
The Geometry of Intersecting Tubes Applied to Controlling a Robotic Welding Torch

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Abstract: The question of how to control a robotic welding torch to trace the joint between two cylindrical pipes can be reduced to a problem in algebra. Maple can be utilised to derive the parametric equations describing the curve of intersection between the two cylinders, and then to explore the solution graphically for various physical parameters. This problem can serve as an excellent introduction to the use of Maple for simplifying and solving systems of equations, and offers several straightforward extensions that increase its applicability to more advanced mathematics courses.

Keywords: pipe welding, cylinders, intersections, geometrical modeling

Introduction: A welding problem

Many problems arising in engineering and physics have a mathematical description that is too complicated for consideration in lower-level mathematics courses. It is often possible to make simplifying assumptions that reduce the problem to manageable proportions, but it is often the case that these simplifications are made at the expense of physical significance. Consequently, it is very rewarding for an educator to find an application that can serve as a useful illustrative example, while still retaining its full physical interpretation. The situation presented here is one such application.

This is a problem that arose from an actual consulting contract with the welding industry. The question is this: *How does one control a robotic welding torch so that the welder follows precisely the joint between two cylindrical pipes?* A typical pipe weld is pictured in the photograph in Fig. 1, where the joint between the two pipes is clearly seen to describe a closed curve. In the most general situation, the company must be able to form welds between pipes with differing cross-sectional radius and arbitrary joint angle.

The problem of controlling the welding machine reduces mathematically to finding the equation of the curve in 3-space describing the joint between two cylinders. The solution is an exercise in geometry and requires only a knowledge of basic trigonometry, parametric equations, and rotations in three dimensions. It is therefore suitable for presentation in a calculus (or pre-calculus) course, and is an excellent introduction to the use of Maple in solving systems of equations. Owing to the simple geometric interpretation of the solution, it is also very easy to motivate students to experiment graphically with the solution in Maple. Furthermore, there are several straightforward extensions of the problem that lend themselves naturally to more advanced mathematics, including multi-variable calculus, and numerical integration. The Maple code presented in this paper is available in worksheet form from the following Web site:

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Figure 1: A photograph of a welding joint between two cylindrical pipes intersecting at an angle of $\frac{\pi}{2}$.

<http://www.math.sfu.ca/~jms/weld/>

It is important at this juncture to point out the added value of using symbolic algebra to derive a closed-form expression for the curves of intersection, rather than a CAD (or Computer-Aided Design) software package, which is already designed to deal with exactly this type of problem. After defining the cylinder surfaces, a CAD package typically approximates the curve of intersection by splines. The spline coefficients can then be coded in the software used to control the robotic welding arm. The disadvantage to this CAD-based approach is that the spline coefficients need to be regenerated for every change in pipe geometry (radius or weld angle), and the control algorithms modified and retested. On the other hand, if we can use Maple to obtain a single expression that includes the geometric parameters, then the robot arm can be reprogrammed on-the-fly for any desired pipe weld.

The computation of curves of intersection between cylinders and other surfaces of revolution is one that is vital to other

fields as well. Similar calculations are required in computer graphics [5, 4] where simple geometric shapes such as planes, cones and cylinders are used as the building blocks for constructing approximations of more complicated surfaces. There are numerous applications in engineering, such as stress analysis of shells [10], where determination of surface intersections is an important first step in analysing the problem. While the discussion in this paper is interpreted in terms of welding cylindrical pipes, the results can be applied to a wide range of industrial situations.

The mathematical formulation

The first step in solving any engineering problem is to define the geometry and relevant physical parameters, and equally important, to clearly identify what constitutes a solution to the problem.

Let us assume that the computer software that controls the welding arm does so using a parametric representation of the curve describing the joint. Because of the cylindrical symmetry of the pipes, it is natural to describe the welding joint as a curve written in terms of polar coordinates, (r, θ) , as

$$(x(r, \theta), y(r, \theta), z(r, \theta)). \quad (1)$$

The three component functions will form the inputs to the control algorithm.

