(I) Graphs and Feynman graphs

(1) Feynman graphs combinatorially

- Feynman graphs tell stories about particles
- QED examples
- Scalar field theory examples
- What are the key features – get them to think about it
  - vertices from some fixed set of possibilities
  - internal and external edges
  - directed and undirected edges
  - edges from some fixed set of possibilities

→ point these will be associated to integrals, but first some more discrete math

(2) Graphs axiomatically

- Usually in graph theory we say a graph is

$G = (V, E)$ V a set of vertices

$E \subseteq V \times V$ a set of pairs of vertices representing edges

for a directed graph E is instead a set of ordered pairs of vertices.

for a multigraph (multiple edges allowed) E is a multiset rather than a set.

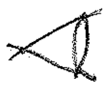
- egs
- what is missing for us?
  - external edges
  - a mix of directed and undirected edges
  - edge types

- so we axiomatize by half edges
  eg

→ but usually too much trouble so use usual graph theory vocabulary.

③ Spanning trees and related objects

- def of cycle, say as a set of edges which can be formed into a sequence.
- def of subgraph ⇝ pay attention to 
  external edges as hooks

- def of tree
- def of spanning tree (what about external edges ⇝ doesn't matter so ignore)
- eg  , 

- def of E verts × edges with an arbitrary orientation

  eg

- def of L and try its determinant on some egs
  $" EE^t$ (verts × verts) before and after removing ...
  figure out what it does (some people may know)
  call it $\tilde{L}$

- enumerative philosophy of variables to mark things
- def of Ψ_G
- $L' = E \Lambda E^t \qquad M = \begin{bmatrix} \Lambda & E^t \\ -E & 0 \end{bmatrix}$

$\tilde{L}'$, $\tilde{M}$ (remove matching row + col (from E part))

try det of each → what do the terms count?

④ The matrix-tree theorem

Def let V be a vector space
let F be a function taking n vectors as input
(output could be another vector or a scalar ~ could be in any vector space itself)

Then F is <u>multilinear</u> if

$$F(\ldots, \underset{\downarrow}{\overset{i}{v_1+v_2}}, \ldots) = F(\ldots, \overset{i}{v_1}, \ldots) + F(\ldots, \overset{i}{v_2} \ldots)$$

and $F(\ldots, \underset{i}{\lambda v}, \ldots) = \lambda F(\ldots, \underset{i}{v}, \ldots)$

for all indices i, all scalars λ, and all vectors v_1, v_2, v.

Def let V be a vector space and F as above
Then F is <u>alternating</u> if

$$F(\ldots, \underset{i}{v_i}, \ldots, \underset{j}{v_j}, \ldots) = -F(\ldots, \underset{i}{v_j}, \ldots, \underset{j}{v_i}, \ldots)$$

for all indices $i < j$ and all vectors v_i, v_j

How did you define the determinant when you learned it
- cofactor expansion?
- product of pivots?

Prop The determinant, viewed as a function taking the columns (or rows) of an $n\times n$ matrix to the scalar value of the determinant, is

① multilinear

② alternating

what the proof will look like will depend on what you want way of looking at the determinant you like best.

lets do it by cofactor expansion because although that isn't beautiful it is well-known:

① by induction

base case $n=1$ $\quad \det[a+b] = a+b$ ✓

assume the result holds for $(n-1)\times(n-1)$ matrices let

$$A = [c_1, \dots, c_i + c_i', \dots], \quad c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \dots$$

then $\det A = a_{11} \det[\tilde{c}_2, \dots, \tilde{c}_i + \tilde{c}_i', \dots]$ where $\tilde{c}_j = \begin{bmatrix} a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$

$\quad - a_{12} \det[\tilde{c}_1, \tilde{c}_3, \dots \tilde{c}_i + \tilde{c}_i', \dots]$

$\quad + \dots (-1)^{i+1}(a_{1i} + a_{1i}') \det[\tilde{c}_1 \dots \tilde{c}_{i-1}, \tilde{c}_{i+1}, \dots]$

$= a_{11} \det[\tilde{c}_2, \dots \tilde{c}_i, \dots] + a_{11} \det[\tilde{c}_2, \dots \tilde{c}_i' \dots]$ by ind hyp

$\quad - a_{12} \det[\tilde{c}_1, \tilde{c}_3, \dots, \tilde{c}_i \dots] - a_{12} \det[\tilde{c}_1, \tilde{c}_3 \dots \tilde{c}_i', \dots]$

$\quad + \dots$

$\quad + (-1)^{i+1} a_{1i} \det[\tilde{c}_1 \dots \tilde{c}_{i-1}, \tilde{c}_{i+1} \dots] + (-1)^{i+1} a_{1i}' \det[\tilde{c}_1 \dots \tilde{c}_{i-1}, \tilde{c}_{i+1} \dots]$

$= \det[c_1 \dots c_i \dots] + \det[c_1 \dots c_i' \dots]$

The result follows by induction.

