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John Charles Fields: A Sketch of His Life and Mathematical Work

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Abstract

Every four years at the International Congress of Mathematicians the prestigious Fields medals, the mathematical equivalent of a Nobel prize, are awarded. The following question is often asked: who was Fields and what did he do mathematically? This question will be addressed by sketching the life and mathematical work of John Charles Fields (1863 - 1932), the Canadian mathematician who helped establish the awards and after whom the medals are named.

Dedication

To The_Tortoise — keep on plugging along...

Acknowledgments

Many people helped me through the writing of this thesis — to many to list here. You know who you are and I thank you very much for all the help you have given me. However, certain individuals do need specific mention. First, I would like to thank my supervisor, Prof. Tom Archibald. Without his guidance, this document would never have been completed. He also aided me in several tasks of translating German into English, including the reviews quoted in Chapter 5 of Fields' work. Also, I would like to mention two members of my family who have been my support "on the ground", so to speak, here in Vancouver during my studies: my mother, Heather, and my brother, Steven. I would also like to mention the staff in the department of mathematics at Simon Fraser University: they were always kind and helpful throughout my studies there.

I would further like to acknowledge the help of the staff at the University of Toronto Archives and James Stimpert of the Milton J. Eisenhower Library at Johns Hopkins University with help finding material related to Fields. The Milton J. Eisenhower library kindly let me use the portrait of Fields in their collection. Lastly, I would like to thank my examiners for their comments on a draft version of this document.

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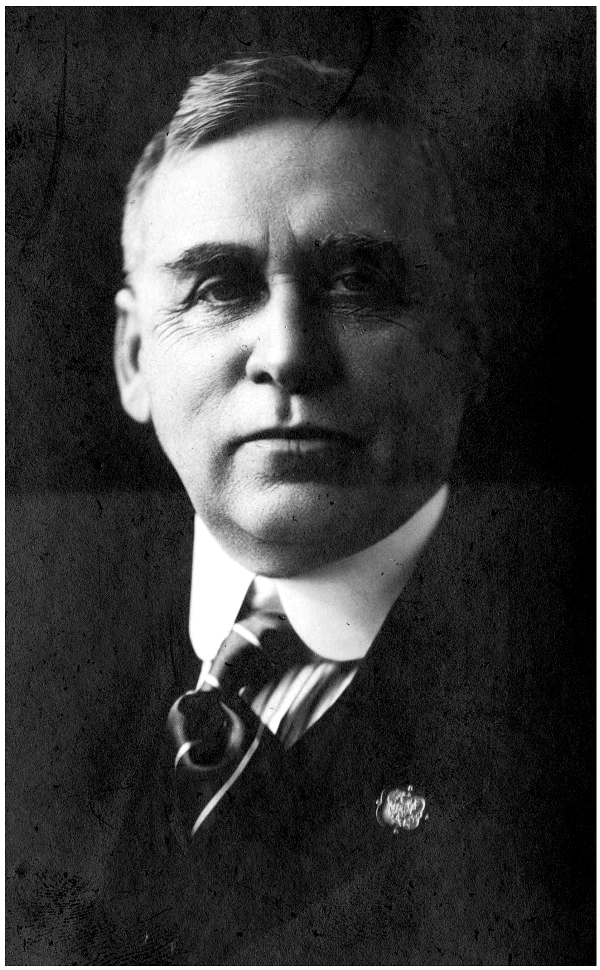

Figure 1: John Charles Fields (1863-1932). © Ferdinand Hamburger Archives of The Johns Hopkins University

Chapter 1

Introduction

Many a mathematician has heard of the foremost award in the mathematical sciences, one often said to be the “Nobel prize” for mathematics, namely, the Fields medal. Few people, mathematicians included, have any idea of exactly who the mysterious Fields was. It may come as a surprise to some that this medal was named after a modest Canadian professor of mathematics at the University of Toronto, John Charles Fields (1863-1932).

Unfortunately, little is known about Fields, except what appears in the standard biographical account of his life, an obituary by J. L. Synge [Syn33]. Much of Fields’ correspondence must be presumed missing or destroyed after it was distributed upon his death in 1932 (many of his possessions were sent to his brother in California [Fie33]). However, the historian is fortunate on some levels. Fields left many notebooks of lectures he either attended or copied from others from a postdoctoral sojourn in Germany in 1890s, spending most of this time in Berlin. There are also files of the minutes of the Toronto International Congress of Mathematics meeting committee held in the archives of the International Mathematical Union, which have allowed scholars like Henry S. Tropp in [Tro76] and Olli Lehto in [Leh98] to detail the early history of the Fields medals and Fields’ role in the organization of the 1924

International Congress of Mathematicians, held in Toronto. For a look at modern mathematics in the light of the Fields medals, see Michael Monastyrsky's book on the subject [Mon97].

Other literature on Fields includes an entry on Fields by Tropp in the Dictionary of Scientific Biography [Tro74]. There is also a small number of general interest accounts of Fields' life and his role in the creation of the Fields medals, and these include articles by C. Riehm [Rie] and by Barnes [Bar03]. Those interested in biographical details of Fields should consult the first part of this thesis, which is primarily biographical in nature. The apparent neglect of work on Fields can be attributed in part to the fact that Fields' research did not use the prevailing methods of his time and would ultimately would be subsumed by other more modern approaches. As a result, his mathematical work has been off the radar of many historians of mathematics.

Regarding Fields' mathematical contributions, it should be noted that Fields' postdoctoral studies in Germany would have a decisive influence on his more mature mathematical work, in particular, Fields' choice to concentrate on algebraic function theory, to which he was given an extensive introduction in Germany by some of the foremost researchers in the area at the time. Fields spent most of his time in Berlin during his postdoctoral stay in Germany. (The standard reference on mathematics in Berlin in the nineteenth century is Kurt-R. Biermann's study [Bie73].) Fields' ideas are an outgrowth of the lectures contained in his notebooks from Germany. This was a bit of an about face for Fields, as he had done his initial research work on differential equations, writing a thesis entitled "Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation $\frac{d^n y}{dx^n} = x^m y$," which was published in the *American Journal of Mathematics* in 1886 [Fie86c]. Following the completion of his doctoral work, he continued along similar lines of thought, using the "symbolic approach in analysis". For more on this topic, the reader should consult Elaine Koppelman's account of the history of the calculus of operations [Kop71], the paper by Patricia R. Allaire and Robert E. Bradley on symbolical algebra in the context of the work of D. R. Gregory [AB02], and I. Grattan-Guiness's chapter on operator methods [GG94], as well as Boole[Boo44], a general account of this method by one of its early proponents, George Boole (1815 - 1864). In the 1890s, Fields wrote on several topics, including a couple of papers in number theory. From 1900 on, Fields wrote almost exclusively on algebraic function theory. This work of Fields can be considered as a study of algebraic curves; thus it comprises an episode in the history of algebraic geometry. Jean Dieudonné in [Die85] has written on algebraic geometry from its prehistory in ancient Greece to its modern theoretical conception in the mid-twentieth century, though Fields' work is not mentioned. Jeremy J. Gray in [Gra98] and [Gra87] has written on the history of the Riemann-Roch theorem, a classical nineteenth century result of theoretical importance to algebraic geometry. Israel Kleiner in [Kle98] has written on the introduction of algebraic ideas, such as ideals and function fields, into algebraic geometry. Fields' mathematical contributions will be discussed chronologically in more detail in part two of this work.

In terms of the context in which Fields functioned as a mathematician during his lifetime, there are several important works that should be read. For an extensive background on progress made on linear differential equations in the nineteenth century, a topic that would occupy the young Fields, one should consult Jeremy J. Gray's volume [Gra99]. Karen Hunger Parshall and David E. Rowe have written a volume on the emergence of the American mathematical research community [PR94]. The account includes a discussion of Johns Hopkins University, the school where Fields chose to do his PhD work, as well as an account of the phenomenon of North American students seeking PhD level training abroad in Germany, something Fields would do himself in the 1890s. Thomas Archibald and Louis Charbonneau have written the most complete preliminary survey of the early history of the Canadian mathematical community before 1945 [AC95]. Fields appears in their account with respect to the establishment of research level mathematics in Canada. For an history of the mathematics department at the University of Toronto, the school where Fields spent most of his professional career, one should consult Robinson's work [Rob79].

We will learn from the sketch of Fields' life and mathematical work, that although he was a modest mathematician in both influence and wealth, Fields ultimately left a very real and important contribution to the modern mathematical landscape, especially for the mathematical community in Canada, by "recognizing the scientific, educational, and economic value" [Abo07] of mathematical research. Fields was a crusader on behalf of this cause, which enabled him (along with the help of others) to secure the first governmental research support for mathematics in Canada. For this reason and others, a world renowned mathematical research institute was named in his honour — The Fields Institute for Research in Mathematical Sciences, located in Toronto, Ontario, Canada.

Part I

A Biographical Sketch of John Charles Fields

Chapter 2

Fields' Youth and Education

2.1 Field's Youth

John Charles Fields was born in Hamilton, Ontario, then Canada West, in 1863, the son of John Charles Fields and his wife, Harriet Bowes. The Fields family lived at 150 King St. East, his father operating a leather shop at nearby 32 King St. West. Both of these buildings no longer exist. There is a Ramada Inn where the family house was located and Jackson Square now occupies the spot where the leather shop once was [Abo07].

J. L. Synge, a friend and colleague of Fields', wrote in his obituary that in his youth, Fields "indulged extensively in sports, baseball, football, hockey, etc.," and while in Australia "learned how to throw a boomerang" [Syn33, 156].

Fields attended secondary school at Hamilton Collegiate where he showed an early talent in mathematics. Following this, in 1880, Fields continued his education by beginning an undergraduate degree at the University of Toronto. In his history of higher education in Canada Harris paints a picture of what the curriculum was like during

Fields' days as a student [Har76, 119-134]. Standard mathematical material likely covered included Euclid's *Elements* books 1-4 and 6, algebra to the binomial theorem, trigonometry, mechanics and hydrostatics. As an honours student in mathematics, Fields was able to go beyond this basic material to cover topics such as conic sections, differential and integral calculus, differential equations, and more advanced topics in applied mathematics [Rob79, 16]. Fields earned a gold medal in mathematics for a distinguished undergraduate career. He graduated in 1884 with a B.A. [Syn33, 153].

2.2 Graduate Study at Johns Hopkins University

At the time of Fields' graduation from the University of Toronto in 1884, it was not possible to obtain a PhD in mathematics at a Canadian institution. It should be noted that at that time, one could get a professorship in mathematics at the university level in Canada without a PhD. Indeed, there were also no British PhDs, yet most professors were British or British trained. So Fields' desire to get a PhD may be seen as an early sign of his feeling that basic scientific research was an important endeavour. To pursue a PhD in the 1880s, there were really two options, either to go to a continental European school or to go to one of the handful of PhD granting institutions in the United States. Fields chose to continue his studies at Johns Hopkins University in Baltimore, Maryland [Syn33, 153].

The school was founded as the result of a bequest of some $7,000,000 by the the Baltimore millionaire, Johns Hopkins (1795-1873). Established in 1876 under the direction of its first president, Daniel Coit Gilman, Hopkins was designed from the start to be a primarily graduate institution where the research productivity of the faculty was of high importance [PR94, 53-54]. Among the 6 initial faculty members was the English mathematician James Joseph Sylvester (1814 - 1897), which was perhaps Gilman's "boldest and riskiest hiring move" [PR94, 58]. Sylvester, whose mathematical career had been frustrated by his Jewish descent, came out of retirement to join the faculty at Johns Hopkins in 1877. As a researcher, Sylvester did important work in invariant theory, number theory, and the theory of partitions. The symbolical or algebraic approach was an important element of his work. At Hopkins, he was set the task of establishing the Department of Mathematics as a place where students could pursue graduate research degrees. He also founded and ran a journal, which would become known as the *American Journal of Mathematics* [Par06, 1-8, 225-277]. Some of Fields' early publications appeared in this journal. Among Sylvester's first batch of graduate students were Fabian Franklin (1853-1939) and Thomas Craig (1855-1900), both of whom would go on to teach Fields at Hopkins, though Craig would teach the majority of graduate courses to Fields. Even though Sylvester left in 1883, a year before Fields attended Johns Hopkins, he would influence Fields' early mathematical work indirectly, as can be seen by the largely symbolical approaches to differential equations and differential coefficients in Fields' work.

Fields arrived at Hopkins in 1884. Fortunately, for many years, lists of classes and their participants were listed in the Johns Hopkins University Circulars. From this source we learn that Fields participated in the following courses and seminars:

First half, 1884-85 [Cir84]

Course:	Instructor/Organizer:
Analytical and Celestial Mechanics	S. Newcomb
Mathematical Seminary	W. Story
Theory of Numbers	W. Story
Modern Synthetic Geometry	W. Story
Mathematical Seminary	T. Craig
Calculus of Variations	T. Craig
Theory of Functions	T. Craig
Problems in Mechanics	F. Franklin

Second half, 1884-85 [Cir85b]

Course:	Instructor/Organizer:
Analytic and Celestial Mechanics	S. Newcomb
Mathematical Seminary	S. Newcomb
Mathematical Seminary	W. Story
Quaternions	W. Story
Modern Algebra	W. Story
Mathematical Seminary	T. Craig
Linear Differential Equations	T. Craig
Theory of Functions	T. Craig

First half, 1885-86 [Cir85a]

Course:	Instructor/Organizer:
Practical and Theoretical Astronomy	S. Newcomb
Mathematical Seminary	W. Story
Finite Differences and Interpolation	W. Story
Advanced Analytic Geometry – Higher Plane Curves	W. Story
Theory of Functions	T. Craig
Linear Differential Equations	T. Craig

Second half, 1885-86 [Cir86b]

Course:	Instructor/Organizer:
Practical and Theoretical Astronomy	S. Newcomb
Mathematical Seminary	W. Story
Theory of Probabilities	W. Story
Advanced Analytic Geometry	W. Story
Mathematical Seminary	T. Craig
Linear Differential Equations	T. Craig
Elliptic and Abelian Functions	T. Craig

First Half, 1886-87 [Cir86a]

Course:	Instructor/Organizer:
Theory of Functions	T. Craig
Abelian Functions	T. Craig

Second half, 1886-87 [Cir87]

Course:	Instructor/Organizer:
Linear Differential Equations	T. Craig

Among the textbooks used for the courses Fields took are treatises on differential equations by Briot, Bouquet, Floquet, and Fuchs; treatises on the theory of functions by Hermite, Broit and Bouquet, Fuchs, and Tannery; and treatises on elliptic and Abelian functions by Cayley and by Clebsch and Gordan. The mathematical seminars were topic based, so that in the second half of the 1884-1885 and 1885-86 years Fields was registered for two different seminars. As can be seen from the tables of courses, Fields took many while pursuing his graduate degree. This pattern of participating in many courses would continue during his study tour in Germany in the 1890s.

Fields received his Ph.D. degree in 1887 with a thesis entitled "Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation $d^ny/dx^n = x^my$." There is no record of who Fields' thesis supervisor was, but it was likely Thomas Craig. There are a couple of reasons to suspect this. First, Craig had research interests in differential equations, the subject of Fields' PhD thesis. Second, Craig was the instructor for most of the courses Fields took while at Hopkins. Other facts that might be pertinent are that Fields' first article published in a mathematical journal was in fact a bibliography on linear differential equations (which he wrote with H. B. Nixon while at Hopkins [FN85]), and that Fields also later wrote a review of Craig's book on linear differential equations [Fie91b]. It is also interesting to note that the topics of many of the courses Fields took with Craig would later be most closely related to Fields' mature research work, namely his work on the theory of algebraic functions. All this said, the issue of the identity of Fields' PhD supervisor is still not fully resolved.

