~~more language: $\{3,1\}^4$ means $\underbrace{3,1,3,1,3,1,3,1}_{4 \text{ times}}$~~

~~lots of identities between zeta values~~
~~the simplest one is $\zeta(2,1) = \zeta(3)$~~
~~(* expand if Mike didn't already talk a lot about this)~~

II MZV Miscellany

(1) Hierarchy of Numbers

We begin with the natural #s $\mathbb{N} = \{0, 1, 2, \ldots\}$ (whatever)

we all know the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(inductive foundation ... $0, s(0), s(s(0))$
then add in negative ones ...)

From the integers we get rational numbers $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$

what does this actually mean?

begin with set of ordered pairs

$$\{(a,b) : a, b \in \mathbb{Z}, b \neq 0\}$$

say two ordered pairs (a_1,b_1), (a_2,b_2) are equivalent if

$$a_1 b_2 = a_2 b_1$$

An equivalence class is a set of all the elements equivalent to some given element

$\mathbb{Q}$ is the set of equivalence classes

then define + and $\cdot$ the way you'd expect

Def A set S is countable if there is a 1-1 and onto map between S and $\mathbb{N}$

- $\mathbb{Z}$ is countable (get them to give the map)
- $\mathbb{Q}$ is countable (get them to give the map)

Def a (cx) number is algebraic if it is a root of a nonzero polynomial in one variable with rational coefficients.

(zero)

fundamentally where do real #s come from: complete $\mathbb{Q}$ (ie make all cauchy sqs converge)

eg $\sqrt{2}$ — give the polynomial

not every algebraic number has an expression in terms of $\sqrt[n]{\ }$s

eg $x^5 - x + 1$

but only need the poly for manipulation

eg golden ratio γ is root of (w is positive) $x^2 - x - 1$

(so it does have a closed form)

$\gamma = \frac{1+\sqrt{5}}{2}$

say I want to know what $\frac{1}{\gamma}$ is

$\gamma^2 - \gamma - 1 = 0$

$\Rightarrow \gamma - 1 - \frac{1}{\gamma} = 0$

$\Rightarrow \frac{1}{\gamma} = \gamma - 1$

only need the polys so do similar things for algebraic #s with no expressions in terms of radicals

is the set of algebraic #s countable? (yes)

— let them try.

Def a (cx) number is transcendental if it is not algebraic

eg e, π

It's really hard to prove things are transcendental (often even hard to prove irrational)

for the Riemann zeta all we know is

Def A *period* is a (real) number which is the value of an absolutely convergent integral of a rational function with rational coefficients over regions in $\mathbb{R}^n$ with bounds given by polynomials with rational coefficients.

(Kontsevich ~~Zagier~~)

(cx periods are cx numbers whose real and imaginary parts are real periods)

eg $\sqrt{2} = \int_{2x^2 \leq 1} dx$ is algebraic and a period

eg $\pi = \iint_{x^2+y^2 \leq 1} dx\,dy$ is tr. and a period.

eg $\log(2) = \int_1^2 \frac{dx}{x}$ is a period

eg $\iiint_{0<x<y<z<} \frac{dx\,dy\,dz}{(1-x)yz}$ is a period

what is this $(*)$ known by Mike ∘ 5:8)
or check by computer.

look iterated integrals!

This returns us to the question

$$\boxed{\text{How many relations are there between Multiple Zeta Values}}$$

We've gotten about as far as we can by hand

so how can we computerize?

its all linear algebra so the answer is matrices

use coordinate vectors

larger than $\mathcal{W}_n$ would be the space where we just think of the MZVs as symbols and take their span — no relations at all

wt	basis of big space $\mathcal{A}_n$	(presumed) basis of $\mathcal{W}_n$
2	$\zeta(2)$	$\zeta(2)$
3	$\zeta(3), \zeta(2,1)$	$\zeta(3)$
4	$\zeta(4), \zeta(2,2)$, $\zeta(3,1)$, $\zeta(2,1,1)$	$\zeta(2,2)$
	$\vdots$	$\vdots$

write the identities in coordinates in terms of the basis of the big space

so $\zeta(3) - \zeta(2,1) = 0 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \zeta(4) - \zeta(3,1) - \zeta(2,2) = 0 \Rightarrow \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

Then $\mathcal{A}_n \xrightarrow[f]{\text{convert to actual MZVs}} \mathcal{W}_n$ $\ker f$ = space of relations is a subspace of $\mathcal{A}_n$

but by the rank nullity thm

$$\dim \mathcal{W}_n + \dim \ker f = \dim A_n$$

. we know the stuff conj for $\dim \mathcal{W}_n$

we can compute $\dim A_n$ (exercise)

but they have found enough ids
— how did they find but they compute
(see things Mike did)

