

# APPLICATIONS OF MULTIZETA VALUES TO MAHLER MEASURE

MATILDE LALÍN

ABSTRACT. These notes correspond to a mini-course taught by the author during the program “PIMS-SFU undergraduate summer school on multiple zeta values: combinatorics, number theory and quantum field theory”. Please send any comments or corrections to the author at [mlalin@dms.umontreal.ca](mailto:mlalin@dms.umontreal.ca).

## 1. PRIMES, MAHLER MEASURE, AND LEHMER’S QUESTION

We start our study by discussing prime numbers. First consider the sequence of numbers  $M_n = 2^n - 1$  for  $n$  natural. We may ask, when is  $M_n$  prime? We can write

$$(2^{rs} - 1) = (2^r - 1)(2^{(s-1)r} + 2^{(s-2)r} + \cdots + 2^r + 1).$$

This implies that  $M_r \mid M_{rs}$  and we need  $n$  prime in order for  $M_n$  to be prime. Notice that  $M_2 = 3, M_3 = 7, M_5 = 31, M_7 = 127$ , but  $M_{11} = 2^{11} - 1 = 2047 = 23 \times 89$ . Therefore the converse is not true.

The primes of the form  $M_p$  are called Mersenne primes. It is unknown if there are infinitely many of such primes, or if there are infinitely many  $M_p$  composite with  $p$  prime. The largest Mersenne prime known to date is with  $p = 57,885,161$  (it has 17,425,170 digits). The search for large Mersenne primes is being carried by the “Great Internet Mersenne Prime Search”: <http://www.mersenne.org>.

**Exercise 1.** Let  $a, p$  be natural numbers such that  $a^p - 1$  is prime, then show that either  $a = 2$  or  $p = 1$

**Exercise 2.** Let  $p$  be an odd prime. Show that every prime  $q$  that divides  $2^p - 1$  must be of the form  $q = 2pk + 1$  with  $k$  integer.

Looking for large primes, Pierce [Pi17] proposed the following construction in 1917. Consider  $P \in \mathbb{Z}[x]$  monic, and write

$$P(x) = \prod_i (x - \alpha_i)$$

then, we look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The  $\alpha_i$  are algebraic integers. By applying Galois theory, it is easy to see that  $\Delta_n \in \mathbb{Z}$ . Note that if  $P = x - 2$ , we get the sequence  $\Delta_n = 2^n - 1$ , the Mersenne numbers. The idea is to look for primes among the factors of  $\Delta_n$ . The prime divisors of such integers must satisfy some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number.

**Exercise 3.** Prove that  $\Delta_n$  is a divisibility sequence, namely, if  $n \mid m$ , then  $\Delta_n \mid \Delta_m$ .

Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1}, \quad p \text{ prime.}$$

Pierce and Lehmer observed that the only possible factors of  $\Delta_n$  are given by prime powers  $p^e$  of the form  $nk + 1$  for some integer  $k$  and  $1 \leq e \leq \deg(P)$ . It is then natural to look for  $P$  that generate sequences that grow slowly so that they have a small chance of having factors. Lehmer [Le33] studied  $\frac{|\Delta_{n+1}|}{|\Delta_n|}$ , observed that

$$\lim_{n \rightarrow \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1, \\ 1 & \text{if } |\alpha| < 1, \end{cases}$$

and suggested the following definition:

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*Key words and phrases.* Mahler measure, higher Mahler measure, polylogarithms.

**Definition 1.1.** Given  $P \in \mathbb{C}[x]$ , such that

$$P(x) = a \prod_i (x - \alpha_i)$$

define the (Mahler) measure \* of  $P$  as

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}.$$

The logarithmic Mahler measure is defined as

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|,$$

where  $\log^+ |\alpha| = \log \max\{1, |\alpha|\}$ .

**Exercise 4.** Prove that for any  $n \in \mathbb{Z}$ ,

$$m(P(x)) = m(P(x^n)).$$

**Exercise 5.** Prove that if  $P(x) = a_d x^d + \dots + a_0$ , then  $|a_i| \leq \binom{d}{i} M(P)$ .

As  $M(P)$  measures the growth of the sequence  $\frac{|\Delta_{n+1}|}{|\Delta_n|}$ , it is natural to ask about the sequences that do not grow: When does  $M(P) = 1$  for  $P \in \mathbb{Z}[x]$ ? We have

**Lemma 1.2.** (Kronecker, [Kr57]) *Let  $P = \prod_i (x - \alpha_i) \in \mathbb{Z}[x]$ , if  $|\alpha_i| \leq 1$ , then the  $\alpha_i$  are zero or roots of the unity.*

**Proof.** Consider the polynomial

$$P_n(x) = \prod_{i=1}^d (x - \alpha_i^n).$$

The coefficients of  $P_n(x)$  are symmetric functions in the algebraic integers  $\alpha_i^n$ , so they are elements of  $\mathbb{Z}$  (all the conjugates of each  $\alpha_i$  are present as roots of  $P(x)$ , since the coefficients are rational). Each of the coefficients is uniformly bounded as  $n$  varies, because  $|\alpha_j| \leq 1$  and the set  $\{P_n\}_{n \in \mathbb{N}}$  must be finite. In other words, there are  $n_1 \neq n_2$  for which

$$P_{n_1} = P_{n_2}.$$

That means,

$$\{\alpha_1^{n_1}, \dots, \alpha_d^{n_1}\} = \{\alpha_1^{n_2}, \dots, \alpha_d^{n_2}\}.$$

Thus, there is a permutation  $\sigma \in \mathbb{S}_d$  such that

$$\alpha_i^{n_1} = \alpha_{\sigma(i)}^{n_2}.$$

If  $\sigma$  has order  $k$ , we get,

$$\alpha_i^{n_1^k} = \alpha_i^{n_2^k},$$

and  $\alpha_i$  is a root of

$$x^{n_1^k} (x^{n_2^k - n_1^k} - 1) = 0.$$

This shows that each  $\alpha_i$  is either zero or a root of unity.  $\odot$

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\*The name Mahler came about 30 years later after the person who successfully extended this definition to the several-variable case.

**Exercise 6.** (a) Give examples to show that in general a polynomial in  $\mathbb{Z}[x]$  may have zeros of absolute value one that are not roots of the unity.

(b) Show that a monic example of (a) can only occur in degree at least 4.

By Kronecker's Lemma,  $P \in \mathbb{Z}[x]$ ,  $P \neq 0$ , then  $M(P) = 1$  if and only if  $P$  is the product of powers of  $x$  and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\dots) = 0.162357612\dots$$

and asked the following (Lehmer's question, 1933):

*Is there a constant  $C > 1$  such that for every polynomial  $P \in \mathbb{Z}[x]$  with  $M(P) > 1$ , then  $M(P) \geq C$ ?*

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

**Exercise 7.** With the help of a computer find the Mahler measures of the following polynomials

(a)  $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ ,

(b)  $x^{10} - x^6 + x^5 - x^4 + 1$ ,

(c)  $x^{14} + x^{11} - x^{10} - x^7 - x^4 + x^3 + 1$ ,

(d)  $x^{14} - x^{12} + x^7 - x^2 + 1$ .

Examples of polynomials with small Mahler measure may be found with the search engine from Mossinghoff's website [M]: <http://www.cecm.sfu.ca/~mjm/Lehmer/search/>.

**Exercise 8.** With the help of a computer investigate the sequences  $\Delta_n$  for  $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$  and for  $x^3 - x - 1$ . Challenge: Find  $\sqrt{\Delta_{4951}}$  for the first polynomial and  $\Delta_{23311}$  for the second polynomial and check that they are prime numbers. Hint: it may be necessary to verify that the  $\Delta_n$  satisfy a recurrence sequence as proved in [Le33], section 8.

The following definition will be used often in the notes.

**Definition 1.3.** Let  $P(x) \in \mathbb{C}[x]$  be a nonzero polynomial of degree  $d$ . We set

$$P^*(x) := x^d \overline{P(x^{-1})}$$

the polynomial with the same coefficients as  $P$ , but in reverse order and conjugated. We say that  $P$  is reciprocal if  $P = \pm P^*$ . Otherwise, we say that  $P$  is non-reciprocal.

We list here some important results in the direction of solving Lehmer's question.

**Theorem 1.4.** (Breusch [Br51], Smyth [Sm71]) *If  $P \in \mathbb{Z}[x]$  is monic, irreducible, non-reciprocal, then*

$$M(P) \geq M(x^3 - x - 1) = \theta = 1.324717\dots$$

**Corollary 1.5.** *If  $P \in \mathbb{Z}[x]$  is monic, irreducible, and of odd degree, then*

$$M(P) \geq \theta.$$

The most general result with a bound involving the degree is given by

**Theorem 1.6.** (Dobrowolski [Do79]) *If  $P \in \mathbb{Z}[x]$  is monic, irreducible and noncyclotomic of degree  $d$ , then*

$$M(P) \geq 1 + c \left( \frac{\log \log d}{\log d} \right)^3$$

where  $c$  is an absolute positive constant.