② let $A = [c_1 \dots c_n]$ $\quad c_i, \tilde{c}_i$ as above

let $e_1 \dots e_n$ be the standard basis vectors

by ① $\quad \det A = a_{11} \det[e_1 c_2 \dots c_n] + a_{21} \det[e_2 c_2 \dots c_n] + \dots + a_{n1} \det[e_n c_2 \dots c_n]$

(*) $\quad = \sum a_{j_1 1} a_{j_2 2} \dots \det[e_{j_1}, e_{j_2} \dots e_{j_n}]$

If we swap two colums of A then we swap the corresponding columns of $[e_{j_1} \dots e_{j_n}]$

but $[e_{j_1} \dots e_{j_n}]$ has exactly one 1 in each row and column and 0 elsewhere

pick two colums $i<j$

cofactor expand using rows which don't touch the 1s in colums i and j

get $\det[e_{j_1} \dots e_{j_n}] = (-1)^{\sim\sim} \det\begin{bmatrix}1&0\\0&1\end{bmatrix}$

or $\det[e_{j_1} \dots e_{j_n}] = (-1)^{\sim} \det\begin{bmatrix}0&1\\1&0\end{bmatrix}$

doing the same on $[e_{j_1} \dots \underset{i}{e_{j_j}} \dots \underset{j}{e_{j_i}} \dots e_{j_n}]$

gives the same power of (-1) because all the steps not involving i and j are the same, but get the opposite matrix here

so $\det[e_{j_1} \dots e_{j_n}] = -\det[e_{j_1} \dots e_{j_j} \cdot e_{j_i} \dots e_{j_n}]$

so returning to $(*)$ $\det A = -\det[c_1 \dots c_j \; c_i \dots c_n]$

In fact the determinant is the only function F taking n vectors, returning a scalar which is

1. multilinear
2. alternating
3. scaled so that $F(e_1 \dots e_n) = 1$

proof Suppose D_1 & D_2 both satisfy the properties

Let $D = D_1 - D_2$.

D is also ① multilinear
② alternating

Note that the argument in ② of the previous proof shows that if we know D on any combination of the e_i then we know D on everything
{using only multilinearity}

by alternation $D(\ldots e_i \ldots e_i \ldots) = -D(\ldots e_i \ldots e_i \ldots)$

so $D(\ldots e_i \ldots e_i \ldots) = 0$

and D on any choice of order for the distinct e_i is determined by alternation

as $(-1)^m D(e_1 \ldots e_n)$

but $D(e_1 \ldots e_n) = D_1(e_1 \ldots e_n) - D_2(e_1 \ldots e_n)$

$= 1 - 1$

$= 0$

So $D = 0$ on any inputs

$\therefore D_1 = D_2$.

Using multilinearity & alternation we can prove a lot of determinant things. Here's one you probably haven't seen

Prop (Cauchy-Binet formula)
Let B be $n \times m$ and C $m \times n$ then

$$\det(BC) = \sum_S \det(B_{n,S}) \det(C_{S,n})$$

where the sum runs over all subsets of size n of $\{1, 2, \ldots, m\}$

Thm (Matrix tree theorem)

Let G be a graph and L the Laplace matrix of G

then the number of spanning trees of G is

$\det \tilde{L}$ for any choice of $\tilde{L}$

Lemma Let G be a graph with n+1 vertices and E its signed incidence matrix

Let $\tilde{E}$ be E with any one row removed

$$\det \tilde{E}_{n,S} = \begin{cases} \pm 1 & \text{if } S \text{ is the edges of a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$

pf Every spanning tree of G has n edges

so every set with n edges that is connected (and meets all vertices) is a spanning tree

think of any isolated vertices as components so just say connected

so S is a sp tree iff S is connected

if S is not connected then order the vertices & edges of G by component then $\tilde{E}_{n,S}$ looks like

$$\begin{array}{cc} & \begin{array}{cc} \text{edges in comp 1} & \text{edges in comp 2} \ \cdots \end{array} \\ \begin{array}{c} \text{verts in compont 1} \\ \text{verts in compont 2} \\ \vdots \end{array} & \begin{bmatrix} * & 0 \\ 0 & * \\ \vdots & \end{bmatrix} \end{array}$$

so $\det \tilde{E}_{n,S} = \det\begin{pmatrix}\text{compont 1}\\ \text{block}\end{pmatrix} \det\begin{pmatrix}\text{compont 2}\\ \text{block}\end{pmatrix} \cdots$

take a component that does not include the vertex corresponding to the removed row then the determinant of that compont block

by every column of this E sums to 0
(+1 for one end of the edge, -1 for the other and nothing else)

so $\operatorname{rank} E < n$ so $\det E = 0$.