② Counting functions w/ pushdown

Let $\dim W_n = p_n$

we know the standard conj $\quad p_n = p_{n-2} + p_{n-3} \quad n \geq 3$

$$p_0 = 1,\ p_1 = 0,\ p_2 = 1\ \ p_3 = 1$$

so (assuming this) what is $W(x) = 1 + \sum_{n\geq 2} p_n x^n$?

$$\text{well } W(x) = 1 + \sum_{n\geq 2} p_n x^n = \cancel{1 + \sum_{n\geq 3} p_n x^n}$$

$$= 1 + \sum_{n\geq 2} p_{n-2} x^n + \sum_{n\geq 2} p_{n-3} x^n$$

$$= 1 + x^2 \sum_{n\geq 2} p_{n-2} x^{n-2} + x^3 \sum_{n\geq 2} p_{n-3} x^{n-3}$$

$$= 1 + x^2 \sum_{n\geq 0} p_n x^n + x^3 \sum_{n\geq 0} p_n x^n$$

$$= 1 + x^2 W(x) + x^3 W(x) \quad \text{since } p_1 = p_0 = 0.$$

$$\text{so} \quad W(x) = \frac{1}{1 - x^2 - x^3}$$

Now having a basis of W doesn't really capture everything,
eg relation between $\zeta(2)$ & $\zeta(4)$ is an alg relation
not a linear relation

How do we capture that — Euler product style.

~~Recall~~ Euler product for Riemann zeta function

- recall geometric sum
- consider $\prod_{p\ \text{prime}} \frac{1}{1-p^{-s}} = \prod_{p\ \text{prime}} (1+p^{-s}+p^{-2s}+\dots)$

what is an elmt of the product
$$2^{-k_1 s}\,3^{-k_2 s}\,5^{-k_3 s}\,7^{-k_4 s} = (2^{k_1}3^{k_2}\dots)^{-s}$$

each such elmt appears exactly once by unique prime factorization

so
$$\prod_{p\ \text{prime}} \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$$

If we want to similarly recover these kinds of product relations for MSVs
let d_n be the # of alg generators - ie generators as a poly alg.

$$\prod_{n\geq 2}\left(\frac{1}{1-x^n}\right)^{d_n} = \sum p_n x^n = \frac{1}{1-x^2-x^3}$$

or $$\prod_{n\geq 2}(1-x^n)^{d_n} = 1-x^2-x^3$$

Another way to write this $$\prod_{n>2}(1-x^n)^{d_n} = \frac{1-x^2-x^3}{1-x^2} = 1-\frac{x^3}{1-x^2}$$

Recall depth of $\zeta(a_1, a_2 \ldots a_k)$ is k

Can we count by depth as well as weight?

Let $d_{n,k}$ = # of alg generators of weight n and depth k

$$\prod_{n\geq 2} \prod_{k>0} (1-x^n y^k)^{d_{n,k}} = ?$$

we need to put in a y for each sum that appears.

One might guess to modify $1-\frac{x^3}{1-x^2}$ to

$$1-\frac{x^3 y}{1-x^2} \qquad \left(\text{so } \prod\left(\frac{1}{1-x^n}\right)^{d_{n,k}} = \frac{1}{1-x^2-x^3y}\right)$$

But this is wrong because by calculation $d_{12,4}=1$ $d_{12,2}=1$

In 1996 Broadhurst + Kreimer conjectured

$$\prod_{n\geq 2} \prod_{k>0} (1-x^n y^k)^{d_{n,k}} = 1-\frac{x^3y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

Why this correction term? (and why not more corrections?)

To see this need to move to alternating sums (Euler sums)

Let $E\begin{pmatrix} s_1, \ldots, s_k \\ \varepsilon_1, \ldots, \varepsilon_k \end{pmatrix} = \displaystyle\sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{\varepsilon_1^{n_1}\ldots\varepsilon_k^{n_k}}{n_1^{s_1}\ldots n_k^{s_k}}$

$\varepsilon_i \in \{\pm 1\}$

(p27)

Then $\zeta(6,4,1,1)$ can be express as a lin comb of depth 2 Euler sums (see datamine p see 10)

but not in terms of depth 2 MZVs

so this guy is new in depth 4 for MZVs

but "shouldn't be"

let $e_{n,k}$ = # of alg gens of Euler sums of wt n w depth k

the conj (aka Broadhurst-Kreimer) is

$$\prod_{n\geq 2}\prod_{k>0}\left(1-x^n y^k\right)^{e_{n,k}} = 1-\frac{x^3y}{(1-xy)(1-x^2)}$$

$$= \frac{1-xy-x^2}{\underset{\uparrow\ \log(2)}{(1-xy)}\ \underset{\uparrow\ \zeta(2)}{(1-x^2)}}$$