**Theorem 1.7.** (Schinzel) *If  $P \in \mathbb{Z}[x]$  is monic of degree  $d$  having all real roots and satisfies  $P(1)P(-1) \neq 0$  and  $|P(0)| = 1$ , then*

$$M(P) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{\frac{d}{2}}$$

with equality iff  $P$  is a power of  $x^2 - x - 1$ .

For the proof, we need the following

**Exercise 9.** For any  $d \geq 1$  and  $y_1, \dots, y_d > 1$  be real numbers, prove that

$$(y_1 - 1) \cdots (y_d - 1) \leq ((y_1 \cdots y_d)^{1/d} - 1)^d.$$

Hint: This is a special case of Mahler's inequality. Apply the arithmetic-geometric mean inequality to  $\left\{ \frac{y_1-1}{y_1}, \dots, \frac{y_d-1}{y_d} \right\}$  and to  $\left\{ \frac{1}{y_1}, \dots, \frac{1}{y_d} \right\}$ .

**Proof.** (Theorem 1.7) Consider

$$E = \prod_{i=1}^d |\alpha_i^2 - 1|.$$

Remark that  $E \geq 1$  since  $P$  is monic and  $\alpha_i \neq \pm 1$ . Note that

$$M(P) = \prod_{|\alpha_i| > 1} |\alpha_i| = \frac{1}{\prod_{|\alpha_i| < 1} |\alpha_i|}.$$

Thus, we may rewrite

$$E = \prod_{|\alpha_i| < 1} |\alpha_i^2 - 1| \prod_{|\alpha_i| > 1} |\alpha_i^2 - 1| = \frac{1}{M(P)^2} \prod_{|\alpha_i| < 1} |\alpha_i^{-2} - 1| \prod_{|\alpha_i| > 1} |\alpha_i^2 - 1|.$$

We apply Exercise 9 in order to obtain

$$\begin{aligned} E &\leq \frac{1}{M(P)^2} \left( \left( \prod_{|\alpha_i| < 1} \alpha_i^{-2} \prod_{|\alpha_i| > 1} \alpha_i^2 \right)^{1/d} - 1 \right)^d \\ &= \frac{1}{M(P)^2} \left( M(P)^{4/d} - 1 \right)^d \\ &= (M(P)^{2/d} - M(P)^{-2/d})^d. \end{aligned}$$

Since  $E \geq 1$ ,

$$(1.1) \quad M(P)^{2/d} - M(P)^{-2/d} \geq 1.$$

Since  $M(P) > 1$ , this implies that  $M(P)^{2/d} \geq \frac{1+\sqrt{5}}{2}$  and we obtain the desired result.  $\odot$

**Exercise 10.** Prove the last assertion of Theorem 1.7.

**Exercise 11.** What happens if we relax the condition of  $|P(0)| = 1$ ?

**Theorem 1.8.** (Bombieri, Vaaler [BV87]) Let  $P \in \mathbb{Z}[x]$  with  $M(P) < 2$ , then  $P$  divides a polynomial  $Q \in \mathbb{Z}[x]$  whose coefficients belong to  $\{-1, 0, 1\}$ .

The last result and Theorem 1.4 suggest that if we want to beat Lehmer's 10-degree polynomial, one should search for reciprocal polynomials having coefficients in  $\{-1, 0, 1\}$ .

The most general result in terms of families is given by

**Theorem 1.9.** (Borwein, Dobrowolski, Mossinghoff [BDM07]) Let  $\mathcal{D}_m$  denote the set of polynomials whose coefficients are all congruent to 1 modulo  $m$ . Si  $f \in \mathcal{D}_m$  has degree  $d$  and no cyclotomic factors, then

$$m(f) \geq c_m \left( 1 - \frac{1}{d+1} \right),$$

where  $c_2 = (\log 5)/4$  and  $c_m = \log(\sqrt{m^2+1}/2)$  for  $m > 2$ .

As a final comment, we remark that Mahler measure is related to classical heights. Generally speaking, a height is a function that measures the size of a mathematical object. For example, the absolute value  $|\cdot|$  measures the size of complex numbers. Another less immediate example is given by the canonical height defined in  $\mathbb{Q}$  as follows. If  $\frac{a}{b}$  is a rational number written in lowest terms (in other words,  $(a, b) = 1$ ), then the height of  $\frac{a}{b}$  is defined as  $\max\{|a|, |b|\}$ . This definition is what allows us to distinguish between the

number 2 and the number  $1.999999999 = \frac{19,999,999,999}{10,000,000,000}$ : while the first has height 2, the second one has height 19,999,999,999.

For a polynomial  $P(x) = a_d x^d + \dots + a_1 x + a_0 \in \mathbb{C}[x]$  define

$$H(P) = \max_{0 \leq i \leq d} \{|a_i|\}, \quad L(P) = \sum_{i=0}^d |a_i|,$$

the height and the length of  $P$ .

**Exercise 12.** Prove the following inequalities:

- (a)  $L(P) \leq 2^d M(P)$ .
- (b)  $H(P) \leq 2^{1-d} M(P)$ .

Mahler [Ma62] considered this construction because he was looking for inequalities of the classical polynomial heights (such as  $L(P)$  or  $H(P)$ ) between the height of a product of polynomials and the heights of the factors. These kinds of inequalities are useful in transcendence theory.  $M(P)$  is multiplicative (that is,  $M(PQ) = M(P)M(Q)$ ) and comparable to the typical heights, and that makes it possible to deduce such inequalities. While investigating this in several variables, he discovered the generalization that we explore in the next section.

## 2. MAHLER MEASURE IN SEVERAL VARIABLES

We will be concerned mostly with the Mahler measure of multivariable polynomials.

**Definition 2.1.** For  $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\times$ , the logarithmic Mahler measure is defined by

$$\begin{aligned} m(P) &:= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}. \end{aligned}$$

The Mahler measure is defined by

$$M(P) := e^{m(P)}.$$

It is possible to prove that this integral is not singular and that  $m(P)$  always exists. This definition appeared for the first time in the work of Mahler [Ma62].

The relationship between Definitions 2.1 and 1.1 is given by the following.

**Theorem 2.2.** (*Jensen's formula*) Let  $\alpha \in \mathbb{C}$ . Then

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta = \log^+ |\alpha|$$

where  $\log^+ x = \log \max\{1, x\}$  for  $x \in \mathbb{R}_{\geq 0}$ .

**Proof.** First assume that  $|\alpha| < 1$ . We have that

$$\begin{aligned} \int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta &= \int_0^1 \log |1 - \alpha e^{-2\pi i \theta}| d\theta \\ &= \int_0^1 \log |1 - \alpha e^{2\pi i \tau}| d\tau \end{aligned}$$

where we have done  $\tau = -\theta$ . We have

$$\begin{aligned} \int_0^1 \log |1 - \alpha e^{2\pi i\tau}| d\tau &= \operatorname{Re} \left( \int_0^1 \log(1 - \alpha e^{2\pi i\tau}) d\tau \right) \\ &= \operatorname{Re} \int_0^1 \left( - \sum_{n=1}^{\infty} \frac{\alpha^n}{n} e^{2\pi i\tau n} \right) d\tau \\ &= \operatorname{Re} \left( - \sum_{n=1}^{\infty} \frac{\alpha^n}{n} \int_0^1 e^{2\pi i\tau n} d\tau \right) \\ &= 0. \end{aligned}$$

We have exchanged the integral and the infinite sum, but this is justified since the sum is absolutely convergent.

Now assume that  $|\alpha| > 1$ . Then, by the previous argument,

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log |\alpha| + \int_0^1 \log |1 - \alpha^{-1} e^{2\pi i\theta}| d\theta = \log |\alpha|.$$

Finally, we are left with the hardest case,  $|\alpha| = 1$ . After multiplying by  $\alpha^{-1}$ , we may assume that  $\alpha = 1$ . Writing  $|1 - e^{2\pi i\theta}| = 2 \sin(\pi\theta)$  for  $0 \leq \theta \leq 1$ , we have

$$(2.1) \quad \int_0^1 \log |1 - e^{2\pi i\theta}| d\theta = \int_0^1 \log \sin(\pi\theta) d\theta + \log 2.$$

Let  $I = \int_0^1 \log \sin(\pi\theta) d\theta$ . The integral exists since  $\sin(\pi\theta) \sim \pi\theta$  for small  $\theta$ .