pf of Thm let $\hat{L}$ be $n \times n$

$$\det \tilde{L} = \det \tilde{E}\tilde{E}^t$$

$$= \sum_S \det \tilde{E}_{n,S} \det \tilde{E}^t_{S,n} \quad \text{by Cauchy Binet}$$

$$= \sum_S (\det \tilde{E}_{n,S})^2$$

$$= \sum_{\substack{T \text{ sp tree} \\ \text{of } G}} (\pm 1)^2 \quad \text{by lemma}$$

$$= \# \text{ of spanning trees}$$

Thm (extended matrix tree)
let G be a graph w L', M as def'd above
then ① $\det \tilde{L}' = \sum_{T \text{ sp tr}} \prod_{e \in T} a_e$
② $\det \tilde{M} = \sum_{T \text{ sp tree}} \prod_{e \notin T} a_e = \psi$

proof of ① is now easy

$$\det \tilde{L}' = \det \tilde{E} \Lambda \tilde{E}^t$$

$$= \sum_S \det \tilde{E}_{n,S} \det (\Lambda \tilde{E}^t)_{S,n}$$

$$= \sum_S \left(\prod_{e \in S} a_e\right) \det \tilde{E}_{n,S} \det \tilde{E}^t_{S,n}$$

$$= \sum_{T \text{ sp}} \prod_{e \in S} a_e$$

the proof of ② involves keeping track of more stuff

here's a fact to make it easy

prop (Schur complement) (slightly backwards form)

let $N = \begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$ be a block matrix with the indicated sizes

suppose A is invertible

then $\det N = \det A \det(D - CA^{-1}B)$

pf idea: block gaussian elimination (but flipped around a bit)

$$N \begin{bmatrix} I_p & -A^{-1}B \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_p & -A^{-1}B \\ 0 & I_q \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 \\ C & -CA^{-1}B + D \end{bmatrix}$$

so $(\det N) \det \begin{bmatrix} I_p & -A^{-1}B \\ 0 & I_q \end{bmatrix} = \det \begin{bmatrix} A & 0 \\ C & -CA^{-1}B + D \end{bmatrix}$

$$\det N \qquad\qquad (\det A)(\det(D - CA^{-1}B))$$

proof of ② we have $\tilde{M} = \begin{bmatrix} \Lambda & \tilde{E} \\ -\tilde{E}^t & 0 \end{bmatrix}$

so $\det \tilde{M} = \det A \det\left(0 + \tilde{E}^t \Lambda^{-1} \tilde{E}\right)$

$= (a_1 \cdots a_{n+1}) \det\left(\tilde{E}^t \Lambda^{-1} \tilde{E}\right)$ but this is $\tilde{L}$ except

$= a_1 \cdots a_{n+1} \sum_{T \text{ sp tree}} \left(\prod_{e \in T} \frac{1}{a_e}\right)$ the variables are all inverted

so $\Lambda^{-1} = \begin{bmatrix} \frac{1}{a_1} & & 0 \\ & \frac{1}{a_2} & \\ 0 & & \ddots \end{bmatrix}$

$= \sum \prod_{e \notin T} a_e$.

(5) Feynman integrals

Feynman graphs represent integrals

- Assign an (arbitrary) orientation to the edges (say the one we used for E)
- Assign to each internal edge a momentum variable in $\mathbb{R}^4$ ⇝ space time (Euclidean)
- Each external edge also gets a momentum
  - view them as known, or as parameters (given by your experiment)
- Require momentum conservation at each vertex

→ play with how many free variables there are.