Let us now move to a mathematical description of the geometry, which is pictured in Fig. 2. From the discussion in the Introduction, we saw that the three main parameters in the problem are the radii of the two pipes, labeled R_1 and R_2 in the diagram, and the angle of intersection, Φ . For the sake of simplicity, assume that the first cylinder (with radius R_1) has an axis of symmetry coincident with the z -axis. We take the second cylinder with its axis of symmetry lying in the xz -plane, and rotated about the y -axis by an angle Φ with respect to the vertical cylinder. We will also assume that the vertical cylinder has the largest radius (i.e. $R_1 \geq R_2$), for reasons which will be made clear later on.

The parametric equations for cylinder 1 are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_1 \cos \theta_1 \\ R_1 \sin \theta_1 \\ z_1 \end{pmatrix}, \quad (2)$$

where θ_1 and z_1 are the two free parameters describing the surface. The equations for the second cylinder can be obtained from that of the first by applying the rotation matrix

$$\begin{bmatrix} \cos \Phi & 0 & -\sin \Phi \\ 0 & 1 & 0 \\ \sin \Phi & 0 & \cos \Phi \end{bmatrix}$$

to Eqs. (2) to obtain

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_2 \cos \theta_2 \cos \Phi - z_2 \sin \Phi \\ R_2 \sin \theta_2 \\ R_2 \cos \theta_2 \sin \Phi + z_2 \cos \Phi \end{pmatrix}, \quad (3)$$

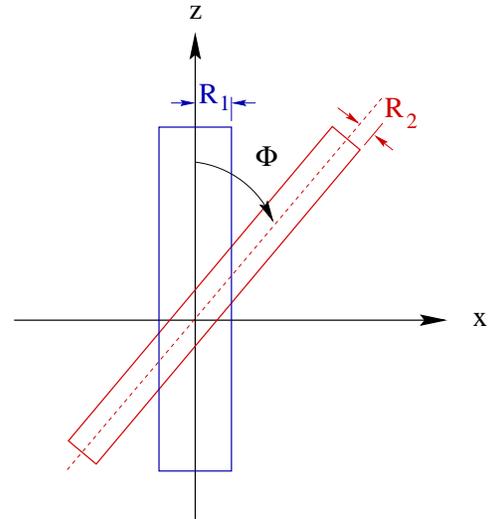


Figure 2: Definition of the physical parameters describing the cylindrical pipe intersection, projected onto the xz -plane.

where θ_2 and z_2 are also free parameters.

The parametric equations for the intersection curve between the cylinders, pictured in Fig. 3, are derived in the following section.

Derivation of the welding curve

Other problems involving the intersection of solids of revolution (such as cylinders, cones or tori) with a plane have been proposed as instructional tools in the ‘‘Classroom Capsules’’ column of the *College Mathematics Journal* [2, 11]. The situation we consider here is a more complicated one, since it involves two solids of revolution. However, the basic solution procedure is quite similar, and so these simpler plane-cutting problems could be introduced first as a preliminary motivation for the pipe welding problem.

We begin by defining the parametric equations for the two cylinders:

```
> cyl1 := { x=R1*cos(theta1),
>          y=R1*sin(theta1), z=z1 };
```

```
cyl1 := {
>          y = R1 sin(theta1), z = z1, x = R1 cos(theta1) }
```

```
> cyl2 := { x=R2*cos(theta2)*cos(Phi)
>          -z2*sin(Phi), y=R2*sin(theta2),
>          z=R2*cos(theta2)*sin(Phi)+z2*cos(Phi) };
```

```
cyl2 := { x = R2 cos(theta2) cos(Phi) - z2 sin(Phi),
>          y = R2 sin(theta2),
>          z = R2 cos(theta2) sin(Phi) + z2 cos(Phi) }
```

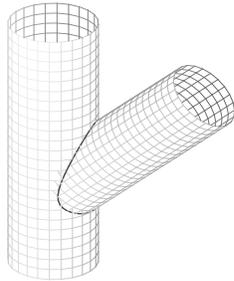


Figure 3: A Maple plot of two cylinders with radii $R_1 = 1$ and $R_2 = \frac{9}{10}$, meeting at an angle of $\Phi = \frac{\pi}{4}$. The intersection curve is plotted as a dark line.

