PROBLEMS, SECTION I

K_n is the complete graph on n vertices. That is it is the graph which consists of n vertices and all possible edges between them.

1. Graphs and Feynman graphs

A connected Feynman graph is 1PI if removing any one internal edge leaves the graph connected. This property is called 2-edge-connected in graph theory.

A graph is internally 6-connected if the only ways to remove less than 6 internal edges and disconnect the graph leave at most one component that is not an isolated vertex.

(1) Draw all graphs with at most 3 loops and at most 3 external edges in QED.
(2) Draw all photon propagator graphs with at most 5 loops in QED. How many are connected? How many are 1PI? How many connected 1PI photon propagator graphs are there at n loops.
(3) Draw all graphs in ϕ^4 with 4 external edges at 4 loops. How many are connected? How many are 1PI?
(4) Write a program to generate all connected 1PI graphs in ϕ^4 with 4 external edges and a given number of loops. How fast is your program? There are some things built in to sage which might help you.
(5) Modify the previous program to generate only internally 6-connected graphs. What proportion of all such graphs are internally 6-connected?

2. Graphs axiomatically

Suppose we have a set S with some structure. An automorphism is a set bijection from S to S which preserves the structure. For example we can view graphs as a set V of vertices with the edges as the added structure. Then an automorphism of a graph is a bijection f from V to V with the property that if v_1 and v_2 were joined by an edge then $f(v_1)$ and $f(v_2)$ are also joined by an edge and vice versa.

(1) Take a small graph like K_3. Write it out in each of the axiomatic forms we discussed. Can you think of more ways to axiomatize graphs?
(2) Draw a graph with no nontrivial automorphisms (axiomatized as in the above example).
(3) Different ways of viewing graphs give different meanings of automorphism. Suppose we view a graph as a set of half edges and the added structure is the pairing of half edges into internal edges along with the collecting of half edges into vertices. Find a graph with a different number of automorphisms this way than the other way. Can you find a relation or inequality between these different ideas of automorphism?
(4) What happens to the idea of automorphism if we view a graph as a set of edges with the structure those sets of edges which form vertices?
(5) Take your hand drawn ϕ^4 graphs from the previous section and calculate the number of automorphisms they each have.

3. Spanning trees

(1) Find all spanning trees of K_4. Find all cycles of K_4.
(2) We commented that $\det L$, $\det L'$, and $\det M$ were all 0. Give an intuitive explanation for this. Prove it.
(3) We built the graph Laplacian L out of E. Give a rule to read L directly off the graph. Extend your rule to construct L' directly from the graph.
(4) Consider random walks on a graph where if at a given step the walk is at vertex i then at the next step we move to one of the neighbours of i with equal probability. Let P be the matrix with

$$p_{i,j} = \begin{cases} 1/\deg(i) & \text{if there is an edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Suppose the walk starts at vertex 1. Explain why the probabilities for where the walk will end after one step are given by

$$P^t \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Explain why the probabilities for where the walk will end after n steps are given by

$$(P^t)^n \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Relate P to the matrices we discussed.
(5) Take a large graph and implement the process from the previous question. What does the result look like? Try different start vectors.
(6) Look up more about random walks on graphs. What are the key results? How are they proved? What are random walks on graphs good for?
(7) Determine how to calculate Ψ in sage. How fast is it?

4. The Matrix-tree theorem

(1) How many different characterizations can you find for the sign of a permutation?
(2) Arguably the nicest formula for the determinant is

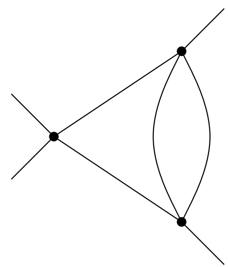
$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2.n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where S_n is the set of permutations of $\{1, 2, \ldots, n\}$. Prove this formula.
(3) How to computers calculate determinants?
(4) Take your favorite determinant fact and prove it just using multilinearity, alternation and normalization. Did you need all the properties? If you don't have a favorite determinant fact try $\det(AB) = \det(A)\det(B)$.

(5) Prove the Cauchy-Binet formula.

5. Feynman integrals

(1) In momentum space, how many variables are left to integrate once momentum conservation is taken into account? Try some examples until you see what's going on.
(2) Write the Feynman integral for the following graph in momentum space, position space, and parametric space.



(3) The matrix which shows up when converting from momentum space to parametric is not the same as the other matrices we've seen so far. Starting with some small examples can you relate this new matrix to one of the old matrices for some other graphs? Does it always work?
(4) Try integrating K_4 parametrically. Do as much as possible symbolically before you work numerically? What number do you get in the end?
(5) Try two parametric integrations on some small graphs. Can you give some meaning to the polynomials which appear in the denominator? What about after three?
(6) Try to integrate the graph from question 2. What goes wrong? Can you characterize the graphs/subgraphs for which this problem will occur?
(7) Redo your characterization from the previous question using only the information of the external edges of the graph (assuming all graphs are in ϕ^4). This is related to the question of renormalizability.