After graduating, Fields became a Fellow at Hopkins (which required a certain amount of undergraduate teaching duties), a position he held until 1889, at which point he was appointed Professor of Mathematics at Allegheny College, a small liberal arts college located in Meadville, Pennsylvania.

He remained in Meadville until 1892, when he resigned his position. The reason for his resignation was that Fields came into a modest inheritance from his father and mother, who died when Fields was 11 and 18 years old respectively [Syn33, 153]. He used his inheritance to pursue post-doctoral studies in Europe. With economical living habits, Fields was able to extend his European studies over 10 years. Fields was well known to be abstemious, "avoiding tea, coffee, alcohol, and condiments, and he did not smoke" [Syn33, 156]. The small amount of money he saved from being so abstemious surely helped him extend his stay somewhat.

With regards to Fields' mathematical output during the years from 1884 to 1893, he produced thirteen publications. These papers include a bibliography of linear differential equation co-authored with fellow Johns Hopkins graduate student H. B. Nixon in 1885 and a paper based on Fields' PhD thesis on symbolic finite and definite integral solutions to the equation $\frac{d^n y}{dx^n} = x^m y$ in 1886. Both papers appeared in the pages of the *American Journal of Mathematics*.

2.3 Post Doctoral Studies in Europe

It is unclear where Fields spent portions of his 10 year study period in Europe. The standard obituary by J. L. Synge states that Fields spent 5 years in Paris and 5 years in Berlin [Syn33, 153]. However, no documentary evidence for his stays in Paris have come to light.[1] It was possible to attend the lectures offered at the Collège de France without having to enrol or pay fees [Arc10, 121]. We do have documentary evidence for Fields' study period in Germany. Fields enrolled in Göttingen in November 1894, where he remained until May 1895. There he had the opportunity to attend lectures by Felix Klein (1849-1925) on number theory, as well as an introductory course on the theory of functions of a complex variable offered by the Privatdozent Ritter. Fields' short stay in Göttingen is interesting, as at the time, Göttingen was the destination of choice for American students pursing studies in mathematics in Germany. For a discussion of this phenomenon, see the fifth chapter of [PR94]. Fields' next stop, Berlin, was much less popular. One possible explanation for Fields' choice of Berlin is the fact that L. Fuchs (1833-1902) and Georg Frobenius (1849-1917) were there. They were both at the forefront of research in linear differential equations, a subject for which Fields showed an early interest.

We are rather well informed on Fields' mathematical activities in Berlin, mostly because a large number of notebooks from this time survive in the Archives at the University of Toronto. Not all these notebooks were taken from lectures Fields attended. Some of the notebooks tend to be more complete, neat, and do not betray signs of wandering attention. These books might be transcriptions from the notes of others. That said, most are arguably first-hand transcriptions.

The notebooks include five courses given by G. Frobenius (1849 - 1917): two on number theory; one on analytic geometry; and two on algebraic equations. There are are notebooks of nine courses given by L. Fuchs (1833 - 1902), including the theory of hyperelliptic and Abelian functions, and topics on differential equations, many related to Fuchs' own work. There are notebooks of six courses given by Kurt Hensel (1861-1941), including algebraic functions of one and two variables, a course on Abelian integrals, and a course on number theory. Hensel's lectures related closely to Fields'

[1] Tom Archibald – personal communication.

later research work. There are notebooks of fifteen courses given by H. A. Schwarz (1843-1921), on such topics as elliptic functions, variational calculus, the theory of functions of a complex variable, synthetic projective geometry, number theory, and integral calculus. Fields also has notes of two courses from G. Hettner (1854-1914) on definite integrals and Fourier series, two courses from J. Knoblauch (1855-1915) on curves and surfaces, and a course from E. Steinitz (1871-1928) on Cantor's theory of transfinite cardinals.[2] In addition to these notebooks we have almost an entire series of lectures by M. Planck, an early edition of what would later be his famous course on theoretical physics; two courses on inorganic chemistry; and one on the history of philosophy.

In the notebooks, Fields took his notes in a mixture of English and German. Some notebooks are rather sketchy, while others, such as those on Schwarz's course on the calculus of variations, are rather complete. And as always, there are the usual signs of a student's mind wandering (for example, see Figures 2.1 and 2.2).

Though the notebooks provide a useful historical resource, Fields makes regrettably few remarks as to his reactions to the material and to his professors. However, Fields' later research work shows the signs of the influence of K. Weierstrass (1815 - 1897) and K. Hensel. Fields' approach to algebraic function theory used several function-theoretic techniques for representing functions along the lines of the work of Weierstrass, but at the same time, Fields' aim was a theory that was arithmetical in nature, the approach favoured by Hensel.

It is interesting to note that Fields' education, both at Johns Hopkins University and during his European study tour, was rich in material that would later become useful for him in his research on algebraic function theory, as a quick glance at the titles of the courses he participated in attests.

[2]The birth and death years for some the instructors of the courses contained in Fields' notebooks have been gleaned from [Sch90].

In terms of Fields' mathematical output during his post-doctoral studies in Europe, Fields did not publish any papers during the years 1894 to 1900, though he seems to have continued to do research, presenting a talk at the meeting of the American Mathematical Society held in Toronto in 1897 on the reduction of the general Abelian integral. He would publish a paper based on this talk in 1901, which would mark the beginning of his mature mathematical research work [Fie01]. It is not really surprising that Fields failed to publish during the years 1894-1900, given the number of courses he apparently attended, as can be ascertained from large number of notebooks full of lecture notes he accumulated during this time.

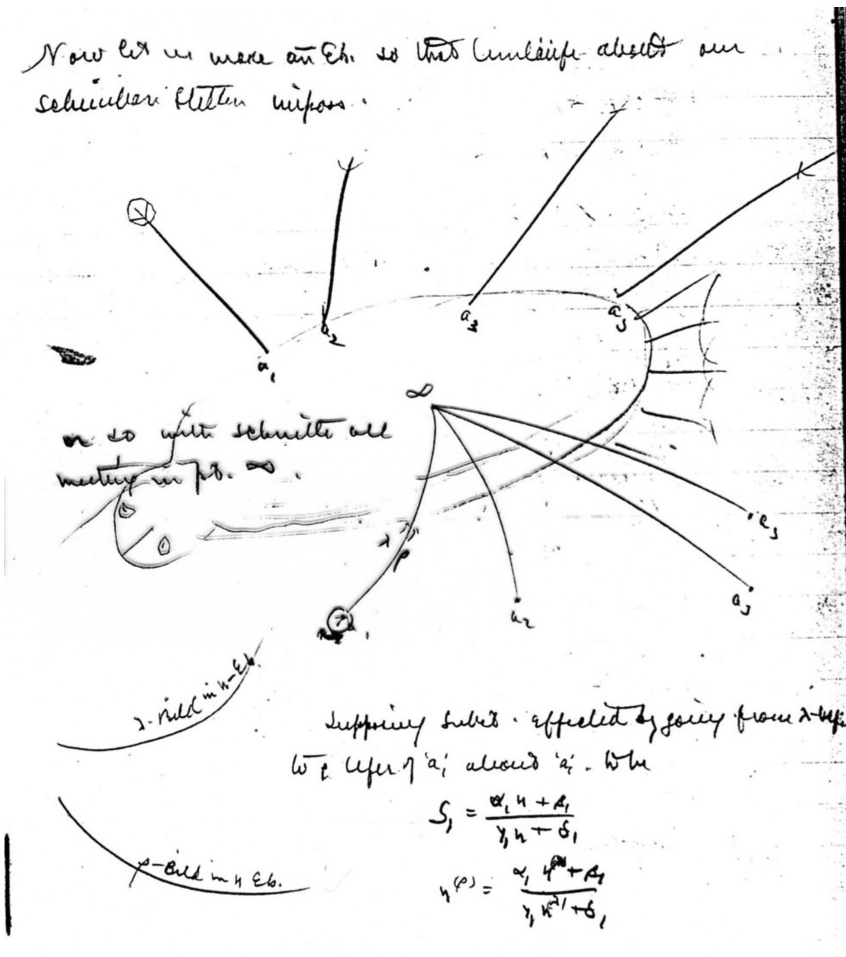

Figure 2.1: A drawing in one of Fields' notebooks showing branch cuts stylized as a bug. UTA (University of Toronto Archives), J. C. Fields, B1972-0024.

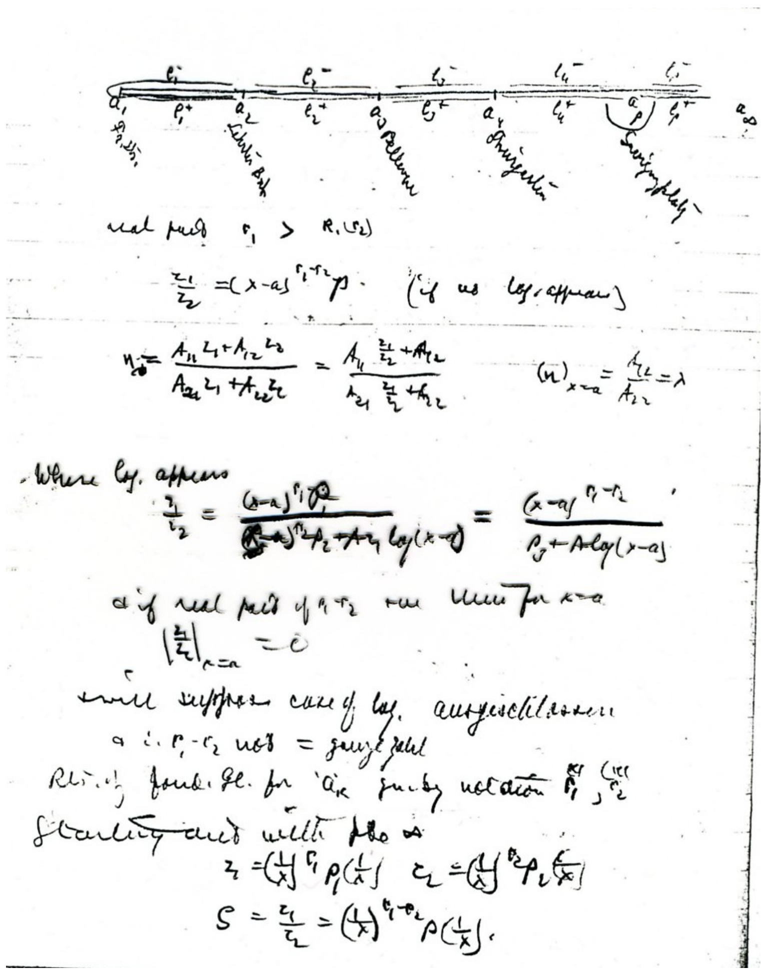

Figure 2.2: A drawing in one of Fields' notebooks showing branch points stylized as a map of the Berlin subway stops in the 1890s. UTA (University of Toronto Archives), J. C. Fields, B1972-0024.

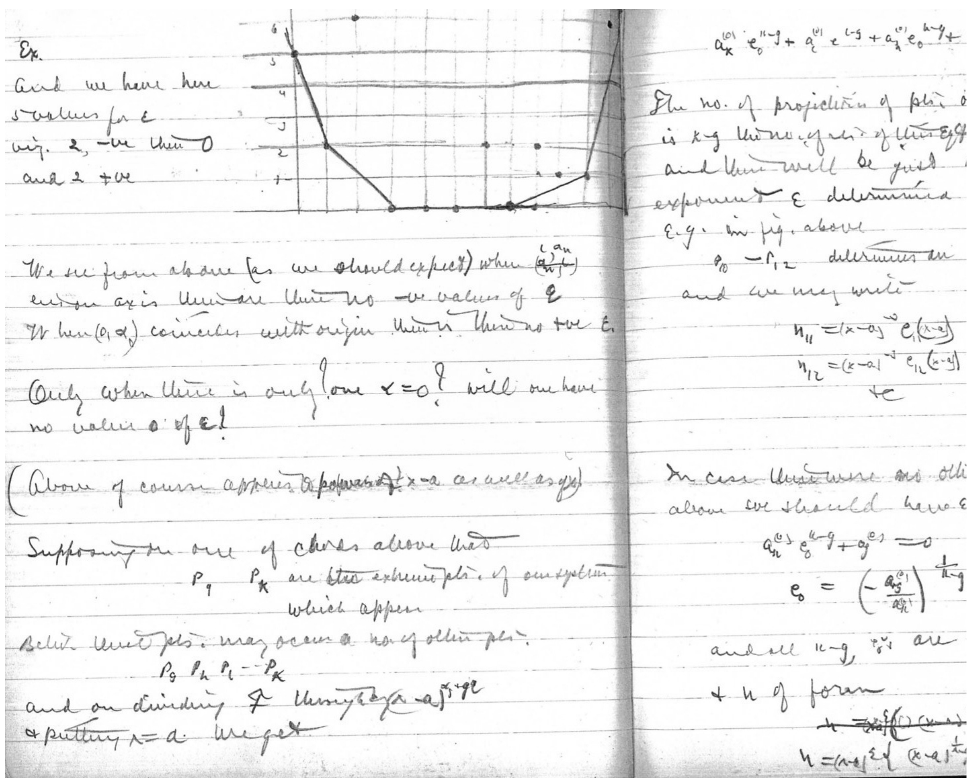

Figure 2.3: A page from Fields' notebook of Hensel's lectures on algebraic functions for 1897-98 showing a diagram of a Newton polygon for an algebraic function. UTA (University of Toronto Archives), J. C. Fields, B1972-0024.

Chapter 3

Professor Fields: 1900-1932

Fields took up a position as special lecturer at the University of Toronto in 1902. At that time the mathematics department had roughly five members including Fields [Rob79, 14]. By 1905 he had gained a regular position as Associate Professor and would later become Professor in 1914 and Research Professor in 1923 [Syn33, 153], 153. Among the honours that Fields received was election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913 [Enr85, 139].

Fields apparently predominantly taught higher level courses. In Robinson's history of the University of Toronto Mathematics Department [Rob79, 18-19] he quotes at length from Norman Robertson, a Toronto lawyer who was born in Orangeville in 1893 and graduated from the University of Toronto in 1914. It is worth quoting parts of this here to give an idea of Fields' teaching style. Robertson reports that in his day in the mathematics and physics program at the university, Fields "did not present any lectures to the earlier years, but lectured to us on differential equations and perhaps quaternions in third and fourth years." Furthermore, Fields "on entering the Lecture Room... armed himself with a large, wet sponge in the left hand, and a piece of chalk in the right hand, and without preliminary, he started at the left panel of the Board, with his back to the class, addressing his remarks to the work he was inscribing on the blackboard. He proceeded right across the front of the room and when he had come to the westerly panel he started back towards the east, erasing all of the work with the wet sponge, and started all over again on the wet first panel!" Fields' lectures were "always formal and were pure theory and delivered without pause or any interval." As well, "it can be understood that it was sometimes difficult to hear him and get the sequence of his argument and in many instances it was difficult to understand the logic of the presentation." However, "if you missed something and went to his room to have it explained, he was most gracious, kind and patient. He would take endless trouble and as much time as was needed to elucidate what had been so ill presented in the lecture room."

On a local level, Fields was active in the life of the university, often giving talks to the mathematics and physics student club [Rob79, 23]. He also successfully lobbied the Ontario legislature for monetary support for scientific research being carried out at the University. The legislature provided an annual grant of $75,000. This figure is quoted from a letter by Robert Alexander Falconer, President of the University of Toronto, to Fields, dated July 2, 1919 [Fal19]. For an idea of how large this sum was for the time, one only need to note that professors only received about $1000 a year in salary [Rie].