The curves of intersection are found by equating the x , y and z components of each surface:

```
> eqns := subs( cyl2, cyl1 );

eqns := { R2 sin(theta2) = R1 sin(theta1),
          R2 cos(theta2) sin(Phi) + z2 cos(Phi) = z1,
          R2 cos(theta2) cos(Phi) - z2 sin(Phi) =
          R1 cos(theta1) }
```

The ensuing equations can then be solved for the unknowns θ_1 , z_1 and z_2 in terms of θ_2 and the cylinder parameters R_1 , R_2 and Φ :

```
> soln := solve( eqns, {theta1, z1, z2} );

soln := { z2 = (-R2 sin(theta2)
          + R2 cos(theta2) cos(Phi) %1 + R1 %1) / (
          sin(Phi) %1), z1 = (
          R2 cos(theta2) sin(Phi)^2 %1
          - cos(Phi) R2 sin(theta2)
          + R2 cos(theta2) cos(Phi)^2 %1
          + cos(Phi) R1 %1) / (sin(Phi) %1),
          theta1 = 2 arctan(%1) }
%1 := RootOf(
          R2 sin(theta2) + R2 sin(theta2) _Z^2 - 2 R1 _Z )
```

(though we could equally well have solved for *any* of the three unknowns in terms of the fourth).

Notice the presence of the `RootOf()` expression appearing in the label `%1`, whose argument is a quadratic polynomial having two roots:

```
> ztemp := R2*sin(theta2) - 2*R1*_Z
> + R2*sin(theta2)*_Z^2:
> zsoln := solve( ztemp, _Z );

zsoln := 1/2 * ( 2 R1 + 2 sqrt(R1^2 - R2^2 sin(theta2)^2) /
              R2 sin(theta2) ,
            1/2 * ( 2 R1 - 2 sqrt(R1^2 - R2^2 sin(theta2)^2) /
              R2 sin(theta2) )
```

Consequently, there are *two distinct solution curves*, corresponding to the fact that cylinder 2 passes through cylinder 1 in two places (refer to Fig. 2). Looking at the functional form of the roots above, it should now be clear why it was necessary to make the earlier assumption of $R_1 \geq R_2$: without this restriction, the square root terms in `zsoln` are undefined for some values of θ_2 in the interval $[0, 2\pi]$.

Our next step is to derive the parametric equations of the welding joint by taking the expressions in `soln` and substituting them into Eqs. (2). Not surprisingly, the resulting expressions are quite lengthy and benefit from some simplification. The following function definition will prove very useful in this regard:

```
> readlib( rationalize );
> trigsimp := proc( a )
>   expand( a, trig );
>   rationalize( " );
>   simplify( " , trig );
> end;
```

The `rationalize()` procedure is employed because of the presence of square roots of trig functions in the denominator of the expressions in `soln`. We now simplify the two welding curves in a `for` loop:

```
> for i from 1 to 2 do
>   subs( RootOf(ztemp) = zsoln[i], soln );
>   subs( " , cyl1 ):
>   weld[i] := trigsimp( " );
> od;
```

The two sets of parametric equations describing the intersection curves on either side of cylinder 1 are

```
> weld[1];
```

$$\left\{ \begin{aligned} z &= \left(R_2 \cos(\theta_2) \right. \\ &\quad \left. + \cos(\Phi) \sqrt{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2} \right) / \\ &\quad (\sin(\Phi)), y = R_2 \sin(\theta_2), \\ x &= -\sqrt{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2} \end{aligned} \right\}$$

and

```
> weld[2];
```

$$\left\{ \begin{aligned} y &= R_2 \sin(\theta_2), z = \left(R_2 \cos(\theta_2) \right. \\ &\quad \left. - \cos(\Phi) \sqrt{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2} \right) / \\ &\quad (\sin(\Phi)), x = \sqrt{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2} \end{aligned} \right\}$$

