PROBLEMS, SECTION II

1. Hierarchy of numbers

Let α be an algebraic number. The degree of α is the minimal degree of a polynomial which has α as a root.

(1) Reproduce `en.wikipedia.org/wiki/File:Algebraic_number_in_the_complex_plane.png`
(2) Prove all algbraic numbers are periods.
(3) Is the set of periods a ring? a field? algebrically closed?
(4) Go to `http://wayback.cecm.sfu.ca/projects/EZFace/`. Try some of the relations we've seen. Can you find anything new?
(5) How many convergent words (that is words either in x, y or in terms of x_k corresponding to multiple zeta values as iterated integrals or as sums respectively which lead to convergent multipe zeta values) are there of weight n.
(6) Make a table of relations between multiple zeta values. How many come from the difference between the two products? What if we also allow divergent words? Try finding more relations with LLL or PSLQ.
(7) Suppose it is the case that all relations between multiple zeta values come from the difference between the two products (suitably generalized to be possibly true, see previous question). Is it then provable that $\zeta(2,6)$ can not be written as a polynomial in single zetas?

2. Counting functions of multiple zeta values

(1) Suppose we have a linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

valid for $n \geq k$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Use the linear recurrence to find a closed form expression for $A(x)$.

(2) Calculate $d_{12,4}$ and $d_{12,2}$ using the naive (incorrect) expression and using the conjectured expression for $\prod_{n>2}\prod_{k>0}(1-x^n y^k)^{d_{n,k}}$.
(3) What do you think would be the most convincing piece of evidence for the Broadhurst-Kreimer conjecture (perhaps some particular coefficient). Can you give a combinatorial interpretation (or some other kind of interpretation) for the function which appears in the Broadhurst Kreimer conjecture.

3. Lyndon words

Let $\mathbb{F}_p$ be the finite field with p elements.

(1) Generate all Lyndon words on 2 letters of length 7.
(2) What is the counting sequence for Lyndon words on k letters. Try some small values of k and look it up on the online encyclopedia of integer sequences.

(3) Let p be a prime. It is known that there are the same number of Lyndon words of length n on p letters as there are irreducible monic polynomials of degree n over $\mathbb{F}_p$. Prove this. There is no known explicit bijection between these objects. Can you find one, or a hint of what one might be? (see `amyglen.files.wordpress.com/2012/03/melbourne_talk_feb2012.pdf`)