Feynman graphs index a sum over all possibilities so need to "sum" over all values of the free variables
need integrate

integrate what? Feynman rules build an integrand out of the Feynman graph

eg massless scalar field theory

to $\xrightarrow{\;p\;}$ associate $\frac{1}{p^2}$ ⟵ physics notation for $\frac{1}{\|p\|^2}$

eg 

eg 

eg massive scalar field theory

same but $\frac{1}{p^2-m^2}$ ⟵ caveat in physics you'll see $\frac{1}{p^2-m^2+i\varepsilon}$

this is to determine which side of the pole to take the limit from

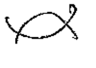
more realistic QFTs like QED are the same idea but much more complicated

But this is not the form we want - it doesn't feel enough like discrete math or like algebraic geometry

For each momentum p^2 appearing in the denominator

use
$$\frac{1}{p^2} = \int_0^\infty e^{-ap^2}\, da$$

check
$$\int_0^\infty e^{-ap^2}\, da = -\frac{1}{p^2} e^{-ap^2}\Big|_0^\infty = 0 - \left(-\frac{1}{p^2}\right) = \frac{1}{p^2}$$

• do it in  eg

note the switching the order of the integrals

Now we need to know how to integrate Gaussian integrals in many vars

So now we need to do some calculus

prop
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$
(even though $\int e^{-x^2} dx$ has no expression in terms of elementary functions)

pf
$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy$$
by Fubini's theorem w since the indefinite integral is abs convergent.

$$= \int_0^\infty \int_0^{2\pi} e^{-r^2}\, r\, d\theta\, dr$$
by polar coordinates

$x = r\cos\theta$
$y = r\sin\theta$

so the change of vars matrix is
$$\begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= 2\pi \int_0^\infty r e^{-r^2}\, dr$$

$$= 2\pi \left. \frac{-e^{-r^2}}{2} \right|_0^\infty$$

$= 2\pi\Big($

which has determinant

Def A map $\langle \cdot , \cdot \rangle$ on a vector sp V, is an <u>inner product</u> if it satisfies

① $\langle x, y \rangle = \langle y, x \rangle$ (or over $\mathbb{C}$ $\langle x, y \rangle = \overline{\langle y, x \rangle}$)
② linear in the first coord (so for $\mathbb{R}$ bilinear, otherwise scalar conjugate in 2nd slot)
③ $\langle x, x \rangle \geq 0$ w $\langle x, x \rangle = 0 \Rightarrow x = 0$

Prop The dot product on $\mathbb{R}^n$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

is an inner product

pf check.

Def • Say two vectors v, w are orthogonal if $\langle v, w \rangle = 0$
• Say the length or norm of a vector is $\sqrt{\langle v, v \rangle}$ written $\|v\|$ (makes sense by ③)

These defs make sense in $\mathbb{R}^n$

• Say θ is the angle between v, w then

$$\cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|}$$

• $\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ = Euclidean length.

Say a set of vectors $\{v_1 \dots v_k\}$ is <u>orthonormal</u>

if $\langle v_i , v_j \rangle = 0$ $i \neq j$

$\langle v_i , v_i \rangle = 1$

· what will happen for us is the same idea but with more variables

let A be real & symmetric

$$\text{prop} \quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\left(-\sum_{1\le i,j\le n} a_{ij}x_ix_j\right)}\, dx_1\ldots dx_n = \sqrt{\frac{(\pi)^n}{\det A}}$$

· pf rewrite $-\sum_{1\le i,j\le n} a_{ij}x_ix_j = -x^t A x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

since A is real & symmetric it can be diagonalized by an orthonormal matrix & $O^t = O^{-1}$ for orthogonal matrices

$$A = O^t \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} O \qquad \& \quad \det A = d_1 \cdots d_n$$

so $e^{-x^tAx} = e^{-x^tO^t\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix}Ox}$

then do the change of vars $x \to y = Ox$

still runs over $\mathbb{R}^n$ & det $= 1$

since O is orthonormal

$$\text{so} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x^tAx}\, dx_1\ldots dx_n = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-y^t\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n\end{bmatrix}y}\, dy_1\ldots dy_n$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-(d_1y_1^2+\ldots+d_ny_n^2)}\, dy_1\ldots dy_n$$

$$= \left(\int_{-\infty}^{\infty} e^{-d_1y_1^2}\right)\cdots\left(\int_{-\infty}^{\infty} e^{-d_ny_n^2}\right)$$

$$= \frac{(\sqrt{\pi})^n}{d_1\ldots d_n}$$

$$= \frac{(\sqrt{\pi})^n}{\det A}.$$

? see next page

real symm matrix lemma

columns are orthogonal & of length 1

① if O is orthonormal then $O^t = O^{-1}$

② if M is real symmetric then all roots of the char poly of M are real

③ if M is real symmetric then eigenvectors corr to distinct eigenvalues are orthogonal