We write  $\sin(\pi\theta) = 2 \sin\left(\frac{\pi\theta}{2}\right) \cos\left(\frac{\pi\theta}{2}\right)$ . Thus,

$$I = \log 2 + \int_0^1 \log \sin\left(\frac{\pi\theta}{2}\right) d\theta + \int_0^1 \log \cos\left(\frac{\pi\theta}{2}\right) d\theta.$$

By making the change  $\tau = \theta/2$  in the first integral and  $\tau = 1/2 - \theta/2$  in the second one, we obtain,

$$I = \log 2 + 4 \int_0^{1/2} \log \sin(\pi\tau) d\tau = \log 2 + 2 \int_0^1 \log \sin(\pi\tau) d\tau = \log 2 + 2I.$$

From this,  $I = -\log 2$  and we obtain the desired result by combining with equation (2.1).  $\odot$

The multivariable Mahler measure is still multiplicative, meaning that for  $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ , we have

$$m(P \cdot Q) = m(P) + m(Q).$$

**Proposition 2.3.** *Let  $P \in \mathbb{C}[x_1, \dots, x_n]$  such that  $a_{i_1, \dots, i_n}$  is the coefficient of  $x_1^{i_1} \dots x_n^{i_n}$  and  $P$  has degree  $d_i$  in  $x_i$ . Then*

(a)

$$|a_{i_1, \dots, i_n}| \leq \binom{d_1}{i_1} \dots \binom{d_n}{i_n} M(P)$$

(b)

$$M(P) \leq L(P) \leq 2^{d_1 + \dots + d_n} M(P)$$

(c)

$$((d_1 + 1) \dots (d_n + 1))^{-1/2} M(P) \leq H(P) \leq 2^{d_1 + \dots + d_n - n} M(P)$$

where

$$H(P) = \max_{0 \leq i_j \leq d_j, j=1, \dots, n} \{|a_{i_1, \dots, i_n}|\}, \quad L(P) = \sum_{j=1}^n \sum_{i_j=0}^{d_j} |a_{i_1, \dots, i_n}|.$$

**Proof.** [(c), lower bound] By an inequality of Hardy–Littlewood–Pólya,

$$M(P) \leq \left( \int_0^1 \dots \int_0^1 |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})|^2 d\theta_1 \dots d\theta_n \right)^{1/2}.$$

Parseval’s formula implies

$$\begin{aligned} \int_0^1 \dots \int_0^1 |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})|^2 d\theta_1 \dots d\theta_n &= \sum_{i_j} |a_{i_1, \dots, i_n}|^2 \\ &\leq (d_1 + 1) \dots (d_n + 1) H(P)^2. \end{aligned}$$

⊙

**Exercise 13.** Prove the rest of Proposition 2.3

**Exercise 14.** If  $P \in \mathbb{C}[x_1, x_2]$  has a constant coefficient  $a$  that in absolute value exceeds the sum of the absolute values of all the other coefficients, prove that  $m(P) = \log |a|$ .

**Exercise 15.** Let

$$P(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x_1, \dots, x_n],$$

where  $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \dots x_n^{m_n}$ . Let  $A$  be an  $n \times n$  integer matrix with non-zero determinant, and define

$$P^{(A)}(\mathbf{x}) := \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{A\mathbf{m}}.$$

Prove that

$$m(P) = m(P^{(A)}).$$

It is also true that  $m(P) \geq 0$  if  $P$  has integral coefficients.

In addition, an analogous of Kronecker’s lemma is true.

**Theorem 2.4.** (Smyth [Sm82]) *For any primitive polynomial (i.e., the coefficients have no nontrivial common factor)  $P \in \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]$ ,  $m(P)$  is zero if and only if  $P$  is a monomial times a product of cyclotomic polynomials evaluated on monomials.*

Let us also mention the following result:

**Theorem 2.5.** (Boyd [Bo81], Lawton [La83]) *For  $P \in \mathbb{C}[x_1, \dots, x_n]$*

$$\lim_{k_2 \rightarrow \infty} \dots \lim_{k_n \rightarrow \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, \dots, x_n)).$$

It should be noted that the limit has to be taken independently for each variable. (Writing this properly would take half a page.)

**Exercise 16.** Explore the limit of Theorem 2.5 in the case of  $1 + x + y$ . Namely, compute several values of  $m(1 + x + x^n)$  and compare them with the value of  $m(1 + x + y) = 0.323065947 \dots$ .

Because of the above theorem, Lehmer’s question in the several-variable case reduces to the one-variable case. In addition, this theorem shows us that we are working with the “right” generalization of the original definition for one-variable polynomials.

The formula for the one-variable case tells us some information about the nature of the values that Mahler measure can reach. For instance, the Mahler measure of a polynomial in one variable with integer coefficients must be an algebraic number.

It is natural, then, to wonder what happens with the several-variable case. Is there any simple formula, besides the integral? (Un)fortunately,<sup>†</sup> this case is much more complicated and we only have some particular examples. On the other hand, the values are very interesting.

<sup>†</sup>If it were that easy this area would not be so interesting!

### 3. EXAMPLES

We show some examples of formulas for Mahler measures of multivariable polynomials.

- Smyth [Sm82]

$$(3.1) \quad m(x + y + 1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1),$$

where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{3} \\ -1 & \text{if } n \equiv -1 \pmod{3} \\ 0 & \text{if } n \equiv 0 \pmod{3} \end{cases}$$

is a Dirichlet  $L$ -function.

- Smyth [Bo81]

$$(3.2) \quad m(x + y + z + 1) = \frac{7}{2\pi^2} \zeta(3),$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Riemann zeta function.

- Boyd & L. (2005)

$$m(x^2 + 1 + (x + 1)y + (x - 1)z) = \frac{1}{\pi} L(\chi_{-4}, 2) + \frac{21}{8\pi^2} \zeta(3).$$

- L. [La03]

$$m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \left(\frac{1-x_2}{1+x_2}\right) (1+y)z\right) = \frac{93}{\pi^4} \zeta(5).$$

- In fact, there are known formulas ([La06]) for

$$m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_n}{1+x_n}\right) (1+y)z\right).$$

- Rogers & Zudilim [RZ11]

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2} L(E_{15}, 2) = L'(E_{15}, 0)$$

where  $E_{15}$  is an elliptic curve (of conductor 15) that happens to be the algebraic closure of the zero set of the polynomial.

Roughly speaking, an elliptic curve is the zero set of a polynomial of the form

$$Y^2 = X^3 + AX + B$$

where  $4A^3 + 27B^2 \neq 0$ .

The polynomial  $x + \frac{1}{x} + y + \frac{1}{y} + k$  generally corresponds to an elliptic curve under the following transformation

$$x = \frac{kX - 2Y}{2X(X-1)} \quad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X \left( X^2 + \left( \frac{k^2}{4} - 2 \right) X + 1 \right).$$

(It is then easy to eliminate the term of  $X^2$  by completing the cube.)

The  $L$ -function of an elliptic curve is a function similar to the Riemann zeta function with coefficients that encode the number of points of the curve over finite fields.

- Boyd (2005)

$$m(z + (x + 1)(y + 1)) \stackrel{?}{=} 2L'(E_{15}, -1).$$

The question mark stands for an identity that has been verified up to 20 decimal places, but for which no proof is known.



- Examples with  $K3$  surfaces, mostly due to Bertin. These includes polynomials in the family  $x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + k$ .

How do we get such formulas? Some of them are very difficult to prove. To be concrete, we are going to show the proof of the first example by Smyth (from [Bo81]):

**Proof.** (Equation (3.1)) By Jensen's formula,

$$m(1+x+y) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log |1+e^{it}+e^{is}| dt ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \max\{|1+e^{it}|, 1\} dt = \frac{1}{2\pi} \int_{-2\pi/3}^{2\pi/3} \log |1+e^{it}| dt.$$

Now we write

$$\log |1+e^{it}| = \operatorname{Re} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{int}.$$

The series does not converge absolutely but it converges uniformly in  $t \in [-2\pi/3, 2\pi/3]$ , since we are far from the singularity at  $t = \pm\pi$ . It follows from

$$\int_{-2\pi/3}^{2\pi/3} e^{int} dt = \frac{2}{n} \sin \frac{2n\pi}{3} = \frac{\sqrt{3}}{n} \chi_{-3}(n)$$

that

$$(3.3) \quad m(1+x+y) = \frac{\sqrt{3}}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \chi_{-3}(n)}{n^2} = \frac{\sqrt{3}}{2\pi} \left( \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^2} - 2 \sum_{n=1}^{\infty} \frac{\chi_{-3}(2n)}{(2n)^2} \right)$$

and use that  $\chi_{-3}(2n) = \chi_{-3}(2)\chi_{-3}(n) = -\chi_{-3}(n)$  to obtain the initial formula.  $\odot$

**Exercise 17.** Prove Smyth's formula (3.2). Hint:

$$m(1+x+y+z) = m(1+x+z(1+y)) = \int_0^1 \int_0^1 \log \max\{|1+e^{2\pi i\theta_1}|, |1+e^{2\pi i\theta_2}|\} d\theta_1 d\theta_2.$$

Some of the formulas explored in this section can be proved and better understood by using polylogarithms.