Fields was involved with scientific organization on the national level in several ways. He was President of the Royal Canadian Institute from 1919 to 1925. The Royal Canadian Institute, which was founded in 1849 by Sandford Fleming (1827-1915), is the oldest scientific society in Canada; and its Royal Charter of Incorporation, granted in 1851, charged the Institute with the "encouragement and general advancement of the Physical Sciences, the Arts and Manufactures...and more particularly for promoting...Surveying, Engineering and Architecture... " [RCI07].

Fields published thirty-nine mathematical papers in total, and also editing the proceedings of the Toronto International Congress of Mathematicians. Fields had several other publications that were printed versions of addresses he gave to the Royal Canadian Institute. These include his presidential address on "Universities, Research and Brain Waste" which was delivered on November 8th, 1919 at a meeting of the Royal Canadian Institute. In the talk he laments the state of university level research in Canada and the waste of the potential of Canadian scientists and engineers. He discusses what can be done to help rectify the situation by using examples from European schools.

On the international level, Fields was Vice-President of both the British Association for the Advancement of Science in 1924 and the American Association for the Advancement of Science, Section A in 1924. The goal of both organizations is roughly to advance public understanding, accessibility and accountability of the sciences and engineering. More importantly than these presidencies, Fields brought the International Mathematical Congress of 1924 to Toronto.

The account that follows of the International Mathematical Union and the Toronto International Congress of Mathematicians is based on Olli Lehto's volume [Leh98, 23-37] on the history of the Union. In 1919 in Brussels in the wake of World War One, the International Research Council was established. The aim was to be a kind of central control of the international scientific community with responsibility for overseeing international meetings as part of its mandate. The first president of the Council was the mathematician Emile Picard (1856 - 1941). It was at the constitutive meeting in Brussels that the International Mathematical Union was established, the initial executive being C. de la Vallée Poussin (1866 - 1962) and W. H. Young (1863 - 1942). Included among the statutes of the IMU was that the Union was to provide the organization for the ICMs. The statutes of both the International Research Council and the International Mathematical Union excluded the central powers of Germany, Austria-Hungary, Bulgaria, and Turkey due to the animosities resulting from the first great war. As a result, no scientists from these countries could participate in international scientific meetings held under the auspices of the Council, a situation that persisted until June 29, 1926, when an Extraordinary General Assembly of the Council was held in Brussels and the exclusionary statutes on membership were deleted ([Leh98], 40).

The first ICM after World War I was held in Strasbourg, France, in 1920. The Congress was soaked in political controversy because of the exclusionary clauses of the Union. Many mathematicians were loud in their opposition to the exclusionary clauses of the IMU, such as G. H. Hardy (1877 - 1947), stating that "All scientific relationships should go back precisely to where they were before [the war].... " (Quoted from [Leh98, 30]). Gösta Mittag-Leffler was also among the more prominent and influential mathematicians opposed to the exclusion of the "central powers". This political controversy carried over to the International Congress in Toronto.

The American delegates to the Strasbourg General Assembly of the Union offered to host the 1924 Congress without first consulting the American Mathematical Society. The Society was not fully supportive of holding the Congress in the United States because of the political controversy. By 1922 it was felt that the political climate would have made financial backing unattainable in the United States because of the restrictions on participation in the Congress as a result of the exclusionary clauses of the Union. The American Mathematical Society withdrew its support for organizing the Congress.

Fields apparently jumped at the opportunity to organize the Congress, which was to be held in Toronto in 1924 despite the political controversy [Leh98, 33-37]. A colleague of Fields', Professor J. L. Synge of the University of Toronto, wrote: "I do not think that he himself approved strongly of the prohibitory clauses in the regulations of the Union; indeed, I feel confident that he would have welcomed the opportunity of organizing a truly international congress in Canada" [Syn33, 154].[1] However, his desire to have the meeting in Canada apparently outweighed his reservations. Fields took it upon himself to help promote the conference. He travelled across the Atlantic several times before the congress in order to arrange details personally and to help gain the participation of important figures. Incidentally, Fields travelled overseas many times during his adult life. There is an amusing anecdote about an incident during one such trip, which appeared in print in *The Star* on August 10, 1932, in a short article entitled "Death Claims Noted Savant at University". The anecdote is as follows: "Dr. Fields was a bachelor, and an amusing story is told of his 'love affairs' abroad. On one occasion, while abroad, he arranged to meet Professor Love at a certain hotel. When Dr. Fields arrived there, the girl clerk presented him with a telegram reading: `Sorry, I cannot meet you, Love`. The doctor treasured this evidence of his 'love affairs.'"[2]

The Toronto congress was quite successful with 444 people registered, which is over double the first post-war meeting in Strasbourg. For comparison, 574 mathematicians had been in attendance at the last Congress before World War One, held in Cambridge, England in 1912, an event Fields attended [Leh98, 15]. Of the 444 registrants of the Toronto Congress, approximately 300 were from North America. Part of the Congress was an excursion across Canada, from Toronto to Victoria on Vancouver Island. Fields put much work into guaranteeing that the Toronto Congress would be a success.

After the 1924 Congress, Fields' health began to deteriorate. With the help of colleagues, Fields was able to complete the proceedings of the Toronto Congress, which

[1] Fields wrote in 1930 that he "visited a number of European mathematicians during the Summer with a view to learning more of the international situation as it affects mathematics" and that he "happens to be persona grata to the Germans as well as to the French" and that it occurred to him "that it might be possible to do something towards smoothing things out and bringing a rapprochement" [Fie30, 4].

[2] Augustus Love (1863 - 1940), a British mathematician, held the Sedleian chair of natural philosophy at Oxford starting in 1899. He did important research on the mathematical theory of elasticity and geodynamics.

appeared in two large volumes in 1928, though as Fields shared in the prefatory note in the first volume of the proceedings, it “was an unexpected and none too welcome task which fell to the lot of the undersigned [i.e., Fields] when circumstances determined that he should edit the Proceedings” [Fie28]. The work fell mostly on Fields’ shoulders due to the fact his colleague, Professor J. L. Synge, General Secretary of the Congress, was leaving Toronto for a job as chair at Trinity College, Dublin and it was felt that the duty of editing the proceedings should be undertaken by someone in Toronto who could see the proceedings through the press [Fie28].

On February 24, 1931, in the minutes of a meeting of the organizing committee, it was first reported that, after meeting the expenses of holding the Congress and publishing the Proceedings, there was a positive balance of $2,700. The committee then continued its report by stating its intention to use most of this money to establish two medals to be awarded at, and in connection with, successive International Mathematical Congresses [Tro76, 170].

At the next recorded meeting of the committee (with Fields still chairman) on January 12, 1932, Fields reported that the major mathematical societies of the U.S., France, Germany, Switzerland, and Italy had indicated support for the introduction of an international medal in mathematics. A memorandum entitled “International Medals for Outstanding Discoveries in Mathematics” attached to the minutes of the meeting and signed by Fields, outlined the idea of the medals. In this memorandum, it was stated that the medals “should be of a character as purely international and impersonal as possible.” And further that “there should not be attached to them in any way the name of any country, institution or person [Tro76, 174].” Obviously, Fields’ wish that the medals should not be named after a person was ignored. The memorandum further set out the nature of the awards, that one should “make the awards along certain lines not alone because of the outstanding character of the achievement but also with a view to encourage further development along these lines [Tro76, 174].”

Figure 3.1: J. C. Fields' gravestone:"John Charles Fields, Born May 14, 1863, Died August 9, 1932." © Carl Riehm. Used with permission.

It was Fields' successful fund-raising for the Congress that was to make the initial steps towards the establishment of the Fields medals. There was a further source of initial funds: Fields organized not only the ICM, but also a second conference, a meeting of the British Association for the Advancement of Science, held in September 1924 in Toronto. In the end, part of the surplus money went to the Royal Canadian Institute to fund research (Toronto *Mail*,, May 4, 1925). Eventually $2500 of the surplus would go towards the medals [Tro76, 170].

Fields died in Toronto on August 9, 1932, apparently from a stroke. He was buried in Hamilton Cemetery, which overlooks the western end of Lake Ontario (at "Cootes Paradise", where McMaster University also sits). His simple gravestone is about 22 inches by 16 inches and is set flat on the ground. It reads "John Charles Fields, born May 14, 1863, died August 9, 1932." A passer-by, reading this very modest stone, would have no idea of Fields' influence on the mathematical and scientific community. Upon his death, Fields left an estate of $45071, a large part of which went towards the medals.[3] He left his brother with a small annuity, and his maid Julia Agnes Sinclair, widow, a small pension as long as she remained unmarried. He left the money to be used towards the awards in the hands of Professor Synge and the prime minister of Canada.

[3] *Toronto Star*, Sept. 30, 1932.

Part II

A Sketch of John Charles Fields' Mathematical Contributions

As previously noted, Fields published thirty-nine mathematics-related publications [Syn33, 159- 160]. Fields' published mathematical work can be divided into two periods: from 1885 to 1893 and from 1901 to 1928. The year 1885 marks the year of his first publication, a bibliography on linear differential equations that was published in the *American Journal of Mathematics*. The year 1928 marks a publication in the proceedings of the International Congress of Mathematicians in Bologna, Italy. After 1928, Fields' health was poor and his mathematical output more or less ended. The years 1894 to 1900 mark Fields' post-doctoral studies in Europe. That Fields failed to publish anything during this time can be accounted for by the many courses he apparently took, as his large collection of lecture notes attests. That said, he apparently did some research during his European study tour, presenting some of this work at the American Mathematical Society meeting in Toronto held in 1897. The work presented at the meeting would later mark the beginning of the publication of his more mature mathematical work in 1901 [Fie01, 49].

The period 1885-1893 will concern us in Chapter Four. A generalization of Riccati's equation was the topic of Fields' PhD thesis, which we will examine in section 4.1. In addition to this work, Fields published proofs of some well known theorems such as the elliptic function addition-theorem and the fundamental theorem of algebra. These two papers plus an another one on Euler and Bernoulli numbers will be discussed in section 4.2. Fields published other work during the years 1885 to 1893, such as his papers entitled "A Simple Statement of Proof of Reciprocal-Theorem" [Fie91c], "Transformation of a System of Independent Variables" [Fie92], and "The numbers of sums of quadratic residues and of quadratic non-residues respectively taken n at a time and congruent to any given integer to an odd prime modulus p" [Fie93], but we will not have occasion to discuss them here.

After Fields' post-doctoral studies in Europe, he concentrated almost entirely on the theory of algebraic functions and the related theory of Abelian integrals. The period 1901-1928, as noted earlier, marks the publication of Fields' more mature mathematical work. The theory of algebraic functions and Fields' conception of the theory will the the focus of Chapter Five.

In the final chapter, we will try to answer the question of whether Fields' mathematical work had any sort of influence or lasting legacy. It will be argued that despite the fact that Fields' approach ultimately did not gain wide use in further research within the mathematical community, his work did play another and important role for mathematics, and especially for the developing Canadian mathematical community.[4]

[4] S. Beatty has written that Fields "by his insistence on the value of research as well as by the importance of his published papers, has, perhaps, done most of all Canadians to advance the cause of mathematics in Canada" [Bea39, 109].

Chapter 4

Skill in manipulations

In this chapter we will look at some of Fields' early research work. At this point in his career, Fields was still trying to establish himself as a research mathematician and he still had not narrowed in on what would become his main research program.

Fields published his first paper in 1885, written with H. B. Nixon, a fellow graduate student in the mathematics department while Fields was at John Hopkins University, in the *American Journal of Mathematics*, which had just been established in 1878 as the first research level journal in North America devoted solely to mathematics. The paper was an extensive bibliography of work on linear differential equations [FN85]. Fields would go on to publish several other papers that would appear in this journal. As Fields' student S. Beatty would observe in his summary of the mathematical contribution of his former teacher, Fields' early papers "offer simplifications of existing treatments, while others give extensions, secured chiefly by skill in manipulation" [Syn33, 156].

4.1 Fields' PhD Thesis

Fields' PhD thesis, entitled "Symbolic Finite Solutions and Solutions by Definite Integrals of the Equation $d^ny/dx^n = x^my$", was among those of his papers published in the *American Journal of Mathematics*. Finite solutions are in closed form, unlike those expressed in series or as integrals. The meaning of the term "symbolic" solution is not so easy to grasp. We will try to untangle the meaning in section 4.1.1, primarily by example. Solutions by definite integrals, also known as solution by "quadratures," are solutions by integrals where boundary conditions on the integrals have been specified. For a survey of the history of ordinary differential equations, one may consult Kline ([Kli90a], 468-501, 709-738), Archibald [Arc03], and Gilain [Gil94].

The equation $\frac{d^ny}{dx^n} = x^my$, which is the focus of Fields' thesis, is similar to certain Riccati equations. Jacopo Riccati (1676-1754), an Italian nobleman and mathematician, studied certain second order differential equations. The differential equation

$$\frac{dy}{dx} = Ay^2 + Bx^n,$$

A and B constant, became known as Riccati's equation [BD77, 7][Gil94, 441-442]. In generalized form it is usually written as

$$\frac{dy}{dx} = a_0(x) + a_1(x)y + a_2(x)y^2.$$

L. Euler (1707 - 1783) showed that by a suitable change of function, the generalized Riccati equation and the second-order homogeneous linear differential equation are equivalent [Gil94, 442]. This amounted to showing that the general Riccati equation

$$\frac{dw}{dx} = q_0(x) + q_1(x)w + q_2(x)w^2,$$

leads to the second-order linear homogeneous equation

$$q_2(x)y'' - \left[q_2'(x) + q_1(x)q_2(x)\right]y' + q_2^2(x)q_0y = 0,$$

via the transformation $w = -y'/yq_2$ [BD77, 112]. If $q_2(x) = 1$ and $q_1(x) = 0$, then this last equation reduces to

$$\frac{d^2y}{dx^2} + q_0(x)y = 0. \tag{4.1}$$

This clarifies the connection to Fields' equation

$$\frac{d^ny}{dx^n} = x^m y$$

and (4.1). Rather than taking a second-order equation, we take an nth order equation, and we set $q_0(x) = -x^m$ for some m.

As Fields points out at the beginning of his thesis, the finite solutions obtained are analogous to the symbolic solutions of Riccati's equation. As J. W. L. Glaisher notes in his 1881 survey article on Riccati's equation, many symbolic solutions to Riccati's equation had been given, by (for instance) R. L. Ellis, Boole, Lebesgue, Hargrave, Williamson, Donkin, and Gaskin [Gla81, 762]. Glaisher discusses some of these symbolic solutions in the sixth section of his survey. It should be pointed out that Fields' symbolic solutions to the equation under investigation in his thesis are rather more complicated than those of the Riccati equation presented in [Gla81], in large measure because Fields considers an nth order equation instead of simply a second-order equation.

In the last half of the nineteenth century, linear differential equations was a topic that received much attention from some of the world's leading research mathematicians, though the main breakthroughs were generated by considering the differential equations in terms of functions of complex variables, rather than simply of real variables as in Fields' work [Gra99]. Fields would go on to hear lectures by some of these researchers in Berlin during his European study tour in the 1890s, including lectures by L. Fuchs, Frobenius, and Schwarz. He also attended a lecture course by Klein while spending a year in Göttingen and seems to have at least communicated with H. Poincaré, another researcher who made fundamental progress on the theory [Gra99]. In a letter to D.C. Gilman, President of Johns Hopkins University, Baltimore, Fields writes that “Poincaré has also in his hands at present a paper on the ‘change of any system of independent variables’, which he has kindly promised to present before the Société Mathématique de France” [Fie]. The paper of Fields’ that he was referring to was [Fie92].