In practice, a pipe weld consists of a single joint (as pictured in Figs. 1 and 3), and so only one of the two sets of equations (either `weld[1]` or `weld[2]`) is required. With this in mind, we next generate a plot of a single pipe weld, consisting of surface plots of cylinder 1 and the “physical half” of cylinder 2, and a curve plot of the intersection `weld[1]`, for representative parameter values $R_1 = 1$, $R_2 = \frac{9}{10}$ and $\Phi = \frac{\pi}{4}$:

```
> with(plots): plotsetup( x11 ):
> vars := { R1 = 1, R2 = 9/10, Phi = Pi/4 }:
> plot1 := plot3d(
>   subs( subs( vars, cyl1 ), [x,y,z] ),
>   theta1=0..2*Pi, z1=-2..5 ):
> plot2 := plot3d(
>   subs( subs( vars, cyl2 ), [x,y,z] ),
>   theta2=0..2*Pi, z2=0..5 ):
> plot3 := spacecurve(
>   subs( subs( vars, weld[1] ), [x,y,z] ),
>   theta2=0..2*Pi, thickness=3, color=red ):
> display( { plot1, plot2, plot3 },
>   orientation=[110,60],
>   scaling=CONSTRAINED );
```

This Maple code was used to generate the plot pictured in Fig. 3.

GRAPHICAL EXPERIMENTATION

Students should be encouraged to explore the changing shape and orientation of the welding joint curves as the parameters R_1 , R_2 and Φ are varied. Perhaps the easiest way to visualise the geometry is with Maple’s `animate` facility. For example, the following code produces an animated plot of the cylinders as the radius of the second cylinder ranges between 0 and 1 and the other parameters are held fixed at $R_1 = 1$ and $\Phi = \frac{\pi}{4}$:

```
> with(plots): plotsetup(x11):
> vars4 := {R1=1,Phi=Pi/4}:
> animate3d( { subs( subs( vars4,
>   subs( {theta1=theta2, z1=2*z2-5/2},
>     cyl1 ) ), [x,y,z] ),
>   subs( subs(vars4,cyl2), [x,y,z] ),
>   subs( subs(vars4,weld[1]), [x,y,z] ),
>   subs( subs(vars4,weld[2]), [x,y,z] ) },
>   theta2=0..2*Pi, z2=0..5/2, R2=1/4..1,
>   scaling=CONSTRAINED, frames=12,
>   view=[-2..2,-2..2,-3..3] );
```

Both welding curves are included in these plots so that the shape of the intersections can be more easily visualised. With only minor modifications to the above code, we can fix the angle Φ and instead visualise the effect of changes in the radius R_2 on the weld.

Of particular interest is the case when $R_2 = R_1$, where the welding joints, which are normally two distinct curves, actually merge together into a single branched curve as pictured in Fig. 4(a). The branch points are also points where the two cylinders are tangent to each other.

On the other hand, when R_2 is taken significantly smaller than R_1 , then the portion of the surface of cylinder 1 on which the welding curve lies is locally quite flat in the region covered by the curve. As a result, the joints are approximately circular in shape, with the deformation from a circle becoming less significant as R_2 is reduced. Fig. 4(c) depicts the welding curves when $R_1 = 1$ and $R_2 = \frac{1}{3}$, and shows that even for this only moderately small value of R_2 , the curves are nearly circular. Consequently, when R_2 is reduced, the pipe weld approaches the solution of the plane–cylinder intersection problem derived in [2].

Further investigation

1. Compare the above expressions for the welding curves to those given in [2] for a cylinder intersecting a plane. Are they the same in the limit as $\frac{R_2}{R_1} \rightarrow 0$?

2. Can you perform the same analysis when one or both of the pipes has the shape of an elliptic cylinder? Or a tapered cylinder?

Extensions

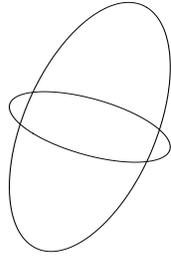
There are two additional aspects of the welding problem that can be studied to increase its applicability to more advanced mathematics courses:

- (i) finding an “optimal” angle at which the torch should be held to minimise interference with the pipe surfaces, drawing some basic formulas from multi-variable calculus,
- (ii) determining the length of the welding joint, which requires the arclength integral formula and numerical integration techniques.

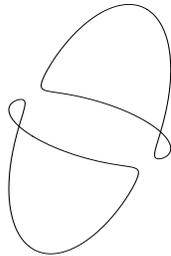
These problems are considered in the next two sections.