3 Lyndon words

let Ω be a finite set call it the alphabet.

let Ω^* be the set of finite words on letters from Ω

let ε denote the empty word.

suppose we have a (total) order on Ω

eg $\Omega = \{a,b,c\}$ $a<b<c$

then

def the lexicographic order on Ω^* is given by

$w_1 < w_2$ if $w_2 = w_1 v$ $v \in \Omega^*$

or $w_1 = v_1 \alpha v_2$, $w_2 = v_1 \beta v_3$ with $v_1, v_2, v_3 \in \Omega^*$, $\alpha, \beta \in \Omega$, $\alpha < \beta$.

eg

def a word $w \in \Omega^*$ is a <u>Lyndon word</u> if

when $w = u, v$, with u, v nonempty

then $u < v$

eg Lyndon words on $\{0,1\}$ up to length 5. (don't forget 00101)

def Given a word $w \in \Omega^*$, $w = \alpha_1 \alpha_2 \cdots \alpha_n$

then the first rotation of w, $R(w) = \alpha_2 \cdots \alpha_n \alpha_1$

def A <u>necklace</u> is an equivalence class of words under the equivalence

$w_1 \sim w_2$ iff $w_1 = R^i(w_2)$ for some i

def ~~a word is aperiodic~~

write w^k to mean the k-fold concatenation of w

$$w^k = \underbrace{www \cdots w}_{k \text{ times}}$$

a word is **aperiodic** if it is not the power of a shorter word

a necklace is **aperiodic** if none of its representatives are powers of a shorter word

eg

prop each aperiodic necklace has a unique Lyndon word representative

pf suppose $w_1 \neq w_2$ represent the same ~~[illegible]~~ necklace and are both Lyndon words

then $w_1 = uv$ $w_2 = vu$ v, u both non empty

then by the Lyndon property $u < v$ and $v < u$ cont

Given a necklace one of its distinct rotations is lexicographically less than all others, call it w. Claim ~~[illegible]~~ w is a Lyndon word

suppose not then $w = uv$ with $u \geq v$

now w not periodic so $u \neq v$

$\therefore$ $u > v$

but vu is a rotation of w and $vu < uv$ cont.

return to egs of Lyndon words

How to count Lyndon words on Ω. Say $|\Omega| = k$

let $\ell_k(n)$ = # of Lyndon words on Ω of length n

How can we build all words out of Lyndon words?

all the rotations of a Lyndon word are distinct by aperiodicity

so far $n\,\ell_k(n)$ words

Now consider words coming from periodic necklaces -

$w = v^d$

all rotations are also periodic with the same period eg

~ are made out of copies of the rotations of v there are $\frac{n}{d}$ of these

so wlog v is a Lyndon word

~ get $\frac{n}{d}\ell_k\left(\frac{n}{d}\right)$

so all words are counted by $\sum_{d|n} \frac{n}{d}\ell_k\left(\frac{n}{d}\right)$

but naively counting there are k^n words in Ω^* of length n so $k^n = \sum_{d|n} \frac{n}{d}\ell_k\left(\frac{n}{d}\right)$

This implicitly determines $\ell_k(n)$ and in fact can invert by möbius inversion

so $$\boxed{\ell_k(n) = \frac{1}{n}\sum_{d|n}\mu(d)\,k^{\frac{n}{d}}}$$ where $$\mu(n) = \begin{cases} 1 & n=1 \\ (-1)^i & n = p_1\cdots p_i \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Prop take $w \in \Omega^*$

w can be uniquely written as

$$w = l_1 l_2 \cdots l_n$$

with the l_i Lyndon words w $l_1 \geq l_2 \geq \cdots \geq l_n$

eg abaacaab = (ab)(aac)(aab)

Cor take $w \in \Omega^*$ w decomps $w = l_1^{k_1} l_2^{k_2} \cdots l_i^{k_i}$ as above with l_j distinct Lyndon words

then
$$\frac{\underbrace{l_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup l_1}_{k_1 \text{ times}} \sqcup\!\sqcup \underbrace{l_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup l_2}_{k_2 \text{ times}} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \underbrace{l_i \sqcup\!\sqcup \cdots \sqcup\!\sqcup l_i}_{k_i \text{ times}}}{k_1!\, k_2! \cdots k_i!}$$

$= w$ + words less than w

and so the Lyndon words with shuffle generate Ω^* as a polynomial alg

pf of cor

first part just do it by decomp everything else as Lyndons

second part certainly the Lyndon words with shuffle generate

and is free by uniqueness.

instead of proving the prop

lets look at an alg to find it

https://www.ics.uci.edu/~eppstein/

PADS/Lyndon.py