4.1.1 Finite Symbolic Solutions.

The basic idea of the symbolic method is to treat the operations of analysis, such as differentiation, as if they were operators or symbols. These operators can then be manipulated in ways analogous to the way variables are treated in basic algebra. Fields uses operator methods extensively in his thesis, as well as in some of his early research papers, such as [Fie89].

The operator idea goes back at least as far as the work of Gottfried Wilhelm Leibniz (1646 - 1716) on the differential d operating on a variable x to produce the infinitesimal dx. In fact, Koppelman in [Kop71, 158-159] traces the the history of the “calculus of operations” back to a letter dated 1695 by Leibniz, addressed to Johann Bernoulli, that discusses the analogy between raising a sum to a power and the differential of a product. The development of the basic results in the calculus of operators is mainly of French origin. Joseph Louis Lagrange (1736 - 1813) stated many of the initial results. Lagrange’s results were later proven by P. S. Laplace in 1776 [Kop71, 159-160]. Lagrange’s work helped spark other work in this area by French mathematicians, including that by L. F. A. Arbogast (1759 - 1803), B. Brisson (1777 - 1828), and Augustin Louis Cauchy (1789 - 1857). In 1807, Brisson pioneered the idea of thinking of a differential equation as a differential operator on a function from which one could use the algebra of operators to produce solutions. Later, Cauchy founded operator based procedures similar to Brisson’s [GG94, 545-546][Kop71, 158-175].

Operator based methods were particularly attractive to English and Irish mathematicians in the nineteenth century. Building on the work in the 1830s by R. Murphy and D. F. Gregory (1813 - 1844), George Boole's paper "On a General Method in Analysis," which appeared in the Philosophical Transactions of the Royal Society in 1844, examined in depth the basic ideas of the differential operator method [Boo44]. He noted that these operators obey three properties: commutativity, distributivity, and the index law (that is, that $D^n D^m = D^{m+n}$, m and n positive integers), though care had to sometimes be taken with respect to commutativity.

Many mathematicians used differential operator methods to solve differential equations. For example, Pierre Simon Laplace (1749 - 1827) used such methods in his research into the shape of the Earth. By the end of the nineteenth century, basic methods were included in textbooks, such as R. D. Carmichael's *A Treatise on the Calculus of Operations* (1855) and Boole's *A Treatise on Differential Equations* (1859) and *A Treatise on the Calculus of Finite Differences* ([GG94], 548). It should be noted that Fields was likely familiar with Boole's text. In Robinson's history of the University of Toronto mathematics department appears a list of material taught during Alfred Baker's days there, and among the texts listed is Boole's text on differential equations [Rob79, 16].[1] One might venture a guess that Fields came across some of the material in these books during his early mathematical education, since using textbooks from England was the norm in Canada.

The principal hope with using differential operator methods to solve differential equations was to be able to express solutions in finite form rather than as infinite series. The overall success of the differential operator approach was somewhat limited however, since many important differential and difference equations were not easily solved using these methods [GG94, 548].

[1]Baker joined the staff at the University of Toronto in October 1875 as a Mathematical Tutor. So it is likely he taught the undergraduate Fields. Baker went on to become a professor of mathematics at the university and a senior colleague of Fields' [Rob79, 16].

To help illustrate the differential operator method as applied to differential equations, we will look at the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0, \tag{4.2}$$

which is discussed briefly in [AB02, 414-415] in the context of a more general discussion of D. F. Gregory's "symbolical algebra", which is essentially the same set of methods as we have been referring to as the symbolical method or the operational calculus.

The basis of the symbolical method in analysis is the technique of the *separation of symbols*. This technique consisted of distinguishing between two essential types of symbols, corresponding to the modern notion of functions and linear operators. The later were to be separated from the former in order to help derive and prove theorems in analysis [AB02, 403]. When we apply the technique of separation of symbols to (4.2), we get the auxiliary equation

$$(D^2 + 3D + 2)y = 0,$$

where were have denoted the differential operator $\frac{d}{dx}$ by D. Note that in the last equation we have separated the differential operator from the function y. Now this expression can be factored to give us

$$(D + 1)(D + 2)y = 0,$$

from which we recognize the particular solutions $y = e^{-x}$ and $y = e^{-2x}$ to be the solutions to $(D + 1)y = 0$, $(D + 2)y = 0$. The general solution is then the linear combination of the two particular solutions [AB02, 414-415]. Of course this is but a simple example and the operational calculus is in general much more difficult, but this example highlights the basic technique, which when applied skilfully, can generate results like those discussed in Fields' doctoral thesis. For a more thorough look at symbolical methods in analysis, one should consult Boole's writings [Boo44] and [Boo59], as well as the papers by Koppelman [Kop71] and Allaire and Bradley [AB02] which give a historical perspective on the method.

Having now seen an example of a basic application of symbolical methods to the solution of differential equations, we will set forth two basic results that Fields uses during the course of his dissertation, to which we will have chance to refer later on.

Result 4.1 *The general solution to an nth order homogeneous differential equation with constant coefficients,*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

whose characteristic roots $\lambda_1, \ldots, \lambda_n$ *are all distinct, is*

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

for some constants $c_1, \ldots, c_n$.

The proof of this result and related results for when the characteristic roots are not distinct can be found in any standard text on differential equations, for example ([Bro93], chapter 9).

Result 4.2 *Given a function* f *in terms of* x, *we have*

$$e^{\frac{d}{dx}} f(x) = \left(1 + h\frac{d}{dx} + \frac{1}{1 \cdot 2}h^2 \frac{d^2}{dx^2} + \cdots\right) f(x) = f(x+h).$$

As Boole points out [Boo59, 389] no direct meaning can be given to the expression $e^{\frac{d}{dx}} f(x)$, but if one were to treat the exponent $\frac{d}{dx}$ as a quantity, the result follows from Taylor's theorem. We can interpret the symbolical terms in the parenthesis as performing an operation on the function f. It should be noted that each of these operational symbols can actually be carried out, even though doing so may be very labour intensive. The symbol $e^{\frac{d}{dx}}$ is often dealt with very much like the symbol e^{At}, where A is a square matrix, as treated in modern theories of linear operators such as that presented in [DS58], namely,

$$e^{At} = I_n + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n t^n.$$

4.1.2 Finite Solutions of $\frac{d^3y}{dx^3} = x^m y$.

Fields' thesis work is good example of what "skill in manipulations" can achieve [Syn33, 156]. However, this makes reading his thesis research rather long and tiresome, as one must go through every single detail of those quite lengthy manipulations. Fields breaks his discussion of symbolic finite solutions of $\frac{d^n y}{dx^n} = x^m y$ into two parts, one for the case $n = 3$ in order to give a more concrete impression of his technique, the second for the equation of the nth order. Luckily for Fields, his technique for $n = 3$ generalizes quite nicely to the nth order case. For sake of the exposition, we will abbreviate his argument for $n = 3$. Throughout his thesis, Fields considers the symbols such as $\frac{d}{d\Delta}$ and $\left(\frac{d}{d\Delta}\right)^{-1}$ which obey the commutative and distributive laws.

The first step in finding a solution when $n = 3$ is to suppose that

$$\frac{d^3y}{dx^3} = x^m y \tag{4.3}$$

is satisfied by the series $y = \sum a_n x^{n\alpha}$ where $\alpha = m + 3$. Using standard series methods, this gives

$$y = \sum_{n=0}^{\infty} \frac{\alpha^{-3n} x^{n\alpha}}{n!(1+\nu_1)\cdots(n+\nu_1)(1+\nu_2)\cdots(n+\nu_2)}, \tag{4.4}$$

where $\nu_1 = \frac{-1}{\alpha}$, and $\nu_2 = \frac{-2}{\alpha}$.

At this point, Fields introduces the symbol $\Delta = \left(\frac{d}{dz}\right)^{-1}$ so that

$$\Delta^n z^m = \frac{z^{m+n}}{(1+m)\cdots(n+m)}.$$

By setting $z = \alpha^{-3} x^{\alpha}$ in (4.4), Fields is able to find that

$$y = \sum \frac{z^{-\nu} \Delta^n z^{\nu_2}}{n!(1+\nu_1)\cdots(n+\nu_1)} \tag{4.5}$$

$$= z^{-\nu_2} \Delta^{-\nu_1} \sum \frac{\left(\frac{d}{d\Delta}\right)^{-n}}{n!} \cdot \Delta^{\nu_1} = z^{-\nu_2} \Delta^{-\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{\nu_1} \cdot z^{\nu_2}, \tag{4.6}$$

where "the functional symbol $\Delta^{-\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{\nu_1}$ is supposed to operate upon z^{ν_2}" [Fie86c, 368]. Fields does not explain the nature of the $\frac{d}{d\Delta}$ operator. One possible way to make sense of it is to note that since $\Delta = \left(\frac{d}{dz}\right)^{-1}$, we have that

$$\frac{d}{d\Delta} = \frac{d}{d\left(\frac{d}{dz}\right)^{-1}} = \frac{d}{d\left(\frac{dz}{d}\right)} = \frac{d}{dz}.$$

What exactly does one do in order to operate upon z^{ν_2} with $\Delta^{-\nu_1} e^{\left(\frac{d}{d\Delta}\right)^{-1}} \Delta^{\nu_1}$? One first applies Δ^{ν_1} to z^{ν_2}, to the result $\Phi(z)$ of which, one then applies $e^{\left(\frac{d}{d\Delta}\right)^{-1}}$, which means roughly, "expand using Taylor's theorem" to get

$$\left(1 + \left(\frac{d}{d\Delta}\right)^{-1} + \frac{1}{2}\left(\frac{d}{d\Delta}\right)^{-2} + \cdots\right)\Phi(z).$$

This expansion follows from (Result 4.2). One then applies $\Delta^{-\nu_1}$, which can also be written as $\left(\frac{d}{dz}\right)^{\nu_1}$, to the preceeding, which roughly means "differentiate the expression term by term ν_1 times."

This settled, Fields then shows that if the right hand side of (4.6) is "known and finite" [Fie86c, 368] for given values ν_1, ν_2 (arbitrary), then the expression also holds for all values of ν_1, ν_2 differing from those given values by integers. Thus if the new values are $\nu_1 - i$, $\nu_2 - k$, then one can simply apply the operation

$$z^{i-\nu_1}\left(\frac{d}{dz}\right)^{i} z^{\nu_1-\nu_2+k}\left(\frac{d}{dz}\right)^{k} z^{\nu_2}$$

to the known function containing the values ν_1 and ν_2 to get new values of the function [Ham07].

At this point, Fields narrows in on actual solutions to $\frac{d^3y}{dx^3} = x^m y$. He is able to do this by noting that in the previous equation, if $m = 0$, then y is known, so either $\nu_1 = -\frac{1}{3}$, $\nu_2 = -\frac{2}{3}$ or $\nu_1 = -\frac{2}{3}$, $\nu_2 = -\frac{1}{3}$. In the first case, we must have that

$$-\frac{1}{3} - i = \frac{1}{-\alpha} = -\frac{1}{m+3},\quad -\frac{2}{3} - k = -\frac{2}{\alpha},$$

from which one can conclude that $m = \frac{-9i}{3i+1}$. Similarly, in the second case, we have that $m = \frac{-3(3i+1)}{3i+2}$. In both cases, the differential equation is solvable in finite terms for any whole number i [Ham07].

We may summarize Fields' results from the first portion of his thesis as follows. The general solutions of (4.3) are

Case 1: $m = \frac{-9i}{3i+1}$

$$y = x\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^{i} x^{1+\frac{3i}{3i+1}}\left(x^{1-\frac{3}{3i+1}}\frac{d}{dx}\right)^{2i} x^{-\frac{2}{3i+1}} \tag{4.7}$$

$$\times\left(C_1 e^{-(3i+1)\lambda_1 x^{\frac{1}{3i+1}}} + C_2 e^{-(3i+1)\lambda_2 x^{\frac{1}{3i+1}}} + C_3 e^{-(3i+1)\lambda_3 x^{\frac{1}{3i+1}}}\right); \tag{4.8}$$

Case 2: $m = \frac{-3(3i+1)}{3i+2}$

$$y = x\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^{i} x^{1-\frac{3i}{3i+2}}\left(x^{1-\frac{3}{3i+2}}\frac{d}{dx}\right)^{2i+1} x^{-\frac{1}{3i+2}} \tag{4.9}$$

$$\times\left(C_1 e^{-(3i+2)\lambda_1 x^{\frac{1}{3i+2}}} + C_2 e^{-(3i+2)\lambda_2 x^{\frac{1}{3i+2}}} + C_3 e^{-(3i+2)\lambda_3 x^{\frac{1}{3i+2}}}.\right) \tag{4.10}$$

where in the above, the C_is are arbitrary constants and the λ_is are here the cube roots of unity.

Having dealt with $\frac{d^n y}{dx^n} = x^m y$ for the case $n = 3$, Fields moves on in the next section of his thesis to deal with nth order case. Luckily, the techniques used by Fields in the case $n = 3$ generalize well, but things get messier. But again, with the assumption that the solution to (4.3) is known for $m = 0$, Fields is able to show that (4.3) is solvable in finite form for all values of

$$m = \frac{-n(ni + k - 1)}{ni + k},$$

where k is a whole number less than and relatively prime to n, and i is an arbitrary integer [Ham07].

In the next section of his dissertation, using the methods suggested by the ideas developed in the first part of his thesis, Fields solves a couple of simpler differential equations, the details of which need not occupy us here.

4.1.3 Solutions by Definite Integrals

A solution to a differential equation is said to be a solution by definite integrals if the solution (a function) is written as a definite integral (possibly a multiple definite integral). This form of solution also sometimes referred to as solving a differential equation by quadratures, especially in older literature. Solving differential equations by definite integrals arguably has been used since the foundational work by G. Leibniz (1646 - 1716) and I. Newton (1643 - 1727) on the calculus, specifically with their work on the fundamental theorem of calculus.[2] The methods of Leibniz in particular were expanded upon by Johann and Jakob Bernoulli, and then shortly thereafter by L. Euler (1707 - 1783). These three individuals studied physical phenomena via differential equations and as a result contributed many of the elementary results of differential equations, including solutions by definite integrals. For more on this see [Jah03], [Arc03], and [Gil94].

Fields' results in this section are a generalization of results obtained by E. Kummer (1810 - 1893) and Spitzer for the solution of $\frac{d^n y}{dx^n} = x^m y$ by definite integrals. Furthermore, by using some of the methods from the first part of his thesis, Fields is able to give particular solutions in definite integrals for given values of m.

In order to get a better grip on what Fields accomplishes in this section, it will be worthwhile to look at the work of Kummer's that Fields cites, namely [Kum39]. Kummer's paper builds on the work of R. Lobatto in [Lob37] who posed $\frac{d^n y}{dx^n} = x^m y$ as an object of further research in mathematical analysis [Kum39]. Kummer only considers the case where m is a positive integer. Spitzer finds the general solution in definite integral form when m has absolute value greater than $2n$.

Kummer's results, stated right at the beginning of his paper are the following. If

[2]Much has been written on Leibniz and Newton's work on the calculus. For more on this, one should consult [Gui03], [Gui94], and Newton's collected mathematical papers with notes by D. T. Whiteside [New81].

$z = \Psi(x)$ is a general solution of

$$\frac{d^{n+1}z}{dx^{n+1}} = x^{m-1}z, \tag{4.11}$$

then the general solution of the equation

$$\frac{d^n y}{dx^n} = x^m y \tag{4.12}$$

can be expressed by the definite integral

$$y = \int_0^\infty u^{m-1} e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\ du, \tag{4.13}$$

where the $n+1$ constants of integration satisfy a certain relation and where the limits of integration have been chosen to "make things work out."