#### 4. POLYLOGARITHMS

Many examples should be understood in the context of polylogarithms.

**Definition 4.1.** The  $k$ th polylogarithm is the function defined by the power series

$$\operatorname{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1.$$

This function can be continued analytically to  $\mathbb{C} \setminus [1, \infty)$ .

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [Z91] considers the following version:

$$\mathcal{L}_k(x) := \operatorname{Re}_k \left( \sum_{j=0}^k \frac{2^j B_j}{j!} (\log |x|)^j \operatorname{Li}_{k-j}(x) \right),$$

where  $B_j$  is the  $j$ th Bernoulli number,  $\operatorname{Li}_0(x) \equiv -\frac{1}{2}$  and  $\operatorname{Re}_k$  denotes  $\operatorname{Re}$  or  $\operatorname{Im}$  depending on whether  $k$  is odd or even.

This function is one-valued, real analytic in  $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$  and continuous in  $\mathbb{P}^1(\mathbb{C})$ . Moreover,  $\mathcal{L}_k$  satisfies very clean functional equations. The simplest ones are

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1} \mathcal{L}_k(x) \quad \mathcal{L}_k(\bar{x}) = (-1)^{k-1} \mathcal{L}_k(x).$$

There are also lots of functional equations which depend on the index  $k$ . For instance, for  $k = 2$ , we have the Bloch–Wigner dilogarithm,

$$D(x) := \operatorname{Im} (\operatorname{Li}_2(x)) + \arg(1-x) \log |x|$$

which satisfies the well-known five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0.$$

The dilogarithm can be also recovered in terms of an integral.

$$-2 \int_0^\theta \log |2 \sin t| dt = D(e^{2i\theta}) = \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}$$

More generally, recall the definition for polylogarithms.

**Definition 4.2.** Multiple polylogarithms are defined as the power series

$$\text{Li}_{n_1, \dots, n_m}(x_1, \dots, x_m) := \sum_{0 < k_1 < k_2 < \dots < k_m} \frac{x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}}{k_1^{n_1} k_2^{n_2} \dots k_m^{n_m}}$$

which are convergent for  $|x_i| < 1$ . The weight of a polylogarithm function is the number  $w = n_1 + \dots + n_m$ .

When  $n_m > 1$  the above series converges for  $|x_i| \leq 1$ . We can then find multizeta values by setting  $x_i = 1$ :

$$\text{Li}_{n_1, \dots, n_m}(1, \dots, 1) = \zeta(n_1, \dots, n_m).$$

**Exercise 18.** (a) Express  $\text{Li}_n(-1)$  in terms of zeta functions.

(b) Express  $\text{Li}_n(e^{2\pi i/3}) - \text{Li}_n(e^{-2\pi i/3})$  in terms of  $L(\chi_{-3}, n)$ .

(c) What is  $\text{Li}_n(i) - \text{Li}_n(-i)$ ?

**Definition 4.3.** Hyperlogarithms are defined as the iterated integrals

$$\text{I}_{n_1, \dots, n_m}(a_1 : \dots : a_m : a_{m+1}) := \int_0^{a_{m+1}} \underbrace{\frac{dt}{t - a_1} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_1} \circ \underbrace{\frac{dt}{t - a_2} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_2} \circ \dots \circ \underbrace{\frac{dt}{t - a_m} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_m}$$

where  $n_i$  are integers,  $a_i$  are complex numbers, and

$$\int_0^{b_{k+1}} \frac{dt}{t - b_1} \circ \dots \circ \frac{dt}{t - b_k} = \int_{0 \leq t_1 \leq \dots \leq t_k \leq b_{k+1}} \frac{dt_1}{t_1 - b_1} \dots \frac{dt_k}{t_k - b_k}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and  $a_{m+1}$  on  $\mathbb{C} \setminus \{a_1, \dots, a_m\}$ . To be concrete, when possible, we will integrate over the real line.

It is easy to see that,

$$\text{I}_{n_1, \dots, n_m}(a_1 : \dots : a_m : a_{m+1}) = (-1)^m \text{Li}_{n_1, \dots, n_m} \left( \frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m} \right)$$

$$\text{Li}_{n_1, \dots, n_m}(x_1, \dots, x_m) = (-1)^m \text{I}_{n_1, \dots, n_m}((x_1 \dots x_m)^{-1} : \dots : x_m^{-1} : 1)$$

which gives an analytic continuation to multiple polylogarithms. For instance, with the above convention about integrating over a real segment, simple polylogarithms have an analytic continuation to  $\mathbb{C} \setminus [1, \infty)$ .

**Exercise 19.** Prove the previous equations.

**Exercise 20.** Prove that

$$\zeta(n_1, \dots, n_m) = \zeta(\underbrace{1, \dots, 1}_{n_m - 2}, 2, \dots, \underbrace{1, \dots, 1}_{n_1 - 2}, 2).$$

**4.1. Applications of polylogarithms to Mahler measure.** Many of the Mahler measure formulas can be proved via polylogarithms. For instance, formula (3.3) may be rewritten as

$$m(1+x+y) = \frac{1}{2\pi} (D(e^{2\pi i/3}) - D(e^{-2\pi i/3})) = \frac{1}{2\pi i} (\text{Li}_2(e^{2\pi i/3}) - \text{Li}_2(e^{-2\pi i/3})).$$

Similarly, the Riemann zeta function arises as a special value of the polylogarithm:

$$\zeta(n) = \text{Li}_n(1)$$

and this identity appears also in Mahler measure formulas.

Here is a more detailed example.

**Theorem 4.4.** For  $a \in \mathbb{R}_{>0}$ ,

$$m((1+x) + a(1-x)y) = -\frac{i}{\pi} (\text{Li}_2(ia) - \text{Li}_2(-ia)).$$

**Proof.** We apply multiplicativity of the Mahler measure and then make the change of variable  $x = e^{i\theta}$  and notice that  $\frac{1-x}{1+x} = -i \tan\left(\frac{\theta}{2}\right)$ .

$$\begin{aligned} \pi m((1+x) + a(1-x)y) &= \pi m(1+x) + \pi m\left(1 + a \frac{1-x}{1+x} y\right) \\ &= \frac{1}{2i} \int_{\mathbb{T}^1} \log^+ \left| a \frac{1-x}{1+x} \right| \frac{dx}{x} \\ &= \frac{1}{2} \int_0^{2\pi} \log^+ \left| a \tan\left(\frac{\theta}{2}\right) \right| d\theta. \end{aligned}$$

Now make  $w = \frac{1}{a \tan(\frac{\theta}{2})}$ , then  $d\theta = \frac{2a dw}{1+a^2 w^2}$ . The previous equation equals

$$\begin{aligned} & - \int_0^1 \log w \frac{2a dw}{w^2 a^2 + 1} \\ &= i \int_0^1 \int_w^1 \frac{ds}{s} \left( \frac{1}{w + \frac{i}{a}} - \frac{1}{w - \frac{i}{a}} \right) dw \\ &= i \left( \text{I}_2\left(-\frac{i}{a} : 1\right) - \text{I}_2\left(\frac{i}{a} : 1\right) \right) \\ &= -i (\text{Li}_2(ia) - \text{Li}_2(-ia)). \end{aligned}$$

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**Exercise 21.** Prove that for  $a \in \mathbb{C}^\times$ ,

$$m(1+x+ay+az) = \begin{cases} \frac{2}{\pi^2} (\text{Li}_3(|a|) - \text{Li}_3(-|a|)) & |a| \leq 1, \\ \log |a| + \frac{2}{\pi^2} (\text{Li}_3(|a|^{-1}) - \text{Li}_3(-|a|^{-1})) & |a| \geq 1. \end{cases}$$

**Exercise 22.** Prove that

$$m((1+x_1)(1+x_2) + (1-x_1)(1-x_2)y) = \frac{7}{\pi^2} \zeta(3).$$

Polylogarithms play a crucial role in these type of formulas, relating Mahler measure to special values of zeta functions and  $L$ -functions. These relationships provide examples of very general conjectures (Beilinson's conjectures) which play a central role in number theory. These statements include, for example, the Birch–Swinnerton-Dyer conjecture, one of the seven Millenium Prize Problems posed by the Clay Institute with a prize of one million dollars for the solution of each.