TORCH ANGLE

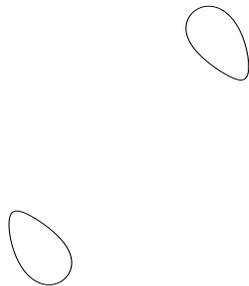
The tip of a usual welding torch has a certain taper angle, and so it is necessary to approach the welding joint from a suitable direction in order to avoid a situation such as that depicted in Fig. 5 where the tip of the torch impinges on the surface of one of the cylinders rather than tracing the joint itself. Furthermore, the quality of the weld generated is generally higher when the torch follows the joint at an angle which is not too sharply inclined to either surface being welded. It is apparent that the width of the torch places a physical limitation on the size of the intersection angle (precluding Φ from being too close to 0 or π), though we have ignored this difficulty in our previous discussion.



(a) Self-intersecting joint: $R_1 = R_2 = 1$.



(b) Intermediate case: $R_1 = 1, R_2 = \frac{99}{100}$.



(c) Nearly circular joint: $R_1 = 1, R_2 = \frac{1}{3}$.

Figure 4: Welding curves for an intersection angle of $\Phi = \frac{\pi}{3}$.

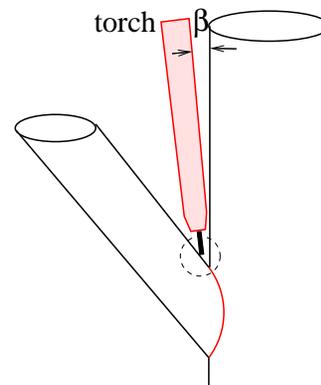


Figure 5: Definition of the torch angle, β , showing the torch can impinge on the pipe surface if β is taken too small.

The torch angle, therefore, is another important input parameter to the algorithm controlling the welding arm. A reasonable strategy for choosing the angle at which the torch approaches the cylinders is the following: *the axis of the torch*

should bisect the angle between the two cylinders at each point along the weld joint. We next formulate this criterion mathematically.

The angle between the two cylinders at any point along their intersection is simply the angle between the normal vectors to each surface at that point. We know from multi-variable calculus that the normal to a surface defined in polar coordinates as in Eq. (1) is given by

$$\vec{n} = \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right).$$

To compute the angle between the normals, we will use the cosine formula for the dot product, which states that the angle between two unit vectors, \vec{x} and \vec{y} is given by $\cos \theta = \vec{x} \cdot \vec{y}$. If \vec{n}_1 and \vec{n}_2 are the unit outward-pointing normals to cylinders 1 and 2 respectively, we can write the optimum torch angle as

$$\beta = \frac{1}{2} [\pi - \cos^{-1}(\vec{n}_1 \cdot \vec{n}_2)]$$

(where the extra π term accounts for the fact that the normals are defined as *outward-pointing*).

The outward-pointing normal vector to cylinder 1 is

```
> diff( subs( cyl1, [x,y,z] ), R1 );
      [cos(theta1), sin(theta1), 0]
```

which in terms of the variable θ_2 , is given by

```
> n1 := trigsimp(
>   subs( RootOf(ztemp)=zsoln[1],
>   subs( soln, " ) ) );
```

$$n1 := \left[-\frac{\sqrt{R1^2 - R2^2 + R2^2 \cos(\theta_2)^2}}{R1}, \frac{R2 \sin(\theta_2)}{R1}, 0 \right]$$

The second normal vector is

```
> n2 := diff( subs( cyl2, [x,y,z] ), R2 );
```

$$n2 := [\cos(\theta_2) \cos(\Phi), \sin(\theta_2), \cos(\theta_2) \sin(\Phi)]$$

from which we can use the `linalg[dotprod]` procedure to compute the angle between two cylinders as

```
> beta := ( Pi -
>   arccos( linalg[dotprod](n1, n2) ) ) / 2;
```

$$\beta := \frac{1}{2} \pi - \frac{1}{2} \arccos \left(-\frac{\sqrt{R1^2 - R2^2 + R2^2 \cos(\theta_2)^2} \cos(\theta_2)}{\cos(\Phi) / R1 + \frac{R2 \sin(\theta_2)^2}{R1}} \right)$$

We can then plot a series of torch angle curves using the Maple `seq()` function:

```
> plotsetup(x11): i := 'i':
> plot( { seq( subs( { R1=1, R2=1/3,
>   Phi=i*Pi/12 }, beta ), i=1..6 ) },
>   theta2=0..2*Pi, 'beta'=0..Pi/2 );
```

A series of torch angle curves is shown in Fig. 6 for the two situations $R_2 = \frac{1}{3}$ and $R_2 = 1$.