In order to demonstrate his result, Kummer begins thus. First differentiate (4.12) to get

$$\frac{d^{n+1}y}{dx^{n+1}} = x^m \frac{dy}{dx} + mx^{m-1}y. \tag{4.14}$$

Substituting into this last equation the expressions for y, $\frac{dy}{dx}$ and $\frac{d^{n+1}y}{dx^{n+1}}$, and observing that equation (4.11) gives us

$$\frac{d^{n+1}\Psi(xu)}{dx^{n+1}} = x^{m-1}u^{m+n}\Psi(xu),$$

we get

$$x^{m-1} \int_0^\infty u^{2m+n-1} e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\ du \tag{4.15}$$

$$= x^m \int_0^\infty e^{-\frac{u^{m+n}}{m+n}} \Psi'(xu)\ du + mx^{m-1} \int_0^\infty u^{m-1} e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\ du. \tag{4.16}$$

As Kummer points out, one can verify the preceding equation by first differentiating $u^m e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)$ in terms of u to get

$$d\left(u^m e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\right) = mu^{m-1} e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\ du$$
$$-u^{2m+n-1} e^{-\frac{u^{m+n}}{m+n}} \Psi(xu)\ du + mu^m e^{-\frac{u^{m+n}}{m+n}} \Psi'(xu)\ du.$$

By now multiplying this equation by x^{m-1} and integrating from 0 to ∞, we get (4.15)-(4.16).

Thus the expression given in (4.13) for y is the general solution of (4.14), and as a consequence also expresses the general solution of (4.12), if the $n+1$ constants of integration satisfy a certain relation.

At this point, Kummer works out the condition on the constants of integration, case by case, successively using his result above and the solution of the equation $\frac{d^n z}{dx^n} = z$. Designating the solution of $\frac{d^n z}{dx^n} = z$ by $\Psi(n, x)$, we get the well known result that

$$\Psi(n,x) = Ce^x + C_1 e^{xe^{\frac{2\pi i}{n}}} + C_2 e^{xe^{\frac{4\pi i}{n}}} + \cdots + C_{n-1} e^{xe^{\frac{2(n-1)\pi i}{n}}}.$$

Thus the main result of Kummer paper states that the solution to

$$\frac{d^n y}{dx^n} = xy$$

is given by

$$y = \int_0^\infty e^{-\frac{u^{n+1}}{n+1}} \Psi(n+1, xu)\, du.$$

From the fact that $\frac{d^n y}{dx^n} = 0$ when $x = 0$, we find the relation among the $n+1$ constants of integration, namely

$$C + e^{-\frac{2\pi i}{n+1}} C_1 + e^{-\frac{4\pi i}{n+1}} C_2 + \cdots + e^{-\frac{2n\pi i}{n+1}} C_n = 0.$$

For the case $m = 2$, the solution of

$$\frac{d^n y}{dx^n} = x^2 y$$

is

$$y = \int_0^\infty \int_0^\infty v e^{-\frac{u^{n+2}+v^{n+2}}{n+2}} \Psi(n+2, xuv)\, du\, dv,$$

and since $\frac{d^n y}{dx^n} = 0$ and $\frac{d^{n+1}}{dx^{n+1}} = 0$ when $x = 0$, one gets two equations giving relations among the constants of integration, namely

$$C + e^{-\frac{2\pi i}{n+2}} C_1 + e^{-\frac{4\pi i}{n+2}} C_2 + \cdots + e^{-\frac{2(n+1)\pi i}{n+2}} C_{n+1} = 0$$
$$C + e^{-\frac{4\pi i}{n+2}} C_1 + e^{-\frac{8\pi i}{n+2}} C_2 + \cdots + e^{-\frac{4(n+1)\pi i}{n+2}} C_{n+1} = 0.$$

In the same manner, one can find the relations among the constants of integration for $m = 3, 4, \ldots$.

Spitzer's result is based on Kummer's method above, but is slightly modified in order to show that if $\Psi(x)$ is the general solution of

$$x^{m+1}\frac{d^{n+1}z}{dx^{n+1}} = \epsilon z,$$

then the general solution of $x^m \frac{d^n y}{dx^n} = -\epsilon y$ may be expressed by

$$\int_0^\infty u^{m-1} e^{-\frac{u^{m-n}}{m-n}} \Psi\left(\frac{x}{u}\right)\, du,$$

again with a certain relation holding among the $(n+1)$ constants of integration. Thus as Fields points out, Spitzer has found the definite integral form of the solution of $\frac{d^n y}{dx^n} = x^m y$ for all negative integer values of m with absolute value greater than $2n$ [Fie86c, 383].

Fields expresses the results of Kummer and Spitzer in combination as follows. If $\Psi(x)$ is the general solution of $\frac{d^{n+1}}{dx^{n+1}} = bx^{m-1}z$, then the general solution of $\frac{d^n}{dx^n} = ax^m y$ may be expressed as

$$y = \int_0^\infty u^{m-1} e^{-\frac{b}{a}\frac{u^{m+n}}{m+n}} \Psi(xu)\, du, \tag{4.17}$$

where there is a certain relation holding among the $n+1$ constants of integration where $m+n$ and m are of the same sign, and $\frac{b}{a}$ is positive or negative depending on whether this sign is a plus or minus [Fie86c, 384].

Fields points out that this result may be easily verified by differentiating $\frac{d^n}{dx^n} - ax^m y = 0$ and then substituting in the expression for $\frac{d^{n+1}}{dx^{n+1}}$, the expression for y' and y from (4.17).

Summarizing the known results, Kummer's solution of $\frac{d^n y}{dx^n} = x^m y$ covers the case where m and $m+n$ are positive, and Spitzer's the case where m and $m+n$ are negative. However, both Kummer and Spitzer assume that that n must be a positive integer. But as Fields remarks, in the verification of (4.17), there is no requirement that n must be positive. So n may be negative as long as m and $m+n$ still satisfy Kummer's or Spitzer's conditions. What this means then, is that from the solution of $\frac{d^{-n+1}z}{dx^{-n+1}} = x^{m-1}z$, one may derive the solution of $\frac{d^{-n}y}{dx^{-n}} = x^m y$. By substituting $x^{m-1}z = v$ into the two previous equations, we get $x^{-m+1}v = \frac{d^{n-1}v}{dx^{n-1}}$ and $x^{-m}u = \frac{d^n u}{dx^n}$. Thus, we see that from the solution of $\frac{d^{n-1}v}{dx^{n-1}} = x^{-m+1}v$ we may derive the solution of $\frac{d^n u}{dx^n} = x^{-m}u$, where in stating this, Fields assumes that in carrying out the operation $\frac{d^{-n}y}{dx^{-n}}$, the constants of integration will always be set equal to zero, so that the solutions of $y = \frac{d^n}{dx^n}(x^m y)$ are the solutions of $\frac{d^{-n}y}{dx^{-n}} = x^m y$.

In the following, Fields makes use the (then well known) result that the solution of $\frac{d^n y}{dx^n} = x^{-2n}y$ is $y = x^{n-1}\sum C_r e^{-\frac{\mu_r}{x}}$, and the result that the solution of $\frac{d^{-n}}{dx^{-n}} = x^{2n}z$ is $z = x^{-2n}y = x^{-n-1}\sum C_r e^{-\frac{\mu_r}{x}}$, where the μs are the nth roots of unity and the Cs are arbitrary constants. Starting with the equation

$$\frac{d^{-n}z}{dx^{-n}} = x^{2n}z,$$

and using (4.17), we may express the solution of $\frac{d^{-n-1}y}{dx^{-n-1}} = x^{2n+1}y$ as

$$y = x^{-n-1}\int_0^\infty u^{n-1}e^{-\frac{u^n}{n}}\sum C_r e^{-\frac{\mu_r}{xu}}\ du.$$

By successively applying (4.17) we may obtain the solution of $\frac{d^{-n-i}}{dx^{-n-i}} = x^{2n+i}y$,

$$y = x^{-n-1}\int_0^\infty\int_0^\infty\cdots\int_0^\infty\prod_{\kappa=1}^{i}u_k^{n+\kappa-2}e^{-\frac{1}{n}\sum_{\kappa=1}^{i}u_k^n}\sum C_r e^{-\mu_r(xu_1\cdots u_i)^{-1}}\ du_1du_2\cdots du_i.$$

By substituting n for $(n+i)$ into the previous equation we get that the solution of $\frac{d^{-n}y}{dx^{-n}} = x^{2n-i}y$ may be expressed as

$$y = x^{-n+i-1}\int_0^\infty\cdots\int_0^\infty\prod_{\kappa=1}^{i}u_\kappa^{n-i+\kappa-2}e^{-\frac{1}{n-i}\sum_{\kappa=1}^{i}u_\kappa^{n-i}}\sum C_r e^{-\mu_r(xu_1\cdots u_i)^{-1}}\ du_1du_2\cdots du_i, \tag{4.18}$$

where by $\prod_{\kappa=1}^{i}u_\kappa^{n-i+\kappa-2}$ Fields means the product $u_1^{n-i-1}u_2^{n-i}\cdots u_i^{n-2}$ and where i is always a positive integer less than n, and the μ_is are the $(n-i)$th roots of unity.

For $\frac{d^n}{dx^n} = x^{-2n+i}u$ we have $u = x^{2n-i}y$, and therefore we have for the equation

$$\frac{d^n y}{dx^n} = x^{-2n+i}y \tag{4.19}$$

the solution

$$y = x^{n-1}\int_0^\infty \cdots \int_0^\infty \prod_{\kappa=1} iu_\kappa^{n-i+\kappa-2} e^{-\frac{1}{n-i}\sum_{\kappa=1}^{i} u_k^{n-i}} \sum C_r e^{-\mu_r (xu_1\cdots u_i)^{-1}}\ du_1\cdots du_i. \tag{4.20}$$

Hence, by deriving (4.20), Fields has derived a definite integral solution of the equation $\frac{d^n y}{dx^n} = x^m y$ for all negative integer values of m between n and $2n$, thus extending the results of Kummer and Spitzer. Fields notes that the solution he gives is not the most general one because the constant of integration when operating by $\frac{d^{-1}}{dx^{-1}}$ was always set equal to zero, but that the solution still contains $-(m+n) = n-i$ arbitrary constants.

Having established the more general solution, Fields now takes some time to find particular solutions of $\frac{d^n y}{dx^n} = x^m y$ for any value of m. In particular, he establishes that the equation

$$\frac{d^n y}{dx^n} = x^{-m}y \tag{4.21}$$

has a solution given by

$$y = x^{n-1}\int_0^\infty \cdots \int_0^\infty u_2 u_3^2 \cdots u_{n-1}^{n-2} e^{\frac{1}{n-m}\left(u_1^{m-n}+\cdots+u_{n-1}^{m-n}+(xu_1\cdots u_{n-1})^{n-m}\right)}\ du_1\cdots du_{n-1} \tag{4.22}$$

where m is any positive quantity greater than n.

At this point, Fields remarks that we can find similar solutions for other values of m by starting with $\frac{d^{-1}y}{dx^{-1}} = x^m y$, where $m-1$ is any negative quantity, and then by successively using formula (4.17) with $\frac{a}{b} = -1$.

Using some of his results from the first part of his thesis on finite symbolic solutions of the nth order equation $\frac{d^n y}{dx^n} = x^m y$, Fields is now able to establish that if we proceed along similar lines by starting from some equation whose solution is known, either finitely or as a definite integral, then we can derive solutions for $\frac{d^n y}{dx^n} = x^m y$ for any real value of m expressed as integrals in various forms [Ham07].

4.2 Some of Fields' Other Early Mathematical Work

Before moving on to discuss Fields' more mature work on the theory of algebraic functions, we will pause here briefly to look at a small selection of Fields' other early publications. In many of his early publications, Fields' proves known results. There are a couple of reasons why he may have done this. As any young academic knows, being able to say that you are a published scholar helps. So Fields may have written these papers to help establish his reputation as a mathematician. It is also conceivable that, as a graduate student and then fellow of the Johns Hopkins University, he was helping to fill the pages of the young *American Journal of Mathematics*, which only began publication in 1878 under the auspices of the university and the editorial leadership of Johns Hopkins based mathematicians.[3]

4.2.1 A Proof of the Theorem: The Equation $f(z) = 0$ Has a Root Where $f(z)$ is any Holomorphic Function of z

This paper, which appeared in the pages of the *American Journal of Mathematics* in February 1886 [Fie86b], was written while Fields was still at Johns Hopkins University. The theorem, when restricted to polynomials, is basically a statement of the fundamental theorem of algebra, which says that every polynomial equation $p(z) = 0$ has a complex number solution. With this theorem, one is able to factor polynomials into linear factors over the complex numbers, or equivalently, as a product of linear and irreducible quadratic factors over the real numbers. This result was used without

[3]For more details on the early history of the American Journal of Mathematics, see [PR94, 88-94].

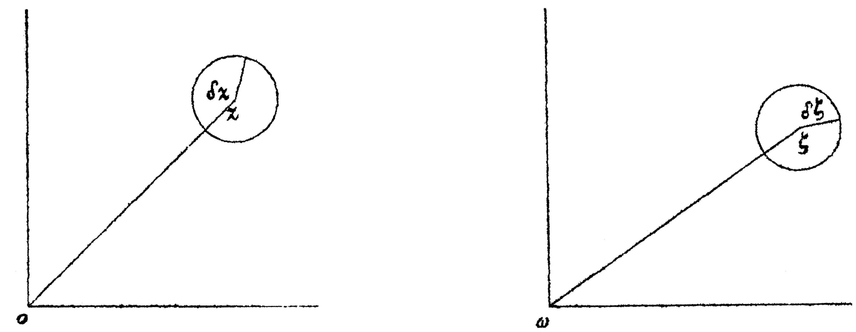

Figure 4.1: Figure from ([Fie86b], 178).

proof for a long time. The first mathematicians who tried to prove the theorem rigorously were Jean Le Rond d'Alembert (1717-1783) and C. F. Gauss (1777 - 1855), though their initial attempts are far from satisfactory from a modern perspective ([Sti89], 196). Fields' proof is but one of many different proofs of the fundamental theorem of algebra. For others, see [FR97].

Fields begins by letting z be a point in one complex plane A with origin o and $f(z) = \zeta$ a point in another complex plane B with origin ω. Recall that a holomorphic function is synonymous with an analytic function, a function which can be represented by a convergent Taylor series expansion at every point. If $f(z) = \zeta$ cannot become equal to zero for any value of z, then Fields observes that there is some minimum distance from the origin ω within which ζ cannot fall. As he points out, we can suppose this, since $f(z)$ becomes infinite as z becomes infinite, and thus the required minimum must occur for a finite value of z, and therefore "can be reached" [Fie86b, 178]. Fields now tries to come up with a contradiction.

We will use the modern concept of neighbourhood to simply the exposition slightly. Fields does not use this terminology in his paper, but what he does write amounts to basically the same thing.

As indicated in Figure 4.1, about z draw a δz - neighbourhood (where δz denotes an increment of z). The corresponding $\delta\zeta$-neighbourhood of ζ (see Figure 4.1) is given by $\delta\zeta = \frac{f^r(z)}{r!}(\delta z)^r$, where $f^r(z)$ is the first successive derivative of $f(z)$ which does not vanish for the value of z. The closed curve describing the $\delta\zeta$-neighbourhood about ζ comes between the point ζ and the origin ω. But this means that the point ζ is not at the minimum distance to the origin as first supposed. This is a contradiction. Hence the function $f(z) = \zeta$ can become zero and hence the equation $f(z) = 0$ has a root. Fields finishes off the paper by giving a slight variation on the final statements of the proof.

This paper must have been somewhat of a disappointment for Fields, as there is an addendum at the end of the paper stating that just as the note was to go to press, Fields discovered that practically the same proof had been given by Hoüel in his *Cours de Calcul Infinitésimal*. Fields had been scooped.