④ if M is $n \times n$ real symmetric then there is a basis of $\mathbb{R}^n$ consisting of orthonormal eigenvectors of M
so M is diagonalisable by an orthonormal matrix

pf

① let $O = \{v_1 \cdots v_n\}$

$$O^t O = \begin{bmatrix} v_1^t v_1 & v_1^t v_2 & \cdots \\ \vdots & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I$$

and O is square
$\therefore O^t = O^{-1}$

② let M be $n \times n$
let $z \in \mathbb{C}^n$ let $q = \bar{z}^t M z$ (a scalar)

then $\bar{q} = z^t \overline{M} \bar{z} = z^t M \bar{z}$ since M is real

$= z^t M^t \bar{z}$ since M symm

$= (Mz) \cdot \bar{z}$

$= \bar{z} \cdot (Mz)$

$= \bar{z}^t M z$

$= q$

so $\bar{q} = q$ so $q \in \mathbb{R}$

suppose the char poly of M had a cx root λ
then λ is an eigenvalue so let $z \in \mathbb{C}^n$ be an eigenvector assoc to λ, let q be as above

③ Let u_1, u_2 be eigenvectors corr to distinct eigenvalues λ_1, λ_2

then $\lambda_1(u_1 \cdot u_2) = (Au_1) \cdot u_2 = u_1^T A^T u_2 = u_1^T A u_2$ since A symm

$= u_1 \cdot (A u_2)$

$= u_1 \cdot (\lambda_2 u_2)$

$= \lambda_2 (u_1 \cdot u_2)$

but $\lambda_1 \neq \lambda_2$ so $u_1 \cdot u_2 = 0$

④ Suppose not, then there is an $n \times n$ matrix M which is real symmetric w/ n as small as possible

(so every $(n-1) \times (n-1)$ or smaller real symmetric matrix is diagonalizable by an orthonormal matrix)

by ② M has n real eigenvalues. Pick one, call it λ

& let u be a unit eigenvector assoc to λ

Let $W = \{w \in \mathbb{R}^n : u \cdot w = 0\} = \text{Nul}(u_1^T)$

$\dim W = \dim \text{Nul}(u_1^T) = n - \text{rank}(u_1^T)$

$= n-1$

choose an orthonormal basis for W, call it $\mathcal{B} = \{v_2 .. v_{n-1}\}$

now if $w \in W$ then $u \cdot (Aw) = u^t A w$

$= u^t A^t w$

$= (Au) \cdot w$

$= \lambda(u \cdot w)$

$= 0$

so $Aw \in W$

$\therefore A$ defines a linear transformation on W

Let B be the matrix of A on W with respect to $\mathcal{B}$

B certainly has real entries

Claim: B is symmetric

Let $e_1 .. e_{n-1}$ be the stan basis vectors in $\mathbb{R}^{n-1}$

so by induction there is a basis $x_1 \ldots x_{n-1}$ of $\mathbb{R}^{n-1}$
consisting of orthogonal eigenvectors of B

then let $w_i \in W$ map to x_i under the $\mathcal{B}$ coords

then w_i is an eigenvector of A say with eigenvalue λ_i
and the w_i remain orthogonal
furthermore they correspond to ~~eigenvals~~ different from λ

So by ③ $\{u, w_1, \ldots, w_{n-1}\}$ is what we want

let $O = [u \; w_1 \ldots w_{n-1}]$

then $AO = \begin{bmatrix} \lambda & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix} O$ so $O^t A O = \begin{bmatrix} \lambda & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{bmatrix}$

Back to our Feynman integrals lets do it schematically
for a graph G we had

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{e \in E} \frac{1}{(\text{Momentum of } e)^2} \right) \prod_{\substack{\text{bases} \\ \text{of} \\ \text{cycles}}} d^4_p$$

in 4-space

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\int_0^{\infty} \cdots \int_0^{\infty} \prod_{e \in E} e^{-a_e (\text{moment of } e)^2} \, da_e \right) \prod_{\substack{\text{bases} \\ \text{of} \\ \text{cycles}}} d^4 p$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-p^t (\,?\,) p + O} \prod d^4 p \right) \prod_{e \in E} da_e$$

stuff for external
to keep things
easy lets set
the external
moments to O

what matrix goes here?

the ij^{th} entry will contain a_e

if p_i, p_j appears in momentum of e

so this is like L
but with cycles in place of vertices.

