Due: Wednesday, October 21st (noon)

Reminders

The midterm will take place in class on Monday, October 26st. It will cover material from class up to Friday, October 23rd.

Math 708 students must select a presentation topic and a date for the presentation. Please consult me if you have not done this. To give you some idea of what might be suitable, topics from the previous edition of the course are listed at the bottom of this assignment.

Reading

From the textbook, reread Section 1.2.2, and then read Section 5.2.4.

Chapter 2 of Applegate, Bixby, Chvátal and Cook, which shows several interesting problems that can be modelled as a TSP, and hence as an integer program.

Problems for Math 408 and Math 708

All problems to be submitted via Canvas. Please submit a single file names `hw3.pdf` containing all your written work, along with files `hw3.dat` and `hw3.mod` for the AMPL question (question 3). Please make sure to write your name on the first page of `hw3.pdf` and in the comments of `hw3.dat` and `hw3.mod`.

1. For each of the following sets, find a valid inequality cutting off the given fractional point:

   a. $\{(x_1, x_2) \in \mathbb{Z}^2_+ \mid x_1 \leq 5, \;\; x_1 \leq 4x_2\} \qquad (x_1^*, x_2^*) = (5, \frac{5}{4})$.

   b. $\{(x_1, x_2, x_3) \in \mathbb{Z}^3_+ \mid x_1 + x_2 - 2x_3 \leq 0, \;\; x_1, x_2, x_3 \leq 1\} \qquad (x_1^*, x_2^*, x_3^*) = (1, 0, \frac{1}{2})$.

   c. $\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4_+ \mid 4x_1 + 8x_2 + 7x_3 + 5x_4 \leq 33\} \qquad (x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, \frac{33}{7}, 0)$.

You should explain how you know that the inequality is valid.

2. Consider the integer programming problem:

$$(IP) \qquad \begin{array}{ll} \text{maximize} & 4x_1 - 6x_2 \\ \text{subject to} & x_1 + x_2 \leq 5, \\ & 2x_1 - 3x_2 \leq 1, \\ & \quad x_1, x_2 \geq 0; \qquad x_1, x_2 \in \mathbb{Z} \end{array}$$

   a. Put the system in standard (equality) form.

   b. Use the simplex method to find a point x^* maximizing the linear programming relaxation of this problem.

   c. Use the optimal basis that you have found to generate a Gomory cut, that is, an inequality cutting x^* but no feasible points of (IP).

3. Exercise 20.3 (a) and (b) from the AMPL book, available at:

`http://ampl.com/resources/the-ampl-book/chapter-downloads/`.

4. Consider solving a maximization problem by branch-and-bound. Suppose that Figure 1 represents part of the branch-and-bound tree. The value inside each node represents the solution to the associated relaxation of the problem; nodes are coloured blue if the relaxed solution is non-integer, green if it is integer, and red if no relaxed solution exists because the problem is infeasible.

a. Which node(s) still need to be expanded?

b. Which node(s) are candidate for the optimal solution?

c. Suppose that before beginning the search, we obtained an integer feasible solution with value 50. Which node(s) would we no longer have to expand?

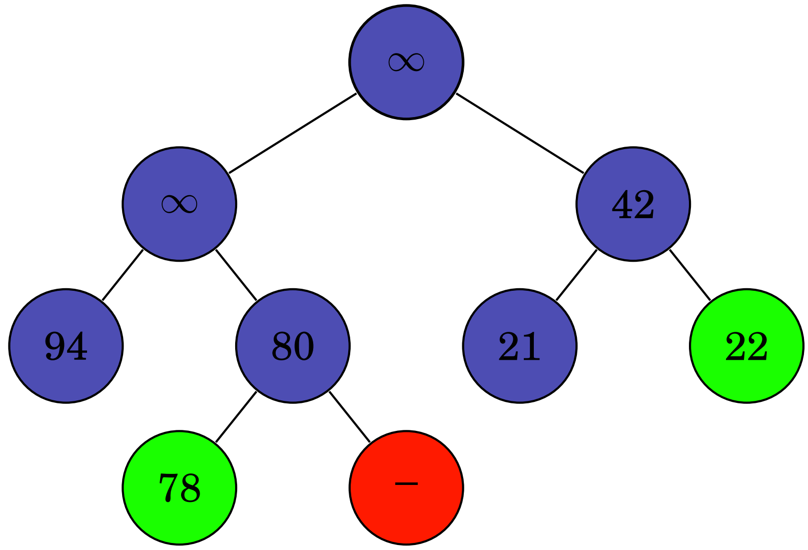

Figure 1: Branch and bound tree for problem 4.

5. Consider the complete graph on 4 vertices. Use the last 6 digits of your student id to weight the 6 edges of the graph.

a. How many Hamiltonian cycles (i.e. cycles connecting all 4 vertices) are in this graph? Give the weight of each cycle.

b. Does your weighting satisfy the triangle inequality?

c. Run the nearest neighbour heuristic the graph, beginning by traversing the minimum weight edge. What is the approximation ratio that you get from this algorithm?

Additional Problems for Math 708

6. Consider the integer program

$$\min x_{n+1} \quad \text{subject to} \quad 2x_1 + 2x_2 + \ldots + 2x_n + x_{n+1} = n \quad \text{and} \quad x \in \{0,1\}^{n+1}$$

Prove that if n is odd, a branch and bound algorithm (without using cuts) will have to examine at least $2^{\lfloor \frac{n}{2} \rfloor}$ candidate problems before it can solve the main problem.

7. Consider the problem of finding a maximum stable set of a graph (a maximum set of vertices with no two vertices sharing an edge). Formulate this problem as:

$$\max \sum_{v \in V} x_v \quad \text{subject to} \quad x_{v_1} + x_{v_2} \leq 1 \quad \forall (v_1, v_2) \in E \quad \text{and} \quad x \in \{0,1\}^{|V|}$$

Show that for any complete subgraph (*clique*) W of G, you can obtain the clique inequality $\sum_{v \in W} x_v \leq 1$ by repeatedly applying rounding cuts.

8. Returning to question 5, what is the worst approximation ratio you personally could get over all possible assignments of weights to edges? What is the worst approximation ratio that any student could get?