4.2.2 A Proof of the Elliptic-Function Addition-Theorem

Elliptic functions can trace their origin to the study of elliptic integrals, that is, integrals of the form

$$\int R\left[x, \sqrt{p(x)}\right]\, dx,$$

where R is a rational function and p is a polynomial of degree 3 or 4. The word "elliptic" refers to a certain class of these integrals that can be interpreted as the arc-length of an ellipse [Coo94, 529]. The idea of inverting elliptic integrals to obtain elliptic functions is due to C. F. Gauss, N. H. Abel (1802 - 1829), and C. Jacobi (1804 - 1851), though Abel was the first to publish. Abel's "Recherches sur les fonctions elliptiques" appeared in *Crelle's Journal* in the fall of 1827. Abel covered much of the same ground as Gauss had in his unpublished work. Gauss noticed this. On March 30 1829, Gauss wrote to W. Bessel (1784 - 1846) that

> It appears that for the time being I won't be able to get back to the work

> on transcendental functions that I have been conducting for many years (since 1798), since I must first finish up many other things. Herr Abel, I notice, has now preceded me and has relieved me of approximately one-third of these things, the more so as he has carried out all the computations elegantly and concisely. He has chosen the exact same route that I took in 1798; for that reason the large coincidence in our results is not surprising. To my amazement this extends even to the form and partly to the choice of notation, so that many of his formulas seem to be exact copies of my own. To avoid any misunderstanding I note, however, that I do not recall ever having discussed these things with anyone (quoted in [Kol96, 138]).

Even though Abel covered some of the same ground as Gauss, Abel developed the theory to a greater extent than Gauss and in a more thorough manner. Abel's paper also contains the first published theory of the inverse functions of elliptic integrals [Kol96, 154].

Around the same time that the first part of Abel's *Researches* was published, a note "Extraits de deux lettres à Schumacher" appeared in the *Astronomische Nachrichten* (Number 123) in 1827 by the German C. Jacobi. He detailed without proof a new result on the transformation of elliptic integrals. In March 1828, Abel read Jacobi's proofs of the results in "Demonstratio theorematis ad theoriam functionum ellipticarum spectantis" in an issue of *Astronomische Nachrichten* (Number 127) in 1827. This was a shock to Abel. There was not much he could do other than quickly publish his own ideas on the transformation of elliptic integrals. He published his theory, for which Jacobi's results are special cases, in "Solution d'un problème général concernant la transformation des fonctions élliptiques" (*Astronomische Nachrichten*, which appeared in two parts, one in volume six in 1828 and one in volume seven in 1829 [Kol96, 156-157]. The priority dispute between Jacobi and Abel would not last long as Abel succumbed to tuberculosis and died in 1829. Jacobi continued to work on elliptic functions for most of his career.

Jacobi's 1829 text *Fundamenta nova functionum ellipticarum* has garnered Jacobi the title of co-founder of the theory of elliptic functions. A novel aspect of the theory of elliptic functions presented in the *Fundamenta nova* was the initiation of the theory of theta functions. Jacobi was to use theta functions to give representations of the three "basic" elliptic functions, later known as sn u, cn u, and dn u, as quotients of theta functions [Kol96, 158-159].

Just like Gauss and Abel, Jacobi started with the elliptic integral of the "first kind," but in what is now know as Legendre normal form,

$$u = F(\phi, k) = \int_0^\phi \frac{dt}{\sqrt{1 - k^2 \sin^2 t}},$$

where k is a parameter with $0 < k^2 < 1$ and $\phi = F^{-1}(u, k)$. Jacobi called the parameter k the modulus and the variable ϕ the amplitude, which he denoted by am u. Substituting $x = \sin\phi$ gives

$$u = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

So for the inverse of the elliptic integral of the first kind, we have $x = \sin \mathrm{am}\ u$, which is read "sine-amplitude of u". In a similar way, Jacobi defined the following essential functions

$$\cos \mathrm{am}\ u = \sqrt{1 - \sin^2 \mathrm{am}\ u}, \qquad \triangle \mathrm{am}\ u = \sqrt{1 - k^2 \sin^2 \mathrm{am}\ u},$$

the cosine-amplitude and the delta-amplitude. In 1838, Christoph Gudermann (1798-1852), who would mentor K. Weierstrass, gave the now common notation — sn u, cn u, and dn u — for the three "Jacobi elliptic functions," in his paper "Theorie der Modular-Functionen und der Modular-Integrale" which appeared in the *J. für Math.*, volume eight, in 1838. Fields uses Gudermann's standard notation in his paper on the addition theorem for elliptic functions.

Recall that the elliptic function addition theorem states that:

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\mathrm{cn}v\mathrm{dn}v + \mathrm{sn}v\mathrm{cn}u\mathrm{dn}u}{1 - k^2\mathrm{sn}^2u\mathrm{sn}^2v}.$$

Fields gives a proof of this result in a paper which appeared in the *American Journal of Mathematics* in July 1886 ([Fie86a]). He does this by integrating the differential equation

$$\frac{d\phi}{\Delta\phi} + \frac{d\Psi}{\Delta\Psi} = 0,$$

where $\Delta\phi = \sqrt{1 - k^2\sin^2\phi}$, etc., using the integrating factor

$$\frac{\Delta\phi\Delta\Psi - k^2\sin\phi\cos\phi\sin\Psi\cos\Psi}{1 - k^2\sin^2\phi\sin^2\Psi}.$$

From this he finds that

$$\frac{\tan\phi\Delta\Psi + \tan\Psi\Delta\phi}{1 - \tan\phi\tan\Psi\Delta\phi\Delta\Psi} = \tan\mu,$$

and observes that since $\mu = \phi$ when $\Psi = 0$, μ is the amplitude of $(u+v)$, where u and v are elliptic functions whose amplitudes are ϕ, Ψ respectively. From this he is able to establish quickly that

$$\mathrm{sn}(u+v) = \sin\mu = \frac{\tan\mu}{\sqrt{1+\tan^2\mu}} = \frac{\sin\phi\cos\Psi\Delta\Psi + \cos\phi\sin\Psi\Delta\phi}{1 - k^2\sin^2\phi\sin^2\Psi},$$

and, using some standard identities, he proves the theorem. Fields finishes the paper by commenting that the addition formulas for $\mathrm{cn}(u+v)$ and $\mathrm{dn}(u+v)$ can be just as readily obtained.

4.2.3 Expressions for Bernoulli's and Euler's Numbers

In this paper, which appeared in the *American Journal of Mathematics* in January, 1891, Fields derives related expressions for the Bernoulli and Euler numbers [Fie91a].

Jakob Bernoulli (1654-1705) in his book *Ars Conjectandi* (1713), a work on the subject of probability, introduced what are now called the Bernoulli numbers. In the course of finding a formula for the sums of the positive integral powers of the integers, Bernoulli gave the formula

$$\sum_{k=1}^{n} k^c = \frac{1}{1+c}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}B_2 n^{c-1} + \frac{c(c-1)(c-2)}{2\cdot 3\cdot 4}B_4 n^{c-3} + \frac{c(c-1)(c-2)(c-3)(c-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}B_6 n^{c-5} + \cdots,$$

where the series terminates at the last positive power of n. The formula was given without proof. The numbers

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66}, \ldots,$$

are the Bernoulli numbers [Kli90a, 451].

The Euler numbers E_{2m} appear in the power series expansion of the secant function:

$$\sec z = \sum_{m=0}^{\infty} (-1)^m \frac{E_{2m}}{(2m)!} z^{2m},$$

where this power series expansion converges for $|z| < \frac{\pi}{2}$, and $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, etc. Euler numbers, often referred to as secant numbers, were supposedly first given their name by the influential English mathematician J. J. Sylvester [Ely82, 337].

Both Bernoulli and Euler numbers appear in the sums of several interesting infinite series that where studied by both Bernoulli and Euler. For example, the Bernoulli numbers appear in the following sums:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{B_2}{2\cdot 2!}(2\pi)^2 = \frac{\pi^2}{6},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = -\frac{B_4}{2\cdot 4!}(2\pi)^4 = \frac{\pi^4}{90}$$

and the Euler numbers appear in the following related formulas:[4]

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{E_0}{2}\frac{\pi}{2} = \frac{\pi}{4},$$

[4]For more on the relationship between these two sets of formulas, see [Mar77, 64-66].

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = -\frac{E_2}{2 \cdot 2!}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{32}.$$

Fields, referring to a result found in Bertrand's *Calcul Différentiel* [Ber64, 141] (which is also derived in an 1889 paper by Fields in the *American Journal of Mathematics*), establishes related expressions for Bernoulli and Euler numbers. The result he refers to is

$$\left(\frac{d}{dx}\right)^n \phi(u) = \sum_{r=1}^{n} \sum_{\rho=r}^{n} \frac{(-1)^{\rho-r} u^{\rho-r} \phi^p(u)}{r!(\rho-r)!} \left(\frac{d}{dx}\right)^n u^r.$$

By setting $u = x^{-ix}$ we get

$$\sec x + \tan x = \frac{2}{u+i} + i = \sum \frac{E_n x^n}{n!},$$

where when n is even we get Euler's number E_n, and when n is odd, we get Bernoulli's number $B_{\frac{n+1}{2}} = \frac{(n+1)E_n}{2^{n+1}(2^{n+1}-1)}$.

By substituting $\theta(u) = \frac{1}{u+i}$ in the formula from the start and by a series of manipulations, Fields is able to find that when n is odd

$$E_n = 2\left(\frac{i}{2}\right)^{\frac{n+1}{2}} \sum_{s=2}^{n+1} \sum_{r=1}^{s-1} i^{s+r} \binom{n+1}{s} r^n.$$

Hence the corresponding expression for Bernoulli's number is

$$B_{\frac{n+1}{2}} = \frac{2(n+1)}{2^{n+1}(2^{n+1}-1)} \left(\frac{i}{2}\right)^{\frac{n+1}{2}} \sum_{s=2}^{n+1} \sum_{r=1}^{s-1} i^{s+r} \binom{n+1}{s} r^n.$$

Fields finishes off the paper by observing that since E_n is real, the imaginary terms in the summations in the last expression for E_n must cancel. He gives an expression for E_n using only real numbers, leaving to readers the exercise of figuring out the corresponding expression for B_n.

4.2.4 Summary of Fields' Early Mathematical Work

In the first part of Fields' mathematical career there is no obvious unifying theme to his work. He had some papers on number theory such as [Fie91c] and [Fie93], papers that use the symbolic method in analysis such as [Fie89] and [Fie92], as well as the papers discussed in sections 4.2.1 – 4.2.3 which provided new proofs to known results. Though Fields must have felt that research and publication was important, it seems that he did not finish his studies at Hopkins with a clear idea of where he was headed mathematically. This was to change after his time in Europe in the mid-1890s, after which almost all of his research efforts focussed on the theory of algebraic functions. It is this aspect of Fields' mathematical career which will occupy us in next chapter.

Chapter 5

Fields' Work in Algebraic Function Theory

The theory of algebraic functions grew out of attempts to generalize the theory of elliptic integrals. Recall from section 4.2.2 that elliptic integrals are integrals of the form

$$\int R\left[x, \sqrt{p(x)}\right]\; dx,$$

where R is a rational function and p is a polynomial of degree 3 or 4. N. H. Abel was the first to make significant progress on studying these types of integrals by studying their inverses. In fact, Abel was able to generalize vastly the theory to integrals I where the function defining the relation between x and y is any polynomial whatsoever. These integrals are now called Abelian integrals in his honour. The generalization of the theory of elliptic integrals turned out to be such a rich area for research that work done on this and related matters can be seen as one of the major themes of nineteenth century mathematics, dominating the attention of some of the most highly regarded mathematicians of the time (e.g., K. Weierstrass).

According to Bliss [Bli24, 96] there are three main approaches to the theory of algebraic functions: the transcendental, the geometric, and the arithmetic. Bliss's perspective on the state of the theory at the turn of the twentieth century is interesting, since, as far as he is concerned, each approach aims at the same thing, but each with its positive and negative features. In fact, in his book on algebraic functions [Bli33], Bliss gives each perspective almost equal consideration.

In the transcendental theory, due in large part to the work of Abel and B. Riemann, Abelian integrals play the key role. The geometrical theory is strongly connected with the study of higher plane curves, and was developed in the work of Clebsch and Gordan, and later by mathematicians such as F. Severi (1879 - 1961). In the arithmetical theories, a slightly misleading name for the approach, the emphasis is on the construction and theory of rational functions and only secondarily on the construction of the Abelian integrals for which these functions are the integrands [Bli24, 96].

The arithmetical approach was first presented by L. Kronecker (1823 - 1891) to the Berlin Academy in 1862, but was not published until 1881. Other theories using the arithmetical approach are those of R. Dedekind (1831 - 1916) and H. Weber (1842 - 1913), K. Weierstrass (1815 - 1897) (developed in his famous lectures but not appearing until the publication of his collected works), as well as later by Hensel and Landsberg (who developed and simplified the methods of Dedekind and Weber), and the theory of our protagonist, J. C. Fields, which was published in book form in 1906 [Fie06][Bli24, 96].

Those who favoured the arithmetical approach often criticized the transcendental and geometrical theories on the grounds that these two theories required the application of an initial birational transformation to a curve described by $F(x, y) = 0$ in order to simplify the singular points. The singular points of such curves can be unwieldy and very complicated in nature [Bli24, 96]. The arithmetical theories were developed in part to side-step these difficulties, by allowing calculations, often with formal power series, to provide all the necessary information about the nature of singular points, though G. Landsberg (1865 - 1912) noted that the power series methods needed to be rendered algebraic. Modern theories of algebraic functions, which were first worked out in the 1920s and 1930 by mathematicians such as O. Zariski (1899 - 1986), are approached mainly using the tools of abstract algebra. The modern methods essentially overshadowed the nineteenth century approaches because of the ability of these theories to generalize to higher dimensions, unlike arithmetical theories such as that of Fields.

In this chapter we will try to give an indication of the scope of Fields' work in algebraic function theory, or what we might now call algebraic geometry. First we will briefly discuss the elements of the theory as understood at the beginning of the twentieth century when Fields undertook the majority of his more mature mathematical researches and then we will briefly discuss Fields' approach to the theory and try to give an indication of how Fields' work relates to other approaches used at the time.

5.1 Elements of Algebraic Function Theory

We will use G. A. Bliss' [Bli33] extensively in this section, as it gives an introduction to both the arithmetic theory and the transcendental theory of algebraic functions. More will be said about the differences between the various approaches to studying algebraic functions at the turn of the twentieth century in the next section.

In what follows, let $f(x, y)$ be a polynomial in y of the form

$$f(x, y) = f_0(x)y^n + f_1(x)y^{n-1} + \cdots + f_n(x)$$

where each coefficient $f_i(x)$ is itself a polynomial in x with complex coefficients.

Definition 5.1 (Algebraic Function) *An algebraic function is a function $y(x)$ defined for values x in the complex x-plane by an equation of the form $f(x, y) = 0$ [Bli33, 24].*

An algebraic function $y(x)$ is in general multi-valued, meaning that for each value x in the complex x-plane for which the function is defined, there is more than one value y in the complex y-plane that corresponds to x. This is contrary to what is observed in elementary analytic geometry, where the points of a curve in the real xy-plane are in one-to-one correspondence with pairs of real values (x, y) satisfying the given algebraic equation $f(x, y) = 0$ (with real coefficients) that defines the curve. The notion of a Riemann surface, the type of surface on which an algebraic function "lives", discussed below, was developed in order to restore a one-to-one correspondence between points of the domain and points satisfying $f(x, y) = 0$ ([Bli33], 75).