Further investigation

3. *Realistically, the robot arm will also have physical constraints on its motion. What are the possible pipe angles (that is, what limits are there on Φ) if, for example, there is a constraint like $TOL \leq \beta \leq \Phi - TOL$? Or $\beta \leq \beta_{max}$? How might such constraints be interpreted physically?*

LENGTH OF WELD

Another characteristic of this problem that is of geometric interest is the *arclength* of the curve of intersection between the cylinders. While this quantity is perhaps less significant to the welding process than the torch angle, the arclength might still be reasonably motivated as a measure of the cost of welding material used (e.g., electrical power, gas or filler metal), or alternatively to gauge the wear and tear on the welding electrode.

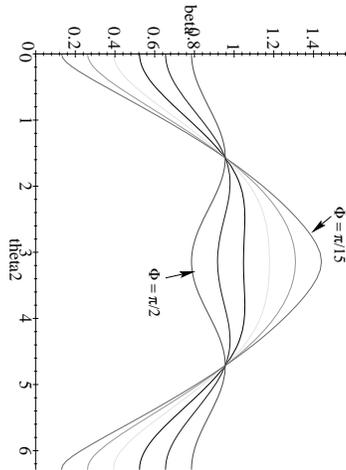
We know from multi-variable calculus that the arclength of a curve, parametrised in terms of θ , is given as

$$L = \int_0^{2\pi} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} d\theta.$$

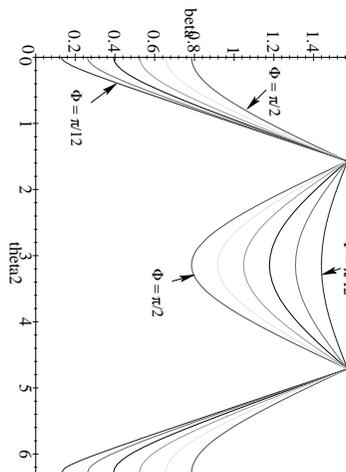
First, differentiate the three components of the weld curve with respect to θ_2

```
> diff( subs( weld[1], [x,y,z] ), theta2 );
```

$$\left[\frac{R2^2 \cos(\theta_2) \sin(\theta_2)}{\sqrt{R1^2 - R2^2 + R2^2 \cos(\theta_2)^2}}, R2 \cos(\theta_2), \left(-R2 \sin(\theta_2) - \frac{\cos(\Phi) R2^2 \cos(\theta_2) \sin(\theta_2)}{\sqrt{R1^2 - R2^2 + R2^2 \cos(\theta_2)^2}} \right) / (\sin(\Phi)) \right]$$



(a) $R_1 = 1$ and $R_2 = \frac{1}{3}$.



(b) $R_1 = 1$ and $R_2 = 1$.

Figure 6: Torch angle plotted as a function of θ_2 . Each curve corresponds to a different value of the angle Φ ranging between $\frac{\pi}{12}$ and $\frac{\pi}{2}$.

and then set up the integrand for the arclength integral:

```
> arcInt := sqrt( "[1]^2 + "[2]^2 + "[3]^2 );
```

$$arcInt := \left(\frac{R_2^4 \cos(\theta_2)^2 \sin(\theta_2)^2}{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2} + R_2^2 \cos(\theta_2)^2 + \left(-R_2 \sin(\theta_2) - \frac{\cos(\Phi) R_2^2 \cos(\theta_2) \sin(\theta_2)}{\sqrt{R_1^2 - R_2^2 + R_2^2 \cos(\theta_2)^2}} \right)^2 / \sin(\Phi)^2 \right)^{1/2}$$

This is an integral that cannot be evaluated explicitly in terms of elementary functions. In the special case when $R_2 = R_1$, the integral can be written in terms of a special function known as an *elliptic integral*. These integrals are very well-known functions (see [1]) that arise in physical situations involving such things as the oscillations of a pendulum and the orbital motion of a planet. When $R_2 \neq R_1$, on the other hand, the integral is not elliptic.

