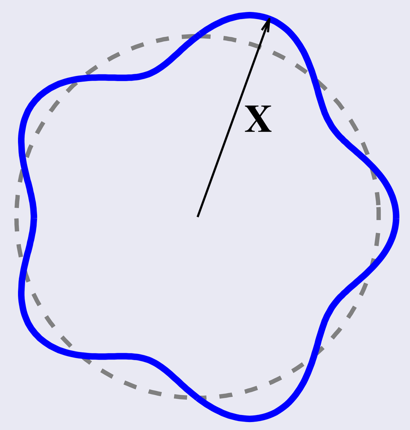
- Assume that the water balloon is a pressurized sphere of radius R at equilibrium. If the membrane is slightly perturbed,

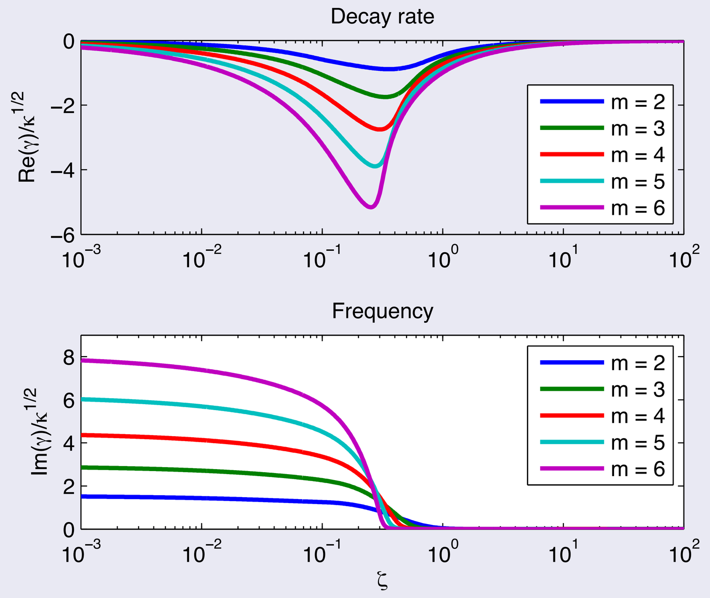
$$\mathbf{X}(\xi,\eta,0) = X^r(\xi,\eta,0)\hat{r} = R(1+\epsilon g(\xi,\eta))\hat{r},$$

  how does it oscillate as it returns to equilibrium?
- We linearize the model equations about the equilibrium state and look for solutions of the form

$$X^r(\xi,\eta,t) = R(1+e^{\gamma t}Y_{m,k}(\xi,\eta)),$$

  where $Y_{m,k}$ is a spherical harmonic of degree m and order k.
- The oscillatory behaviour of the membrane is determined by $\gamma\in\mathbb{C}$.