Definition 5.2 (Ordinary and Singular Points) *A point $x = a$ in the complex x-plane where $f_0(a) \neq 0$ and the discriminant of f at $x = a$ does not vanish is called an ordinary point. A point $x = a$ at which one or both of $f_0(a)$ and the discriminant of f at $x = a$ vanish, is called a singular point [Bli33, 25].*

The point $x = \infty$ is ordinary or singular according to whether 0 is an ordinary or singular point of $f(x', y) = 0$, where $x' = \frac{1}{x}$. The point at infinity is in general treated as a point in the complex plane. In what follows, when we speak of the complex plane, we mean the extended complex plane – the complex plane with the point at infinity adjoined.

Rational functions are functions of the form $P(x, y)/Q(x, y)$, where P and Q are polynomials in the complex variables x and y. These form one of the most important classes of functions in algebraic function theory. Rational functions have no singularities in the complex plane other than poles.

Definition 5.3 (Poles) *Given an analytic function $g(z)$, suppose $g(z_0)$ fails to be analytic at a point $z = z_0$. If we can find a positive integer n such that $\lim_{z \to z_0}(z - z_0)^n g(z) = A \neq 0$, then the point $z = z_0$ is called a pole of order n. If $n = 1$, z_0 is called a simple pole [Spi64, 67].*

For example, the rational function

$$g(z) = \frac{(z-3)^2}{(z-1)^3(z-7)},$$

has a pole of order 3 at $z = 1$ and a simple pole at $z = 7$.

It can be shown that when the discriminant of $f(x, y)$ is non-zero for all but a finite number of values $x = a$, the values of the algebraic function $y(x)$ are distinct and exactly n in number [Bli33, 25]. For example, given an algebraic function $y(x)$ defined by the relation $y^2 = x$, often called the square root function, we see that for every non-zero finite value of $x = a$ correspond two distinct values of y, respectively $y = +\sqrt{a}$ and $y = -\sqrt{a}$. At $x = 0$, the discriminant is equal to zero and the two values of y are not distinct. The two functions $y = +\sqrt{x}$ and $y = -\sqrt{x}$ are called the *branches* of the multi-valued function $y(x)$. More generally, we may consider a multi-valued function as a collection of single valued functions, each member of which is a branch of the function.

In the theory of algebraic functions, it is often useful to represent functions in terms of series expansions. In fact, this is the basis for the arithmetical approaches to the theory, including Fields' own. The following is a basic theorem about series representations of algebraic functions.

Theorem 5.4 *In the neighbourhood of an ordinary point $x = a$ the n values of an algebraic function $y(x)$ are defined by n convergent series*

$$y = b_i + c_{i1}(x-a) + c_{i2}(x-a)^2 + \cdots,$$

where the numbers b_i are the n distinct roots of $f(a, y)$ and $i = 1, \ldots, n$ [Bli33, 25].

It can be shown that similar expansions exist for values of $y(x)$ near a singular point, but that such series may have terms with negative and/or fractional powers of $x - a$ [Bli33, 29]. I. Newton (1643 - 1727) showed in *De methodis serierum et fluxionum* (1671) that any algebraic function $y(x)$ can be expressed as a fractional power series in x and gave a method for generating these series expansions using what are now known as Newton polygons [Sti89, 125-126]. The fractional powers were not well understood until the variables were taken to be complex and it was not until the work of V. Puiseux (1820 - 1883) in 1850 [Pui50] that this was more rigorously clarified [Sti89, 125-126]. One of the more important results proved by Puiseux is the following:

Theorem 5.5 (Puiseux's Theorem) *In a neighbourhood of a point (x_0, y_0) of the domain of an algebraic plane curve $f(x, y) = 0$, $y(x)$ can be expressed by a finite number of fractional power series developments*

$$y - y_0 = a_1(x - x_0)^{\frac{q_1}{q_0}} + a_2(x - x_0)^{\frac{q_2}{q_0}} + \cdots,$$

which converge in some interval about x_0 and all the q_i have no common factors. The points given by each development are called a branch of the algebraic curve [Kli90a, 552-553].

For this reason, fractional power series expansions of algebraic functions are now often referred to as *Puiseux expansions* [Sti89, 125-126].

The reason for the failure of ordinary power series to represent algebraic functions at points that are not ordinary, is that at these points such functions display branching behaviour. This is a result of the multi-valuedness of such functions. Fractional power series are able to capture this behaviour.

Using the simple example of the square root function, we can see this behaviour. Writing our complex numbers in polar form, we have for $x = r^{i\theta}$, $y = \sqrt{r}e^{i\frac{\theta}{2}}$. Consider a point $a = r_1 e^{i\theta_1}$ so that $y = \sqrt{r_1} e^{i\frac{\theta_1}{2}}$. By making a full circuit around the origin, that is by adding 2π to the angle θ_1, we arrive at what should be the same point, but instead we find that $y = \sqrt{r_1} e^{i\frac{\theta_1+2\pi}{2}} = -\sqrt{r_1} e^{i\frac{\theta_1}{2}}$. If we however make another circuit around the origin, that is add 4π to θ_1, we get $y = \sqrt{r_1} e^{i\frac{\theta_1+4\pi}{2}} = \sqrt{r_1} e^{i\frac{\theta_1}{2}}$ [Spi64, 37].

Definition 5.6 (Branch Point, Algebraic Branch Point) *A branch point of an analytic function is a point $x = re^{i\theta}$ in the complex plane whose complex argument θ can be mapped from a single point in the domain to multiple points in the range. An algebraic branch point is a branch point whose neighbourhood of values wrap around the range a finite number of times p as their complex arguments varies from 0 to a multiple of 2π and is said to have order p [Weia], [Weib].*

Referring back to Puiseux's Theorem (5.5), supposing q_1/q_0 is the smallest exponent (written in lowest terms) in the fractional series representation of our algebraic function defined by $f(x, y) = 0$ about a branch point $x = x_0$, then we can simply read off the order of the branch point as q_0, the denominator of this smallest exponent. This is in fact the definition used by Paul Appell and Édouard Goursat in their book on algebraic functions [AG29].

In our example of the square root function, we see that the origin is an algebraic branch point of order 2. A circuit around any other finite point does not lead to different values. So we see that the origin is the only finite branch point of the square root function. So for the square root function, the only finite point about which the function cannot be represented by a ordinary power series is the algebraic branch point $x = 0$. In order to represent the square root function as a power series at the branch point, one needs to admit fractional exponents in the terms of the series expansion.

Since the fractional series expansion of the square root function is so simple, it is worthwhile to look at the series expansion of a more complicated algebraic function about some of its branch points. Let us consider $y^2 - x(x-1)(x-2) = 0$. This relation defines an algebraic function $y(x)$ with two branches, $y_1 = \sqrt{x^3 - 3x^2 + 2x}$ and $y_2 = -\sqrt{x^3 - 3x^2 + 2x}$, which we have labelled with subscripts 1 and 2 for ease of reference. There are three branch points for each branch. The following are three such expansions. If we expand y_1 and y_2 about $x = 0$ we get[1]

$$y_1 = \sqrt{2}x^{\frac{1}{2}} - \frac{3}{4}\sqrt{2}x^{\frac{3}{2}} - \frac{1}{32}\sqrt{2}x^{\frac{5}{2}} - \frac{3}{128}\sqrt{2}x^{\frac{7}{2}} - \frac{37}{2048}\sqrt{2}x^{\frac{9}{2}} - \frac{117}{8192}\sqrt{2}x^{\frac{11}{2}} - \cdots,$$

and

$$y_2 = -\sqrt{2}x^{\frac{1}{2}} + \frac{3}{4}\sqrt{2}x^{\frac{3}{2}} + \frac{1}{32}\sqrt{2}x^{\frac{5}{2}} + \frac{3}{128}\sqrt{2}x^{\frac{7}{2}} + \frac{37}{2048}\sqrt{2}x^{\frac{9}{2}} + \frac{117}{8192}\sqrt{2}x^{\frac{11}{2}} + \cdots.$$

If we expand y_1 about $x = 1$, we get

$$y_1 = i(x-1)^{\frac{1}{2}} - \frac{1}{2}i(x-1)^{\frac{5}{2}} - \frac{1}{8}i(x-1)^{\frac{9}{2}} - \frac{1}{16}i(x-1)^{\frac{13}{2}} - \frac{5}{128}i(x-1)^{\frac{17}{2}} + \cdots.$$

Having briefly discussed branch points and their orders, it is now useful to define another piece of important terminology. A *branch cut* is a portion of a line or curve (with ends possibly open, half-open or closed) that is introduced into the complex plane to distinguish between branches of a multi-valued function. That is, traversing the cut in the domain means one moves from one branch to another in the range. A standard example is the complex logarithm, where there is a countable infinity of values, and the branch cut is usually taken as the negative real axis. Branch cuts are most often taken as lines for convenience. In the example of the square root function, the ray consisting of the origin and the positive real axis of the complex plane can be taken as our branch cut.

As was already mentioned, an algebraic function such as the square-root function is multi-valued. That is, for an algebraic function $y(x)$ defined by the related $F(x, y) = 0$, there is not a unique element in the domain for every element in the range. B. Riemann (1826 - 1866) in his 1851 doctoral dissertation on the foundations of complex

[1]These expansions were calculated using the computer program Maple.

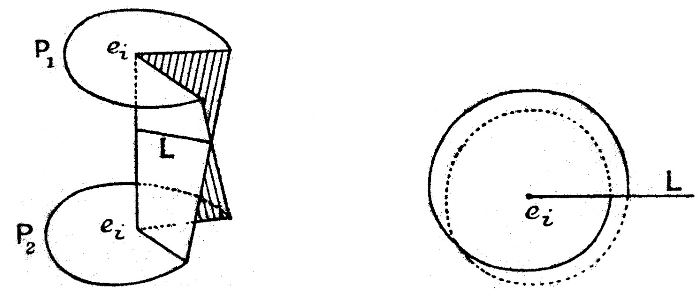

Figure 5.1: An depiction of the Riemann Surface for the square root function. From [AG29, 27].

analysis [Rie53] introduced the idea of what are now called Riemann surfaces. It is on such surfaces that multi-valued functions such as algebraic functions can be said to "live", in the sense that they are single-valued functions on such surfaces [Kli90a, 657-658].

In order to demonstrate the basic concept of a Riemann surface, we will again use the example of the square root function. In order to construct the Riemann surface for the square root function, one first takes two planes that are thought of as lying one above the other, one plane corresponding to the branch $y = \sqrt{x}$ and one plane corresponding to the branch $y = -\sqrt{x}$. These planes are often called "sheets" of the surface. To both planes is added the point at infinity, denoted by ∞. The two planes are then joined at the branch points 0 (i.e., the origin) and ∞. As noted earlier, branch cuts are usually taken to be as simple as possible. In our case, it is convenient to take as our branch cut the positive real axis joining the points 0 and ∞ [Kli90a, 656-657]. In Figure 5.1, we see a visualization of this construction and how one can traverse from one sheet to another as one makes successive rotations of 2π about the origin.

As can be gathered by the discussion so far, to every algebraic function defined by a relation $F(x, y) = 0$ we can associate a corresponding Riemann surface. For example, to the square root function $y^2 - x = 0$ corresponds the Riemann surface in Figure 5.1 and vice versa.

In general, a surface may have boundary curves or may be closed like a sphere or torus. The Riemann surfaces corresponding to algebraic functions are closed surfaces [Kli90a, 660]. The sphere is a so-called *simply connected* surface, in the sense that if one were to describe a closed curve on the sphere, it would divide the sphere into two regions so that it is not possible to continuously describe a path from one region to the other without crossing the dividing curve. On the other hand, the torus is not simply connected. It is possible to describe closed curves on a torus which do not disconnect the torus into separate regions as is the case with a sphere. By introducing a series of branch cuts, one may make any Riemann surface of an algebraic function simply connected. We say a Riemann surface has connectivity N if $N-1$ appropriate branch-cuts are required to to make the surface simply connected [Kli90a, 661]. Thus a sphere has connectivity 1, whereas the surface for the square root function (Figure 5.1) has connectivity 2.

The connectivity of a Riemann surface of q sheets may be expressed in terms of the orders w_i of each of its branch points $r_1, \ldots, r_m$. The key result here states that the connectivity N of a Riemann surface for an algebraic function is given by

$$N = \sum_{i=1}^{m} w_i - 2q + 3 \quad [\text{Kli90a}, 661].$$

It can be shown that the connectivity N of a closed surface with a single boundary is an odd number, which can be written in the form $2p+1$. From this we get that

$$2p = \sum_{i=1}^{m} w_i - 2q + 2. \tag{5.1}$$

The integer p is called the genus of the Riemann surface and of the corresponding relation $F(x, y) = 0$ [Kli90a, 661]. The square root function has branch points at the origin and at infinity, both of order 2, which can be read off from the denominator of the first term in the Puiseux expansion about those points. The Riemann surface for the square root function has two sheets. Then equation (5.1) gives $2p = 4 - 4 + 2$. Hence the genus of the Riemann surface of the square root function is 1 and is thus seen to be topologically equivalent to a torus.

It is natural to ask what kinds of functions exist on a Riemann surface corresponding to $F(x, y) = 0$. We know that y is a single-valued function on the surface. As a result, any rational function $H(x, y)$ which has finitely many poles is also single-valued on the surface, since we may substitute into H the value of y given by the relation $F(x, y) = 0$. The branch points for H will then be the same as those of F, though the poles may differ [Kli90a, 662].

5.2 Fields' Theory of Algebraic Functions

Surveying the scene at the end of the nineteenth century, G. Landsberg (1865 - 1912) wrote in 1898 that the "task of building our theory [of algebraic functions] on purely algebraic grounds, foresaking all function-theoretic or geometric methods, remains today still only partly solved" [Lan98a].[2] Some initial progress was made by R. Dedekind (1831 - 1916) and H. Weber (1842 - 1913) in their memoir "The theory of algebraic functions of one variable" published in volume 92 of *Crelle's Journal*, building on Dedekind's ideal theory. The basic idea of their work was to exploit the analogy between algebraic number fields and algebraic function fields – just as an algebraic number field can be considered as a finite extension of the field of rational numbers, an algebraic function field $K = \mathbb{C}(z)(w)$ can be considered a finite extension of the field $\mathbb{C}(z)$ of rational functions of one variable over the complex numbers, where w is a root of a polynomial $a_0 + a_1\alpha + \cdots + a_n\alpha^n$ and the a_is are in $\mathbb{C}(z)$. Thus an algebraic function $w = f(z)$ is defined implicitly by a polynomial eqution

[2] Translated from the German with the aid of Tom Archibald.

$F(z, w) = a_0 + a_1 w + \cdots + a_n w^n = 0$. It can be shown that all elements of K are algebraic functions [Kle98, 30]. Continuing the analogy between algebraic number fields and algebraic function fields, if R is the "ring of integers" of K over $\mathbb{C}(z)$, which consists of the elements of K that are the roots of monic polynomials over $\mathbb{C}[z]$, then every nonzero ideal of R is a unique product of prime ideals. The meromorphic functions on a Riemann surface form a function field of algebraic functions, with entire functions forming the corresponding ring of integers. With this, Dedekind and Weber were able to give a more rigourous definition of a Riemann surface S of the algebraic function field K. In modern terminology, S is the set of nontrivial discrete valuations on K, that is, to each point of the domain there corresponds exactly one valuation. Finite points on S correspond to ideals of R and to deal with infinite points, the notions of "place" and "divisor" were used. Furthermore, with these notions in place, Dedekind and Weber were able to more rigorously develop many of Riemann's ideas about algebraic functions [Kle98, 30].