However, regardless of whether or not we have an elliptic integral, we are forced to approximate the integral numerically, which may be done using the Maple construct `evalf(Int(...))`. To this end, we define an ‘‘arclength function’’ that depends on R_2 and Φ (with R_1 fixed at 1), and whose third argument is the number of digits of accuracy required in the floating point answer:

```
> arcFunc := (r,p,acc) ->
> Int( subs( {R1=1,R2=r,Phi=p}, arcInt ),
> theta2=0..2*Pi, acc );

arcFunc := (r,p,acc) -> Int(
  subs( { R1 = 1, Phi = p, R2 = r }, arcInt ),
  theta2 = 0..2 * pi, acc)
```

To verify that the arclength formula returns a reasonable result, we perform a calculation with $R_2 = \frac{1}{10}$ being significantly smaller than R_1 :

```
> evalf( arcFunc( 1/10, Pi/2, 6 ) );
.628713
```

In this situation, the welding curve is approximately circular, and so we expect the arclength to be very close to the circumference of a circle of radius $\frac{1}{10}$. This is very easily tested as follows:

```
> evalf( 2*Pi/10, 6 );
.628318
```

Next, we produce a surface plot of the arclength as a function of R_2 and Φ , which is pictured in Fig. 7.

```
> plotsetup(x11);
> plot3d( arcFunc( r2,phi,3 ), r2=1/10..1,
>         phi=Pi/20..Pi/2, grid=[10,10],
>         style=patchcontour, axes=frame );
```

The number of digits of accuracy was chosen to be 3, so that the cost of the computation is kept to a minimum while at the same time being sufficiently accurate for plotting purposes: The basic shape of this surface plot can be justified us-

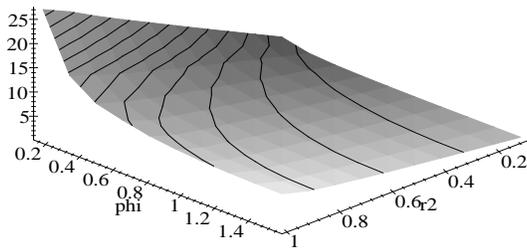


Figure 7: Surface plot of the arclength of the welding joint as a function of R_2 and Φ .

ing simple geometric reasoning: the arclength is largest when $R_2 = 1$ and Φ is close to zero (when the joints form very long, distended curves), and falls off toward zero when the radius of the cylinder 2 becomes very small.

Further investigation

- 4. *Elliptic integrals and their numerical approximation:*
 - (a) Show that when $R_1 = R_2$, the arclength derived above can be written in terms of elliptic integrals.
 - (b) Take the equations of the welding curves when the two radii are equal, and show that they reduce to the equations of two intersecting ellipses, rotated at an angle of Φ with respect to each other. In light of the result in part (a), and the graphical evidence in Fig. 4(a), this should not be surprising, since the term “elliptic integral” derives from the fact that this integral arises in the calculation of the perimeter of an ellipse [3].
 - (c) Maple V includes several functions defining elliptic integrals, via the procedures *EllipticE* and *EllipticK*.

Furthermore, Release 4 includes special *evalf* routines corresponding to these functions that implement efficient numerical techniques for approximating elliptic integrals (see [1] for a discussion). If you are using an earlier release of Maple, the numerical code can be obtained from the Maple share library [9], and has also been made available on the web page <http://www.math.sfu.ca/~jms/weld/>. Use these routines to compute values of the integral from part (a) numerically, and compare the efficiency of this routine to the result obtained with the standard *evalf/Int* function.

Concluding remarks

We have presented a simple geometric problem arising in robotic welding, and have shown that Maple can be used both to solve the problem, and to explore the solution graphically. It is our hope that we have demonstrated the value of this problem as an instructional tool for motivating the applications of solving systems of equations in polar coordinates, and also some introductory concepts from multi-variable calculus. The problem is particularly easy to motivate, due to its simple geometric interpretation and obvious application to modern industry. Class discussions can also be raised regarding more advanced aspects of robotic welding such as intelligent joint tracking and adaptive control of industrial robots (see [7]).

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Biography

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