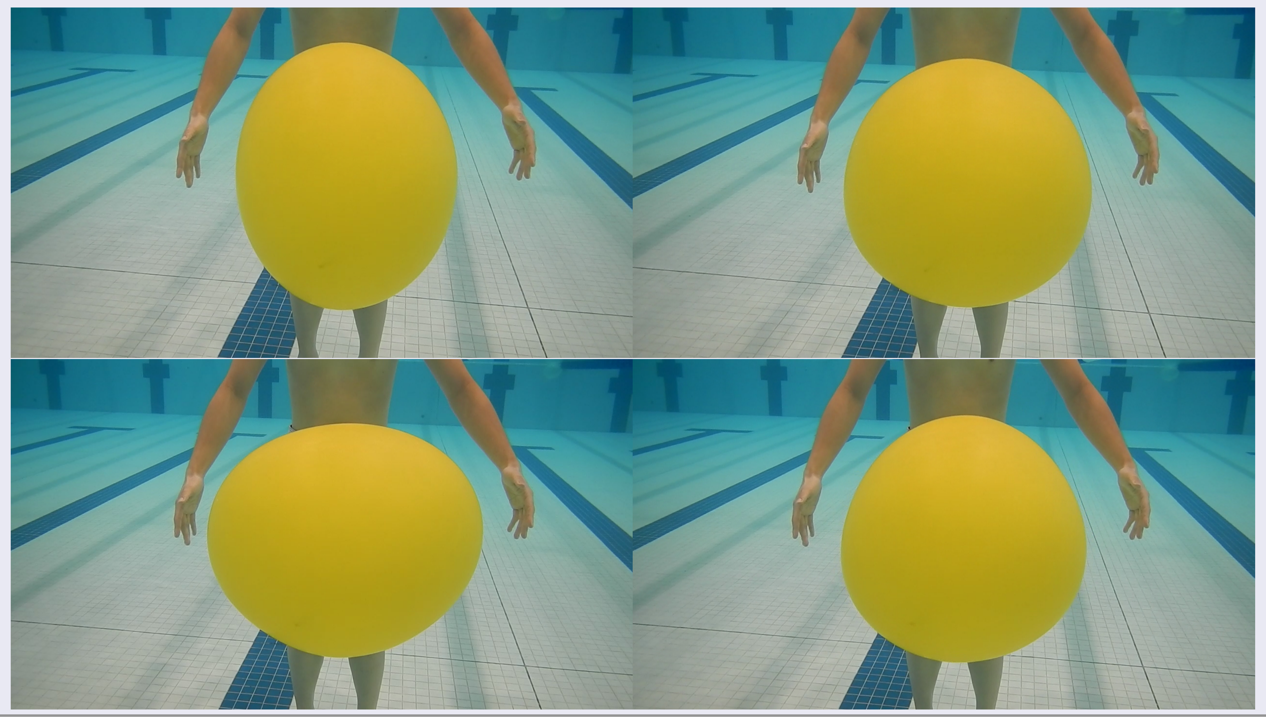




- Dimensionless parameters: $\nu = \frac{\mu}{\rho RU}$, $\kappa = \frac{\sigma}{\rho RU^2}$, $\zeta = \frac{\nu}{\sqrt{\kappa}}$.

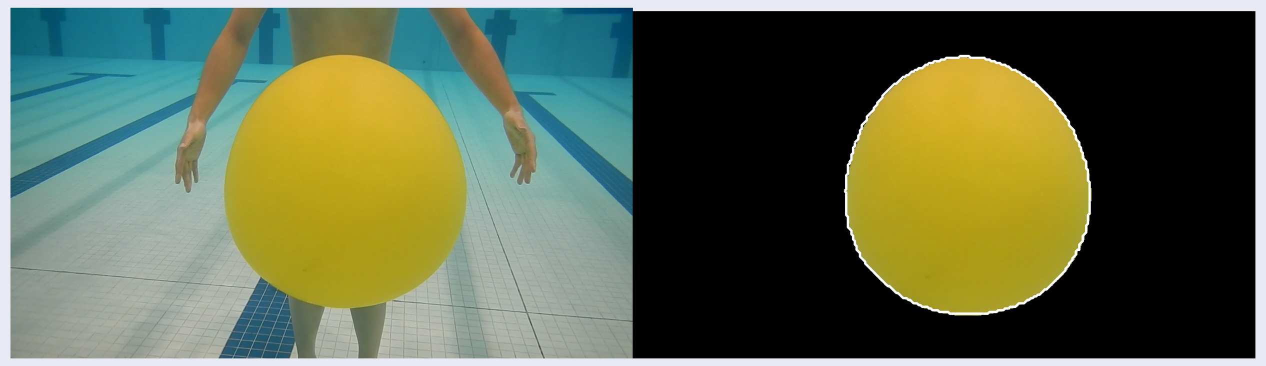
Experiment

- Three types of balloons were purchased for the experiment with inflation radii ranging from 6cm to 45cm.
- All the experiments were performed in the SFU pool.
- The balloons were filled with water and inflated to a desired radius.
- For each experiment, the balloon was lightly squeezed from both sides to excite a 2-mode ($m=2$).
- The oscillations were filmed using a Nikon Coolpix AW120 Waterproof Camera.