It is natural to wonder if these applications of polylogarithms to Mahler measure can be generalized. The key ingredient here is the relationship that can be established from the integral to hyperlogarithms. Inspired

by this, we will explore other types of integrals which should also fit into the statement of Beilinson's conjectures.<sup>‡</sup>

## 5. HIGHER MAHLER MEASURE

**Definition 5.1.** The  $k$ -higher Mahler measure of  $P$  is defined by

$$m_k(P) := \int_0^1 \dots \int_0^1 \log^k |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n.$$

In particular, notice that for  $k = 1$  we obtain the classical Mahler measure

$$m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest example of higher Mahler measure is with the polynomial  $P = 1 - x$  [KLO08]:

**Theorem 5.2.** ([KLO08]) *We have,*

$$\begin{aligned} m_2(1-x) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}, \\ m_3(1-x) &= -\frac{3\zeta(3)}{2}, \\ m_4(1-x) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}, \\ m_5(1-x) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}, \\ m_6(1-x) &= \frac{930\zeta(6) + 180\zeta(3)^2 + 315\zeta(2)\zeta(4) + 15\zeta(2)^3}{8} \\ &= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6, \end{aligned}$$

and similar equations for higher indexes.

Before proceeding into the proof of Theorem 5.2 we recall some properties of the Gamma function

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt.$$

**Exercise 23.** Prove  $\Gamma(1) = 1$  and that  $\Gamma(s+1) = s\Gamma(s)$ . Deduce that  $\Gamma(n+1) = n!$  for  $n$  positive integer.

We notice that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt = 2 \int_0^\infty e^{-s^2} ds = \sqrt{\pi},$$

where we have made the change  $t = s^2$  and used the Gaussian integral.

Let

$$B(r, s) = \int_0^1 t^{r-1} (1-t)^{s-1} dt,$$

the Beta function.

**Exercise 24.** Prove that, whenever the integral for the Beta function converges,

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

Now,

$$B(s, s) = \int_0^1 t^{s-1} (1-t)^{s-1} dt = \frac{1}{2} \int_{-1}^1 \left(\frac{1+u}{2}\right)^{s-1} \left(\frac{1-u}{2}\right)^{s-1} du,$$

---

<sup>‡</sup>The relationship between higher Mahler measure and Beilinson's conjectures is yet to be established. One of the motivations to find examples of higher Mahler measure formulas is precisely the search of a precise formulation of this relationship.

where we have set  $t = \frac{1+u}{2}$ . This yields,

$$\frac{1}{2^{2s-1}} \int_{-1}^1 (1-u^2)^{s-1} du = \frac{1}{2^{2s-2}} \int_0^1 (1-u^2)^{s-1} du.$$

By setting  $v = u^2$ , this equals

$$\frac{1}{2^{2s-1}} \int_0^1 (1-v)^{s-1} v^{-1/2} dv = \frac{1}{2^{2s-1}} B\left(\frac{1}{2}, s\right).$$

Thus,

$$\Gamma(2s) = \frac{2^{2s-1}}{\Gamma\left(\frac{1}{2}\right)} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right).$$

The above is called the duplication formula for  $\Gamma$ .

Finally, we state the Weierstrass product without proof.

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{\frac{s}{n}},$$

where  $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = 0.577216\dots$  is the Euler–Mascheroni constant.

**Proof.** [Theorem 5.2] To prove the above equalities one uses a construction by Akatsuka [A09]. We consider the integral

$$Z(s; x-1) = \int_0^1 |e^{2\pi i \theta} - 1|^s d\theta.$$

(This is called the Zeta Mahler measure.) We first make the change  $t = \sin^2 \pi \theta$ :

$$\begin{aligned} Z(s, x-1) &= 2^{s+1} \int_0^{1/2} (\sin \pi \theta)^s d\theta \\ &= \frac{2^s}{\pi} \int_0^1 t^{\frac{s-1}{2}} (1-t)^{-1/2} dt. \end{aligned}$$

Thus, we obtain the Beta function

$$\begin{aligned} &= \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \\ &= \frac{2^s}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)} \\ &= \frac{2^s}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)}, \end{aligned}$$

where  $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$  is the Gamma function. Hence, by using

$$\Gamma\left(\frac{s+1}{2}\right) = \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} 2^{1-s} \pi^{\frac{1}{2}} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2} + 1\right)} 2^{-s} \pi^{\frac{1}{2}},$$

and

$$\Gamma(s+1)^{-1} = e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}},$$

we obtain

$$\begin{aligned}
Z(s, x-1) &= \prod_{n=1}^{\infty} \frac{\left(1 + \frac{s}{2n}\right)^2}{1 + \frac{s}{n}} \\
&= \exp\left(\sum_{n=1}^{\infty} \left\{2 \log\left(1 + \frac{s}{2n}\right) - \log\left(1 + \frac{s}{n}\right)\right\}\right) \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} \left\{2 \left(\frac{1}{2n}\right)^k - \frac{1}{n^k}\right\} s^k\right) \\
&= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(k) (2^{1-k} - 1) s^k\right) \\
&= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k (1 - 2^{1-k}) \zeta(k)}{k} s^k\right).
\end{aligned}$$

The Zeta Mahler measure yields a Taylor series whose coefficients are given by  $m_k(P)$ .

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{m_k(x-1)}{k!} s^k &= Z(s, x-1) \\
&= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k (1 - 2^{1-k}) \zeta(k)}{k} s^k\right).
\end{aligned}$$

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**Exercise 25.** Prove that

$$m_k(1-x) = \sum_{b_1 + \dots + b_h = k, b_i \geq 2} \frac{(-1)^k k!}{2^{2h}} \zeta(b_1, \dots, b_h),$$

where  $\zeta(b_1, \dots, b_h)$  denotes

$$\zeta(b_1, \dots, b_h) = \sum_{l_1 < \dots < l_h} \frac{1}{l_1^{b_1} \dots l_h^{b_h}}.$$

Hint:

$$\begin{aligned}
\log^k |1-x| &= (\operatorname{Re} \log(1-x))^k = \left(\frac{1}{2}(\log(1-x) + \log(1-x^{-1}))\right)^k \\
&= \frac{1}{2^k} \left(\int_0^1 \frac{dt}{t-x^{-1}} + \int_0^1 \frac{dt}{t-x}\right)^k = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \left(\int_0^1 \frac{dt}{t-x^{-1}}\right)^j \left(\int_0^1 \frac{dt}{t-x}\right)^{k-j}.
\end{aligned}$$

We know how to answer Lehmer's question in the case of the higher Mahler measure.

**Theorem 5.3.** ([LS11]) *If  $P(x) \in \mathbb{Z}[x]$  is not a monomial, then for any  $h \geq 1$ ,*

$$m_{2h}(P) \geq \begin{cases} \left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,} \\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.} \end{cases}$$

Let  $P_n(x) = \frac{x^n - 1}{x - 1}$ . For  $h \geq 1$  fixed,

$$\lim_{n \rightarrow \infty} m_{2h+1}(P_n) = 0.$$

Moreover, this sequence is nonconstant.

A natural generalization for the  $k$ -higher Mahler measure is the multiple higher Mahler measure for more than one polynomial.

**Definition 5.4.** Let  $P_1, \dots, P_k \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$  be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$\begin{aligned} & m(P_1, \dots, P_k) \\ := & \int_0^1 \cdots \int_0^1 \log |P_1(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| \cdots \log |P_k(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n. \end{aligned}$$

This construction yields the higher Mahler measure of one polynomial as a special case:

$$m_k(P) = m(\underbrace{P, \dots, P}_k).$$

Moreover, the above definition implies that

$$m(P_1) \cdots m(P_k) = m(P_1, \dots, P_k)$$

when the variables of  $P_j$ 's are algebraically independent.

The simplest example in this regard is, again, given by linear polynomials.

**Theorem 5.5.** ([KLO08]) For  $0 \leq \alpha \leq 1$

$$m(1-x, 1-e^{2\pi i\alpha}x) = \frac{\pi^2}{2} \left( \alpha^2 - \alpha + \frac{1}{6} \right).$$

**Proof.** Let  $A_\varepsilon = [\varepsilon, \alpha - \varepsilon] \cup [\alpha + \varepsilon, 1 - \varepsilon]$ . By definition,

$$\begin{aligned} m(1-x, 1-e^{2\pi i\alpha}x) &= \int_0^1 \operatorname{Re} \log(1-e^{2\pi i\theta}) \cdot \operatorname{Re} \log(1-e^{2\pi i(\theta+\alpha)}) d\theta \\ &= \int_0^1 \left( -\sum_{k=1}^{\infty} \frac{1}{k} \cos 2\pi k\theta \right) \left( -\sum_{l=1}^{\infty} \frac{1}{l} \cos 2\pi l(\theta+\alpha) \right) d\theta \\ &= \sum_{k,l \geq 1} \frac{1}{kl} \int_{A_\varepsilon} \cos(2\pi k\theta) \cos(2\pi l(\theta+\alpha)) d\theta \\ &\quad + \int_{[0,1] \setminus A_\varepsilon} \left( -\sum_{k=1}^{\infty} \frac{1}{k} \cos 2\pi k\theta \right) \left( -\sum_{l=1}^{\infty} \frac{1}{l} \cos 2\pi l(\theta+\alpha) \right) d\theta. \end{aligned}$$

Because  $\log |1 - e^{2\pi i\theta}| = \log |2 \sin \pi\theta| \sim \log |2\pi\theta|$  for  $\theta$  near zero, and  $\int_0^\varepsilon \log(Kx) dx \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , the last term approaches zero.