constant involving π's

4 ← from the d^4

$\infty \quad \infty$

So now you can forget all that if you didn't like it and we can just say the object we care about is

$$\int \frac{1}{\psi^2}$$

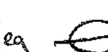
see p 16.5

Some questions → does it converge?
→ if not which subsets of edges diverge
(think about it in the problem sessions)

Answer: power counting — easiest to see in momentum space
each internal edge contributes two powers of integration var to the denom
each indep cycle contributes four dp

Let l_G be the number of indep cycles of G
Let e_G be the number of internal edges of G

if $e_G < 2l_G$ then for large values of momenta the integrand looks like $\displaystyle\int \frac{(dp)^{2 \cdot 2 l_G}}{(p^2)^{e_G}} \sim \left(\int dx\right)^{\text{some tve power}} \to \infty$

(eg  : $e_G = 3$, $l_G = 2$)
$3 < 4$

if $e_G = 2l_G$ then for large values of $\displaystyle\int \frac{(dp^2)^{2l_G}}{(p^2)^{e_G}} \sim \left(\int \frac{dx}{x}\right)^{\text{tve some power}}$
$\sim (\log x)^{\text{some tve power}}$
$\to \infty$

(eg  $e_G = 2$, $l_G = 1$)
$2 = 2$

if $e_G > 2l_G$ ... converges

and the same story for subgraphs.

The graphs we care about most are just barely divergent

def A primitive log-divergent graph has $e_G = 2l_G$ and $e_\gamma > 2l_G$ for every subset of edges

Another way to mod the integral is by Fourier transform

Some Fourier facts: $\displaystyle\int_{\mathbb{R}^4} \frac{d^4x}{(2\pi)^2} \frac{e^{ipx}}{x^2} = \frac{1}{p^2}$ $\displaystyle\int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^2} \frac{e^{-ipx}}{p^2} = \frac{1}{x^2}$

again the physicist conventions are $p^2 = p \cdot p$

$px = p \cdot x$ etc. as 4-vectors

This is special to 4 dims ~ power of 2 a more general formula is

$$\int \frac{d^n x}{(2\pi)^{\frac{n}{2}}} \frac{e^{ipx}}{|x|^k} = \frac{\Gamma(\frac{k}{2})}{|p|^{n-k}}$$

so $n=4$ $k=2$ makes this duality work

One more Fourier fact we need (in 1-dim)

ie Fourier tran of 1

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} 1 e^{-ipx} = \sqrt{2\pi}\, \delta(p)$$

dirac delta ie $\delta(p) = 0$ $p \neq 0$ ~ $\int_{-\infty}^{\infty} \delta(p)\, dp = 1$

Def the position space Feynman integral of a graph G is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_e \frac{1}{(v_e^- - v_e^+)^2} \prod_v d^4 v$$

where G is given an arbitrary orientation, and a var is assoc to each vertex ~ v_e^-, v_e^+ are the vars vertex at the two ends of e.

Note we integrate each vertex var over $\mathbb{R}^4$

p16, 75

prop Position space and momentum space give the same result

pf

$$\int_v \prod_e \frac{1}{(v_e^- - v_e^+)^2} \prod_v d^4 v$$

$$= \int_v \int_{p_e} \prod_e \frac{e^{-i p_e (v_e^- - v_e^+)}}{p_e^2} \prod_e d^4 p_e \prod_v d^4 v$$

$$= \int_{p_e} \prod_e \frac{1}{p_e^2} \prod_v \left(\underbrace{\int_v e^{-i v \cdot (\sum_{e \ni i} \pm p_e)} d^4 v}_{\text{split one var at a time}} \right) \prod_e d^4 p_e$$

$$= \int_{p_e} \prod_e \frac{1}{p_e^2} \prod_v \underbrace{\delta(\sum_{e \ni i} \pm p_e)}_{} \prod_e d^4 p_e$$

this imposes momentum conservation

and so this whole expression

is the momentum space integral

Finally go other way

as Fourier trans is reversible

Note this tells us Fourier trans is keeping

edges the same and swapping vertices for cycles

ie planar dual

— define planar dual. maybe dual in other cases set the last variable to 1

So I will write $\displaystyle\int \frac{1}{\psi^2}$

to mean $$\int_0^\infty \cdots \int_0^\infty \frac{1}{\psi^2}\, da_1 \cdots da_{n-1} \Big|_{a_n=1}$$

where ψ is the Kirchhoff poly of a graph G which is primitive log divergent and has n edges.

What does that have to do with MZVs?

 : $6\zeta(3)$

(*) Bring the table on a computer