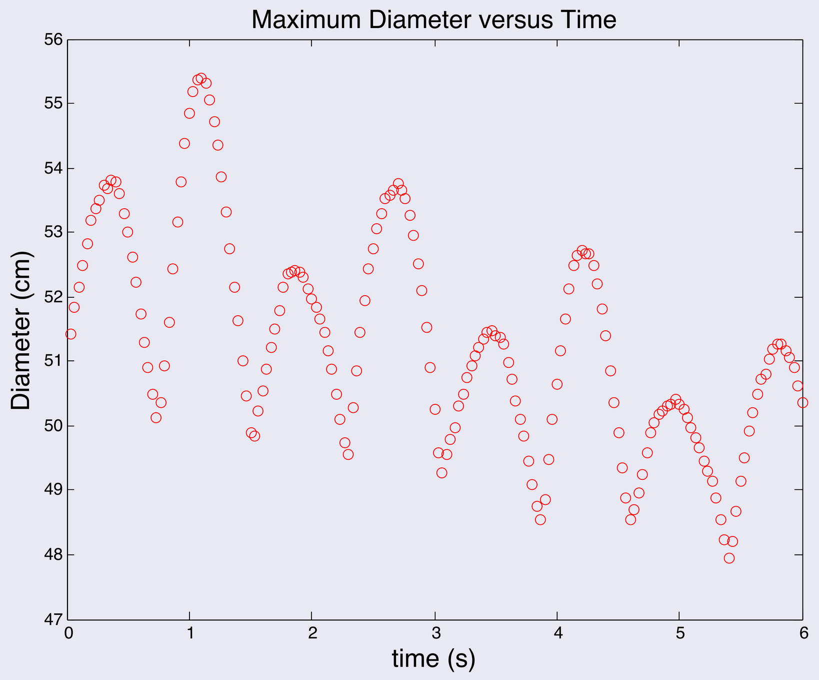


Data Analysis

- Using Matlab, we employed colour segmentation on each frame such that the balloon could be isolated from the rest of the image.



- We used an edge finding tool in Matlab to extract the edge of the balloon, and calculated the maximum diameter for each frame.
- The oscillation period of each experiment is calculated by finding the time between the peaks found with Matlab's `findpeaks()`.



- To calculate the stiffness parameter, σ, a digital manometer was used to measure the pressure difference between the interior of the balloon and the atmospheric pressure.
- We plugged in the pressure differential values into the formula

$$[[p]] = \frac{\sigma}{R},$$

  where $[[p]]$ is the pressure difference.

Preliminary Results

- Comparison between experiment and analysis:

Balloon Type & R (cm)	σ (dyn $\cdot$ cm^{-1})	Experimental Frequency (s^{-1})	Theoretical Frequency (s^{-1})
A,9	197100	18.9	25.4
B,9	227700	15.7	27.3
B,12	260640	11.1	19.0
B,15	307500	9.0	14.8
C,15	297750	8.6	14.5
C,20	299400	5.3	9.5
C,25	313250	4.2	6.9

- From the data above, we observe a trend where the frequency decreases as the balloon size increases, as predicted in the analysis.
- However, we can clearly see that the experimental frequency is higher than the experimental data.

Open Problem

- The oscillation frequency found in the experiments do not match the theory. What could be wrong?
- One possible answer: The force density, $\mathbf{F}(\mathbf{X},t)$, used in our model is too simple (linear) and does not reflect the force induced by the rubber.
- The rubber material actually behaves non-linearly (Mooney-Rivlin type).
- Beatty [2] showed that non-linear elastic materials oscillate differently than linear materials, even under small perturbations.
- We are currently exploring this area to explain the discrepancy in the results.

Summary

- The goal of this project is to conduct experiments of immersed elastic surfaces and compare with the theoretical results.
- A linear stability analysis of the IB model gives a theoretical prediction of the oscillatory behaviour of the membrane.
- We have conducted experiments of elastic membranes oscillating under water using various types and sizes of balloons.
- The experimental videos were analyzed with Matlab's image processing toolbox, and an oscillating frequency was calculated.
- The data shows a discrepancy between experiment and theory, the resolution of this issue is a work in progress.

References

- C. S. Peskin, *The immersed boundary method.* Acta Numerica, pp. 1-39, 2002.
- M. F. Beatty, *Small amplitude radial oscillations of an incompressible, isotropic elastic spherical shell.* Mathematics and Mechanics of Solids, 16(5):492-512, 2011