Notice that

$$\int_{[0,1] \setminus A_\varepsilon} \cos(2\pi k\theta) \cos(2\pi l(\theta+\alpha)) d\theta = \begin{cases} O(\varepsilon) & \text{if } l = k, \\ \frac{O(\varepsilon)}{|k-l|} & \text{otherwise,} \end{cases}$$

and

$$\int_0^1 \cos(2\pi k\theta) \cos(2\pi l(\theta+\alpha)) d\theta = \begin{cases} \frac{1}{2} \cos(2\pi k\alpha) & \text{if } l = k, \\ 0 & \text{otherwise.} \end{cases}$$

By putting everything together we conclude that

$$m(1-x, 1-e^{2\pi i\alpha}x) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k\alpha)}{k^2} = \frac{\pi^2}{2} \left( \alpha^2 - \alpha + \frac{1}{6} \right).$$

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More generally,

$$(5.1) \quad m(1 - \alpha x, 1 - \beta x) = \begin{cases} \frac{1}{2} \operatorname{Re} \operatorname{Li}_2(\alpha \bar{\beta}) & \text{if } |\alpha|, |\beta| \leq 1, \\ \frac{1}{2} \operatorname{Re} \operatorname{Li}_2\left(\frac{\alpha \bar{\beta}}{|\alpha|^2}\right) & \text{if } |\alpha| \geq 1, |\beta| \leq 1, \\ \frac{1}{2} \operatorname{Re} \operatorname{Li}_2\left(\frac{\alpha \bar{\beta}}{|\alpha \beta|^2}\right) + \log |\alpha| \log |\beta| & \text{if } |\alpha|, |\beta| \geq 1. \end{cases}$$

yields the equivalent of a Jensen's formula for multiple Mahler measure:

$$m(1 - \alpha x) = \begin{cases} 0 & \text{if } |\alpha| \leq 1, \\ \log |\alpha| & \text{if } |\alpha| \geq 1. \end{cases}$$

**Exercise 26.** Prove Equation (5.1).

**Exercise 27.** Prove that

$$\begin{aligned} m(1 - x, 1 - e^{2\pi i \alpha} x, 1 - e^{2\pi i \beta} x) &= -\frac{1}{4} \sum_{k, l \geq 1} \frac{\cos 2\pi((k+l)\beta - l\alpha)}{kl(k+l)} \\ &\quad - \frac{1}{4} \sum_{k, m \geq 1} \frac{\cos 2\pi((k+m)\alpha - m\beta)}{km(k+m)} \\ &\quad - \frac{1}{4} \sum_{l, m \geq 1} \frac{\cos 2\pi(l\alpha + m\beta)}{lm(l+m)}. \end{aligned}$$

Multiple higher Mahler measure has applications to the computation of the higher Mahler measure. For example, Theorem 5.3 is proven as a consequence of the following result.

**Theorem 5.6.** ([LS11]) *If  $P(x) \in \mathbb{Z}[x]$  is reciprocal, then*

$$m_2(P) \geq \frac{\pi^2}{12}.$$

This inequality is sharp, with equality for  $x - 1$ .

**Proof.** (Theorem 5.3 from Theorem 5.6) First suppose that  $h = 2$  and  $P(x)$  non-reciprocal. Let  $d = \deg P$ , and consider  $P^*(x) = x^d P(x^{-1})$ . Thus  $P(x)P^*(x) \in \mathbb{Z}[x]$  is reciprocal. Moreover,  $m_2(P) = m_2(P^*) = m(P, P^*)$ , thus,

$$m_2(PP^*) = m_2(P) + 2m(P, P^*) + m_2(P^*) = 4m_2(P).$$

We obtain the desired bound by applying Theorem 5.6 to  $PP^*$ .

The rest of the proof (for  $h > 2$ ) is left as an exercise.  $\odot$

**Exercise 28.** Complete the proof of Theorem 5.3 by proving

- (a) 
$$m_{2h}(P) \geq m_2(P)^h,$$
- (b) 
$$m_{2h}(P) \geq m(P)^{2h}.$$

**Open Question 1.** Is it possible to improve the bounds in Theorem 5.3?

**Exercise 29.** With the help of a computer explore the values of the higher Mahler measure in polynomials. For example, compute  $m_2$  for the polynomials in Exercise 7.



**Open Question 2.** Can you find  $P \in \mathbb{Z}$  non-reciprocal with  $m_2(P) < m_2(x^3 + x + 1) = 0.3275495729\dots$ ? (Notice that  $\frac{\pi^2}{48} = 0.2056167583\dots$ )

**Theorem 5.7.** ([KLO08])

$$\begin{aligned} m_2(1 - x + y(1 + x)) &= \frac{4i}{\pi}(\text{Li}_{2,1}(-i, -i) - \text{Li}_{2,1}(i, i)) \\ &\quad + \frac{6i}{\pi}(\text{Li}_{2,1}(i, -i) - \text{Li}_{2,1}(-i, i)) \\ &\quad + \frac{i}{\pi}(\text{Li}_{2,1}(1, -i) - \text{Li}_{2,1}(1, i)) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi}L(\chi_{-4}, 2) \end{aligned}$$

**Proof.** First notice that

$$(5.2) \quad m_2(1 - x + y(1 + x)) = m_2\left(\left(\frac{1-x}{1+x}\right) + y\right) + 2m\left(\left(\frac{1-x}{1+x}\right) + y, 1+x\right) + m_2(1+x).$$

For the first term, we have

$$m_2\left(\left(\frac{1-x}{1+x}\right) + y\right) = \frac{1}{(2\pi i)^2} \int_{|y|=1} \int_{|x|=1} \log^2\left|\left(\frac{1-x}{1+x}\right) + y\right| \frac{dx dy}{x y}.$$

By applying (5.1), the above line becomes

$$\begin{aligned} &= \frac{1}{2\pi i} \int_{|x|=1, |1-x|\leq|1+x|} \frac{1}{2} \text{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right) \frac{dx}{x} + \frac{1}{2\pi i} \int_{|x|=1, |1-x|\geq|1+x|} \frac{1}{2} \text{Li}_2\left(\left|\frac{1+x}{1-x}\right|^2\right) \frac{dx}{x} \\ &\quad + \frac{1}{2\pi i} \int_{|x|=1, |1-x|\geq|1+x|} \log^2\left|\frac{1-x}{1+x}\right| \frac{dx}{x} \\ &= \frac{1}{2\pi i} \int_{|x|=1, |1-x|\leq|1+x|} \text{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right) \frac{dx}{x} + \frac{1}{2\pi i} \int_{|x|=1, |1-x|\geq|1+x|} \log^2\left|\frac{1-x}{1+x}\right| \frac{dx}{x}. \end{aligned}$$

For the second term in equation (5.2) we obtain

$$m\left(\left(\frac{1-x}{1+x}\right) + y, 1+x\right) = \frac{1}{(2\pi i)^2} \int_{|y|=1} \int_{|x|=1} \log\left|\left(\frac{1-x}{1+x}\right) + y\right| \log|1+x| \frac{dx dy}{x y}.$$

By Jensen's formula respect to the variable  $y$ ,

$$= \frac{1}{2\pi i} \int_{|x|=1} \log^+\left|\frac{1-x}{1+x}\right| \log|1+x| \frac{dx}{x} = \frac{1}{2\pi i} \int_{|x|=1, |1-x|\geq|1+x|} \log\left|\frac{1-x}{1+x}\right| \log|1+x| \frac{dx}{x}.$$

Then equation (5.2) becomes

$$(5.3) \quad \begin{aligned} m_2(1 - x + y(1 + x)) &= \frac{1}{2\pi i} \int_{|x|=1, |1-x|\leq|1+x|} \text{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right) \frac{dx}{x} \\ &\quad + \frac{1}{2\pi i} \int_{|x|=1, |1-x|\geq|1+x|} (\log^2|1-x| - \log^2|1+x|) \frac{dx}{x} \\ &\quad + \frac{\zeta(2)}{2}. \end{aligned}$$

For the first term in (5.3), set  $x = e^{2i\theta}$ ,

$$\begin{aligned} &\frac{1}{2\pi i} \int_{|x|=1, |1-x|\leq|1+x|} \text{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right) \frac{dx}{x} \\ &= \frac{2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \text{Li}_2(\tan^2 \theta) d\theta = \frac{4}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\text{Li}_2(\tan \theta) + \text{Li}_2(-\tan \theta)) d\theta. \end{aligned}$$

Now we make the change of variables  $y = \tan \theta$ .

$$\begin{aligned}
&= \frac{8}{\pi} \int_0^1 (\operatorname{Li}_2(y) + \operatorname{Li}_2(-y)) \frac{dy}{y^2 + 1} \\
&= \frac{4}{\pi} \int_0^1 (\operatorname{Li}_2(y) + \operatorname{Li}_2(-y)) \left( \frac{1}{1 + iy} + \frac{1}{1 - iy} \right) dy \\
&= \frac{4}{\pi} (i\operatorname{Li}_{2,1}(i, -i) + i\operatorname{Li}_{2,1}(-i, -i) - i\operatorname{Li}_{2,1}(-i, i) - i\operatorname{Li}_{2,1}(i, i)).
\end{aligned}$$

For the second term in (5.3),

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{|x|=1, |1-x| \geq |1+x|} (\log^2 |1-x| - \log^2 |1+x|) \frac{dx}{x} \\
&= \sum_{k, l \geq 1} \frac{1 - (-1)^{k+l}}{kl} 2 \int_{\frac{1}{4}}^{\frac{3}{4}} \cos(2\pi k\theta) \cos(2\pi l\theta) d\theta \\
&= \sum_{k, l \geq 1} \frac{1 - (-1)^{k+l}}{2\pi kl} \left( \frac{i^{k+l+1}(1 - (-1)^{k+l})}{k+l} + \frac{i^{k-l+1}(1 - (-1)^{k-l})}{k-l} \right) \\
&= \frac{i}{\pi} \sum_{k, l \geq 1} \frac{(1 - (-1)^{k+l})i^{k+l}}{kl^2} - \frac{i}{\pi} \sum_{k, l \geq 1} \frac{(1 - (-1)^{k+l})i^{k+l}}{(k+l)l^2} \\
&\quad + \frac{2i}{\pi} \sum_{k > l \geq 1} \frac{(1 - (-1)^{k+l})i^{k-l}}{(k-l)l^2} - \frac{2i}{\pi} \sum_{k > l \geq 1} \frac{(1 - (-1)^{k+l})i^{k-l}}{kl^2} \\
&= \frac{i}{\pi} (\operatorname{Li}_1(i)\operatorname{Li}_2(i) - \operatorname{Li}_1(-i)\operatorname{Li}_2(-i) - \operatorname{Li}_{2,1}(1, i) + \operatorname{Li}_{2,1}(1, -i)) \\
&\quad + \frac{2i}{\pi} (\zeta(2)(\operatorname{Li}_1(i) - \operatorname{Li}_1(-i)) - \operatorname{Li}_{2,1}(-i, i) + \operatorname{Li}_{2,1}(i, -i)) \\
&= \frac{i}{\pi} (-i \log 2L(\chi_{-4}, 2) - \frac{\pi i}{16} \zeta(2) - \operatorname{Li}_{2,1}(1, i) + \operatorname{Li}_{2,1}(1, -i)) \\
&\quad + \frac{2i}{\pi} (\zeta(2) \frac{\pi i}{2} - \operatorname{Li}_{2,1}(-i, i) + \operatorname{Li}_{2,1}(i, -i)).
\end{aligned}$$

Putting everything together in (5.3), we obtain the final result

$$\begin{aligned}
&m_2(1 - x + y(1 + x)) \\
&= \frac{4i}{\pi} (\operatorname{Li}_{2,1}(-i, -i) - \operatorname{Li}_{2,1}(i, i)) + \frac{6i}{\pi} (-\operatorname{Li}_{2,1}(-i, i) + \operatorname{Li}_{2,1}(i, -i)) \\
&\quad + \frac{i}{\pi} (-\operatorname{Li}_{2,1}(1, i) + \operatorname{Li}_{2,1}(1, -i)) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi} L(\chi_{-4}, 2).
\end{aligned}$$

☺

This result may be contrasted to Smyth's

$$m(1 - x + y(1 + x)) = \frac{2}{\pi} L(\chi_{-4}, 2).$$

**Exercise 30.** Prove the above formula from Theorem 4.4.

Naturally, we have the following generalization of the Zeta Mahler measure.

**Definition 5.8.** Let  $P_1, \dots, P_k \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$  be non-zero Laurent polynomials. Their higher zeta Mahler measure is defined by

$$\begin{aligned}
&Z(s_1, \dots, s_l; P_1, \dots, P_l) \\
&= \int_0^1 \dots \int_0^1 |P_1(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r})|^{s_1} \dots |P_l(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_r})|^{s_l} d\theta_1 \dots d\theta_r,
\end{aligned}$$

Its Taylor coefficients are related to the multiple higher Mahler measure:

$$\frac{\partial^l}{\partial s_1 \cdots \partial s_l} Z(0, \dots, 0; P_1, \dots, P_l) = m(P_1, \dots, P_l).$$

Similarly to the case of  $x - 1$ , one gets

$$\begin{aligned} Z(s, t; x - 1, x + 1) &= \int_0^1 |2 \sin \pi \theta|^s |2 \cos \pi \theta|^t d\theta \\ &= 2^{s+t+1} \int_0^{1/2} (\sin \pi \theta)^s (\cos \pi \theta)^t d\theta. \end{aligned}$$

Setting  $u = \sin^2(\pi \theta)$ ,

$$\begin{aligned} Z(s, t; x - 1, x + 1) &= \frac{2^{s+t}}{\pi} \int_0^1 u^{\frac{s-1}{2}} (1-u)^{\frac{t-1}{2}} du \\ &= \frac{2^{s+t}}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{s+t}{2} + 1\right)} \\ &= \frac{\Gamma(s+1) \Gamma(t+1)}{\Gamma\left(\frac{s}{2} + 1\right) \Gamma\left(\frac{t}{2} + 1\right) \Gamma\left(\frac{s+t}{2} + 1\right)} \\ &= \prod_{n=1}^{\infty} \frac{\left(1 + \frac{s}{2n}\right) \left(1 + \frac{t}{2n}\right) \left(1 + \frac{s+t}{2n}\right)}{\left(1 + \frac{s}{n}\right) \left(1 + \frac{t}{n}\right)} \\ &= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \left\{ (1 - 2^{-k})(s^k + t^k) - 2^{-k}(s+t)^k \right\}\right) \end{aligned}$$

around  $s = t = 0$ .

Thus,

$$m(\underbrace{x-1, \dots, x-1}_k, \underbrace{x+1, \dots, x+1}_l) = \int_0^1 (\log |2 \sin \pi \theta|)^k (\log |2 \cos \pi \theta|)^l d\theta$$

belongs to  $\mathbb{Q}[\pi^2, \zeta(3), \zeta(5), \zeta(7), \dots]$  for integers  $k, l \geq 0$ .

We obtain, for instance, the following equalities.

$$\begin{aligned} m(x-1, x+1) &= \int_0^1 \log |2 \sin \pi \theta| \log |2 \cos \pi \theta| d\theta = -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24}, \\ m(x-1, x-1, x+1) &= \int_0^1 (\log |2 \sin \pi \theta|)^2 \log |2 \cos \pi \theta| d\theta = 2 \frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}, \\ m(x-1, x+1, x+1) &= \int_0^1 \log |2 \sin \pi \theta| (\log |2 \cos \pi \theta|)^2 d\theta = 2 \frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}. \end{aligned}$$

**Exercise 31.** Prove the following.

(a) For a positive constant  $\lambda$ , we have  $Z(s, \lambda P) = \lambda^s Z(s, P)$ .

(b) Let  $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  be a Laurent polynomial such that it takes non-negative real values in the unit torus. Then we have the following series expansion on  $|\lambda| \leq 1/\max(P)$ , where  $\max(P)$  is the maximum of  $P$  on the unit torus:

$$\begin{aligned} Z(s, 1 + \lambda P) &= \sum_{k=0}^{\infty} \binom{s}{k} Z(k, P) \lambda^k, \\ m(1 + \lambda P) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Z(k, P) \lambda^k. \end{aligned}$$

More generally,

$$m_j(1 + \lambda P) = j! \sum_{0 < k_1 < \dots < k_j} \frac{(-1)^{k_j - j}}{k_1 \dots k_j} Z(k_j, P) \lambda^{k_j}.$$

(c)  $Z(s, P) = Z(\frac{s}{2}, P\bar{P})$ , where we put  $\bar{P} = \sum_{\alpha} \bar{a}_{\alpha} x^{-\alpha}$  for  $P = \sum_{\alpha} a_{\alpha} x^{\alpha}$ . Note that  $P\bar{P}$  is real-valued on the torus.

**Theorem 5.9.** ([KLO08]) *Let  $c \geq 2$ .*

•

$$\begin{aligned} Z(s, x + y + c) &= c^s \sum_{j=0}^{\infty} \binom{s/2}{j} \frac{1}{c^{2j}} \binom{2j}{j} \\ &= c^s {}_3F_2 \left( \begin{matrix} -\frac{s}{2}, -\frac{s}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| \frac{4}{c^2} \right), \end{aligned}$$

where the generalized hypergeometric series  ${}_3F_2$  is defined by

$${}_3F_2 \left( \begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j}{(b_1)_j (b_2)_j j!} z^j,$$

with the Pochhammer symbol defined by  $(a)_j = a(a+1) \dots (a+j-1)$ .

•

$$m_2(x + y + c) = \log^2 c + \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{1}{k^2 c^{2k}}.$$

•

$$m_3(x + y + c) = \log^3 c + \frac{3}{2} \log c \sum_{k=1}^{\infty} \binom{2k}{k} \frac{1}{k^2 c^{2k}} - \frac{3}{2} \sum_{k=2}^{\infty} \binom{2k}{k} \frac{1}{k^2 c^{2k}} \sum_{j=1}^{k-1} \frac{1}{j}.$$

In particular, we obtain the special values

•

$$m_2(x + y + 2) = \frac{\zeta(2)}{2},$$

•

$$m_3(x + y + 2) = \frac{9}{2} \log 2 \zeta(2) - \frac{15}{4} \zeta(3).$$

The previous Theorem may be completed with the trivial statement

$$m(x + y + 2) = \log 2.$$

In fact, the motivation for setting  $c = 2$  is that this is the precise point where the family of polynomials  $x + y + c$  reaches the unit torus singularly. In classical Mahler measure, those polynomials are among the simplest to compute the Mahler measure, and the same is true in higher Mahler measures.

## 6. LOG-SINE INTEGRALS AND MAHLER MEASURE

We consider the works of Borwein, Straub and collaborators [BS12, BBSW12, BS11].

**Definition 6.1.** For  $n$  a positive integer and  $k$  a non-negative integer, the generalized log-sine integral is defined by

$$\text{Ls}_n^{(k)}(\sigma) := - \int_0^{\sigma} \theta^k \log^{n-1-k} \left| 2 \sin \frac{\theta}{2} \right| d\theta.$$

This integral was studied by Lewin who gave several evaluations [Le81]. We use the notation  $\text{Ls}_n(\sigma)$  for the case  $k = 0$ .

Some special values are expressed in terms of polylogarithms, such as

$$-\frac{1}{\pi} \sum_{m=0}^{\infty} \text{Ls}_{m+1}(\pi) \frac{\lambda^m}{m!} = \frac{\Gamma(1 + \lambda)}{\Gamma(1 + \lambda/2)^2},$$

which implies the recurrence

$$\frac{(-1)^n}{n!} \text{Ls}_{n+2}(\pi) = \pi(1 - 2^{-n})\zeta(n+1) + \sum_{k=1}^{n-2} \frac{(-1)^k}{(k+1)!} (1 - 2^{1+k-n})\zeta(n-k)\text{Ls}_{k+2}(\pi),$$

and yields, starting from  $\text{Ls}_2(\pi) = 0$ ,

$$\begin{aligned} \text{Ls}_3(\pi) &= -\frac{\pi^3}{12}, \\ \text{Ls}_4(\pi) &= \frac{3\pi}{2}\zeta(3), \\ \text{Ls}_5(\pi) &= -\frac{19}{240}\pi^5, \\ \text{Ls}_6(\pi) &= \frac{45\pi}{2}\zeta(5) + \frac{5}{4}\pi^3\zeta(3). \end{aligned}$$

More generally, we have the following result from [BS11].

**Theorem 6.2.** ([BS11]) *For  $0 \leq \tau \leq 2\pi$ , and nonnegative integers  $n, k$  such that  $n - k \geq 2$ ,*

$$\begin{aligned} \zeta(\{1\}^k, n-k) - \sum_{j=0}^k \frac{(-i\tau)^j}{j!} \text{Li}_{\{1\}^{n-k-2}, 2+k-j}(\{1\}^{n-k-2}, e^{i\tau}) \\ = \frac{i^{k+1}(-1)^{n-1}}{(n-1)!} \sum_{r=0}^{n-k-1} \sum_{m=0}^r \binom{n-1}{k, m, r-m} \left(\frac{i}{2}\right)^r (-\pi)^{r-m} \text{Ls}_{n-(r-m)}^{(k+m)}(\tau). \end{aligned}$$

Here are some results that were proved using properties of log-sine integrals.

- Let

$$\mathfrak{m}_k(1+x+y_*) := \mathfrak{m}(1+x+y_1, 1+x+y_2, \dots, 1+x+y_k).$$

Then, Sasaki [Sa10] proved

$$\mathfrak{m}_k(1+x+y_*) = \frac{1}{\pi} \text{Ls}_{k+1}\left(\frac{\pi}{3}\right) - \frac{1}{\pi} \text{Ls}_{k+1}(\pi).$$

In particular, [BS12] proved

$$\begin{aligned} \mathfrak{m}_2(1+x+y_*) &= \frac{\pi^2}{54}, \\ \mathfrak{m}_3(1+x+y_*) &= \frac{9}{2\pi} \text{Im}(\text{Li}_4(e^{\pi i/3})), \\ \mathfrak{m}_4(1+x+y_*) &= \frac{6}{\pi} \text{Im}(\text{Li}_{1,4}(1, e^{\pi i/3})) - \frac{\pi^4}{4860}, \end{aligned}$$

and similar formulas for  $k = 5, 6$ .

- Let

$$\mathfrak{m}_k(1+x+y_*+z_*) := \mathfrak{m}(1+x+y_1+z_1, 1+x+y_2+z_2, \dots, 1+x+y_k+z_k).$$

Then, in [BS12]:

$$\mathfrak{m}_k(1+x+y_*+z_*) = \frac{1}{\pi^{k+1}} \int_0^\pi \left( \theta \log\left(2 \sin \frac{\theta}{2}\right) + D(e^{i\theta}) \right)^k d\theta.$$

In particular,

$$\begin{aligned} \mathfrak{m}_1(1+x+y_*+z_*) &= \frac{7}{2\pi^2}\zeta(3), \\ \mathfrak{m}_2(1+x+y_*+z_*) &= \frac{4}{\pi^2} \text{Li}_{1,3}(1, -1) + \frac{7\pi^2}{360}, \end{aligned}$$

- Other results from [BS12] include

$$m(1+x, 1+x+y+z) = \frac{2}{\pi^2} \lambda_4 \left( \frac{1}{2} \right) - \frac{19}{720} \pi^2,$$

$$m(1+x, 1+x, 1+x+y+z) = \frac{4}{3\pi^2} \lambda_5 \left( \frac{1}{2} \right) - \frac{3}{4} \zeta(3) + \frac{31}{16\pi^2} \zeta(5),$$

where

$$\lambda_n(x) := (n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \text{Li}_{n-k}(x) \log^k |x| + \frac{(-1)^n}{n} \log^n |x|.$$

- Examples involving higher Mahler measure were computed in [BBSW12]

$$m_2(1+x+y) = \frac{3}{\pi} \text{Ls}_3 \left( \frac{2\pi}{3} \right) + \frac{\pi^2}{4},$$

$$m_3(1+x+y) \stackrel{?}{=} \frac{6}{\pi} \text{Ls}_4 \left( \frac{2\pi}{3} \right) - \frac{9}{\pi} \text{Im}(\text{Li}_4(e^{\pi i/3})) - \frac{\pi}{4} D(e^{\pi i/3}) - \frac{13}{2} \zeta(3).$$

**Exercise 32.** Find

$$m_2 \left( \frac{1+x+y}{1+x+z} \right).$$

**Open Question 3.** Is it possible to express  $\text{Ls}_n \left( \frac{2\pi}{3} \right)$  in terms of polylogarithms so that the above formulas only contain polylogarithms? (Perhaps using Theorem 6.2?)

**Open Question 4.** Investigate identities between log-sine integrals and polylogarithms with the help of the program developed by Borwein and Straub in [BS11].

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DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL. CP 6128, SUCC. CENTRE-VILLE. MONTRÉAL, QC H3C 3J7, CANADA

*E-mail address:* mlalin@dms.umontreal.ca, <http://www.dms.umontreal.ca/~mlalin>