Further work along the lines laid out by Dedekind and Weber was made by Hensel exactly when Fields was in Berlin [Lan98a][Kle98, 30]. Fields' own approach seems to be an outgrowth of Hensel's push for a purely algebraic approach to the theory of algebraic functions. However, Fields' approach, contrary to that presented by Hensel and Landsberg in their 1904 memoir and contrary to the work of Dedekind and Weber, seems to have retained some of the Weierstrassian function theoretic methods, in that it avoids the use of Riemann surfaces and the theory of divisors entirely [HL02].

Fields first began to publish on the theory of algebraic functions in 1901. P. Appell (1855 - 1930) and E. Goursat (1858 - 1936) on pages 344-345 of their book *Théorie des Functions algébriques et de leurs Intégrales*, sketch a method of C. Hermite's (1822 - 1901) for obtaining a reduced form for a hyperelliptic integral using rational operations and indicate that such a method might also be used for Abelian integrals in general. In a paper entitled "On the Reduction of the General Abelian Integral"

(1901) published in the *Transactions of the American Mathematical Society*, Fields deals with the more general case. This paper is followed by a paper titled "Algebraic Proofs of the Riemann-Roch Theorem and the Independence of the Conditions of Adjointness" published in 1902 in *Acta Mathematica*. In this paper Fields investigates the conditions for expressing the most general rational function subject to an irreducible algebraic curve $F(x, y) = 0$, with certain assumptions made about the multiple points of F, proving a version of the Riemann-Roch theorem, of which more will be said below [Fie02, 167]. It is interesting to note that Fields cites Weierstrass' lectures of 1896 as an inspiration for his approach used in the paper and promises that in upcoming work he will present a more general theory that will enable the proof of the Riemann-Roch theorem "for any arbitrary combination of infinities" [Fie02, 170].

G. Landsberg had also undertaken work to find new algebraic proofs of the Riemann-Roch theorem [Lan98b]. In reviews in the *Jahrbuch über die Fortschritte der Mathematik*, a reviewing and abstracting journal based in Germany, he stated that he had reservations about these works, particularly with regards to simplifying assumptions that were made based on geometric arguments — that is, Fields' approach was not algebraic enough for him [Lan07a] [Lan07b]. Fields' monograph, *Theory of algebraic functions of a complex variable*, published by Mayer and Müller in 1906, seems to have been an attempt to address criticisms such as those by Landsberg. Much of Fields' later work on algebraic function theory, all published in the form of research papers, was an attempt to clarify and expand on the ideas presented in his book, particularly those in Chapter Six. Fields' approach is based on fractional power series expansions of points of an algebraic function and counting arguments. Almost all of his work is based on his concept of "order of coincidence," which we will define shortly after a few preliminaries. Given an algebraic function

$$F(x, y) = y^n + f_{n-1}y^{n-1} + \cdots + f_0 = 0,$$

where the coefficients f_i are polynomials in x and assuming that F has no multiple factors, we may "split up" F into n branches whose equations Fields writes as

$$v - P_1 = 0, v - P_2 = 0, \ldots, v - P_n = 0 \text{ [Fie06, 2]}.$$

The v corresponds to how we have been using y and the P_is can be thought of as short-hand for the series expansions in fractional powers of x at the various branch points.

Because the degree of the function defining the Riemann surface is n, any rational function on that surface may be reduced to one of degree less than n. This means it may be expressed in terms of a basis with n elements, for example $\{1, y, y^2, \ldots, y^{n-1}\}$.

One of Fields' primary goals was to study rational functions subject to the condition defined by $F(x, y) = 0$. We would now describe these as rational functions on a variety.

Definition 5.7 (Order of Coincidence) *The order of coincidence of a rational function $H(x, y)$ with respect to a branch $y - P = 0$ is the smallest exponent of the series expansion for $H(x, P)$.*

As Fields notes, the order of coincidence of a rational function with respect to a branch of an algebraic function can be either positive or negative, integral or fractional [Fie06, 3]. For example, given

$$F(x, y) = y^3 + x^3y + x = 0,$$

the three branch representations for F at $(0, 0)$ are given by

$$y = \omega x^{\frac{1}{3}} + \frac{\omega}{3} x^{\frac{8}{3}} + \cdots,$$

where there is one expansion for each cube root of unity ω. Fields would write these expansions as $y - P_1 = 0$, $y - P_2 = 0$, and $y - P_3 = 0$.

Now, let us consider the rational function $H(x, y) = y^2 + x$ subject to $F(x, y) = 0$. Then the order of coincidence can by found by substituting $y = P_i$ into H, resulting in

$$H(x, P_i) = \left(\omega x^{\frac{1}{3}} + \frac{\omega}{3} x^{\frac{8}{3}} + \cdots\right)^2 + x = \omega^2 x^{\frac{2}{3}} + x + \frac{\omega^4}{9} x^{\frac{16}{3}} + \cdots.$$

Thus the order of coincidence of $H(x, y)$ with respect to the branch $y - P_i$, $i = 1, 2, 3$, is $\frac{2}{3}$, the least exponent in the series expansion of $H(x, P_i)$.

With this basic concept of order of coincidence, Fields is able to by-pass various technical apparatus, such as Riemann surfaces and divisors. The language of divisors would later be commonly used in algebraic geometry. Fields' aim was to build up a coherent theory of algebraic functions on the concept of order of coincidence, which for whatever reason he felt provided a satisfactory "algebraic" basis for the theory. As noted earlier, there were several different approaches to the theory of algebraic functions, so Fields' foundation was one among several competing approaches. Ultimately, these approaches were surpassed by others such as by E. Noether (1882 - 1935), who made use of the language of abstract algebra. It seems to be the general view that this work gained greater clarity with the work of O. Zariski (1899 - 1986) in the 1930s and then later with the work of A. Grothendieck (1928 -), recipient of the 1966 Fields medal, and other notable mathematicians in the 1950s and 1960s [Die85, 59, 91].

One of Fields' aims was to create a theory sufficiently powerful to obtain previously known results, among them the Riemann-Roch theorem, a classical result in algebraic function theory. From one point of view, this is a theorem relating the number of linearly independent rational functions on a Riemann surface to the genus of the surface and the degree of the algebraic function involved in defining the surface. With the theorem one can determine the number of linearly independent rational functions on a Riemann surface (algebraic curve) that have at most a specified finite number of poles. More precisely, let H be a rational function which is single-valued on the Riemann surface of genus p and which has only poles of the first order at points $c_1, \ldots, c_m$. Since H is a function defined on the Riemann surface, it may be reduced in degree to a linear combination of a certain number of functions, up to $n+1$ of them, if the surface is of degree n. The exact number will depend on the genus of the surface and the number of sheets (i.e., the degree of the defining polynomial). If q linearly independent functions (adjoint functions) vanish on $c_1, \ldots, c_m$, then it turns out that H contains $m-p+q+1$ arbitrary constants. Hence H is a linear combination of arbitrary multiples of $m-p+q$ functions, each having $p-q+1$ poles of the first order where $p-q$ of poles are common to each of the functions comprising the linear combination [Kli90a, 665].

Fields' monograph of 1906 on the theory of algebraic functions gives the most broad account of his approach. In chapter 14 of the book, Fields gives several formulations of the Riemann-Roch theorem. In order to give a statement of one of Fields' formulations of the Riemann-Roch theorem, some preliminaries are needed. A curve C' is said to be *adjoint* to a curve C when the multiple points of C are ordinary or cusps and if C' has a point of multiplicity of order $k-1$ at every multiple point of C of order k [Kli90b, 935]. Given a curve $F(x,y)=0$ of order n, the *strength* of a set of Q (multiple) points used in determining an adjoint curve of degree $n-3$ is defined to be the number of q conditions to which the coefficients of the general adjoint curve of degree $n-3$ must be subjected in order that it may pass through these Q points [Fie02, 167]. An algebraic equation $F(x,y)=0$ can be factored into a product of ρ irreducible factors. In stating the Riemann-Roch theorem, Fields uses the following notations. He indicates the poles c_i of the first order by c_i^{-1} and uses the term "coincidences" to indicate singularities such as the c_i's.

Theorem 5.8 (Riemann-Roch) *The most general rational function of* (x,y) *whose infinities are included under a certain set of* Q *infinities* $c_1^{-1}, \ldots, c_Q^{-1}$*, depends upon* $Q-q+\rho$ *arbitrary constants where* q *is the strength of the set of* Q *coincidences* $c_1, \ldots, c_Q$ *[Fie06, 165].*

With this Fields has given a more general version of the Riemann-Roch theorem than developed in his paper of 1902 in *Acta Mathematatica*. There Fields was only able to give a proof for the theorem when $F(x, y) = 0$ is irreducible, that is, when $\rho = 1$. So Fields carried through on his promise to devise a more general theory. However, given that much of Fields' later writings were reworkings of various parts of his monograph, we can surmise that the reception of his monograph was lukewarm. This was even the case with many of his later papers that were reworkings of portions of his book. For example, consider Fields' paper of 1910 entitled "The Complementary Theorem" which appeared in the pages of the *American Journal of Mathematics*. G. Faber of the University of Königsberg, in his review of the paper in the *Jahrbuch* wrote that "the paper purports to give a proof of the so-called 'Weierstrass Preparation Theorem' that the author gave in the 11th chapter of his *Theory of Algebraic Functions*, by a shorter and simple one," however "the proof still seems to me long and hard to understand" [Fab07]. In another review on Fields' paper entitled "Direct derivation of the complementary theorem from elementary properties of the rational functions," which was published in the proceedings of the fifth International Congress of Mathematicians in 1913 [Fie13a], Prof. Lampe of Berlin writes, after quoting Fields' own introduction to a paper in the *Philosophical Transactions of the Royal Society* [Fie13b] where Fields' claims to have achieved simplification, that "perhaps he [Fields] could try for even more simplification" [Lam07]. All things considered, it seems that Fields' mathematical work on algebraic function theory mostly influenced the work of his student Samuel Beatty (1881-1970), who continued to do work on Fields' research program to some extent. Incidentally, Beatty was the first to receive a PhD in mathematics granted by a Canadian Institution [Rob79, 28].

Chapter 6

Conclusion: The Legacy of J. C. Fields

As can be gathered, Fields was a somewhat minor historical figure in terms of his mathematical contributions, his main claim to fame being that he helped establish the Fields medals. So the question arises, why study minor historical figures? Lewis Pyenson, in discussing work by Marc Bloch, noted three elements present in masterful historical work: "textual and historiographic criticism, a synthetic grasp of the broad sweep of history, and concern for the little actor who reflected the mentality of an age" and that the most convincing of historical collective biographies are "perhaps those where all three elements are present" [Pye77, 178]. He further notes that

> The most important problems in science are seen as those defined, debated, and resolved in élite circles of scientists and science boosters. Science is interpreted as a sort of conspiracy, necessarily financed and occasionally sanctioned by the dumb masses. The history of science nods to the role of popular opinion in determining scientific discourse, but few studies have persuasively described the mechanics of such an interaction [Pye77, 179].

Further, to describe such interaction and to "study the ideology of science as a cultural system we need *especially* to consider the ordinary scientist" [Pye77, 179].[1] These remarks, though written in 1977, remain relevant today, especially in the history of mathematics. Thus the goal of providing a preliminary sketch of the life and mathematical work of J. C. Fields, was to contribute towards a collective historical biography of early Canadian research mathematics and how Canadian mathematicians fit within, and contributed towards, the broader international mathematical milieu.

Though we have only been able to give a sketch of the life and mathematical work of Fields, there is more to learn about how Fields' work in algebraic function theory is situated in the varied approaches to the theory of algebraic functions, and of algebraic geometry more generally, in the late nineteenth and early twentieth centuries. A detailed picture of these approaches including that of Fields, their strengths, weaknesses, similarities and differences, would be most welcome.

There are some confusing issues regarding Fields' notebooks – the purported dates the courses were given, the course titles in the Official Index of Lectures compared to his titles, and the courses recorded on Fields leaving certificates from Berlin and Göttingen – that still need to be worked out. What courses did Fields actually attend officially? What notebooks are only transcriptions of other student's notebooks? Also with respect to Fields' post-doctoral European study tour in the 1890s, no documentary evidence has come to light yet with regards to his stay in Paris during the 1890s. It would be interesting to know if he attended any courses while in Paris, as various sources, such as [Syn33] indicate that Fields spent time there during his study tour. There is currently also no documentary evidence for the identity of Fields' PhD supervisor. It should be noted that Fields was known to have corresponded widely with mathematicians around the world, though most of this correspondence is unfortunately missing [Fie30, 4]. Because of Fields' supposedly wide circle of contacts

[1] Emphasis in the original.

around the world, a prosopographic study centring around Fields may help further contribute towards a collective historical biography of early Canadian research mathematics and how Canadian mathematicians fit within and contributed towards the broader international mathematical milieu.

In Synge's standard obituary of Fields is an excellent summary of Fields' mathematical work which is provided by Professor Samuel Beatty, Fields' only PhD student [Syn33, 156-160]. Beatty divides Fields' work into four groups. The first groups consists of Fields' publications from the period 1885-1893. These papers "offer simplifications of existing treatments, while others give extensions, secured chiefly by skill in manipulation" [Syn33, 156]. The second group is composed of six papers that appeared during the years 1901-1904, along with a seventh paper, which was published in 1913 to supply details for one of the earlier papers in this group. As Beatty notes, these "seven papers mark his [Fields'] developing interest in Algebraic Functions" [Syn33, 156]. The third and last group in Beatty's classification involve Fields' publications about algebraic functions, but for which the assumption that "the fundamental equation has been reduced to having all of its singularities of certain simple types" has been eliminated [Syn33, 156]. Papers in the fourth group deal with material covered in the first 12 chapters of Fields' 1906 book, but there is a "saving of space and effort being due in the main to the use of the Lagrange Interpolation Formula" [Syn33, 157], and also summarizes Fields' theory of algebraic functions *im Kleinen* and *im Grossen*. One other paper in this group applies "certain aspects of the Algebraic Function Theory to the Theory of Ideals" [Syn33, 157]. Furthermore, Beatty writes that

> The work of Dr. Fields on Algebraic Functions must be regarded as the development and organization of ideas on the subject which he happened to have at the turn of the century. These had very little reference to prevailing methods. He made it his life work, first to show that they could

> be used to create a satisfactory theory and then to give to the structure thus secured both elegance and generality. His treatment has the great merit of being completely algebraic in character and of meeting every difficulty without an appeal to geometric intuition. The machinery, which he had to invent for the purpose, is simple, and its parts are beautifully coordinated [Syn33, 159].

We see that despite being respectable work, since Fields did not use "prevailing methods," the impact of Fields' mathematical work is hard to judge, especially since the major thrust of his life work, his arithmetical theory of algebraic functions (with its tinge of function-theoretic methods along the lines of Weierstrass [Bli33, iii]), was ultimately to be overshadowed by the modern approaches that used abstract algebra. There are also other confounding factors to consider when trying to judge the impact of Fields' mathematical contributions. As mentioned in the previous chapter, Fields' work seems to have received a lukewarm reception, at least from German mathematicians as can be seen from some of the reviews from the *Jahrbuch*. Also, Fields' papers are hard to follow, especially for the modern reader. The reviews in the *Jahrbuch* certainly indicate that some of his contemporaries found his work hard going as well. So who exactly read his work? Clearly, his student S. Beatty read his work; those who reviewed Fields' work must have read portions of it; the American mathematician G. A. Bliss mentions Fields' monograph in the preface of his book on algebraic functions [Bli33, iii] as well in an earlier expository paper [Bli24, 96] on the theory of Hensel and Landsberg's. However, Bliss fails to use Fields' approach in his book. However, despite the above, Fields' work appears to have been well regarded, as can be seen by his election to the Royal Society in 1913 and to other societies and academies. As mentioned, Fields' work was subsumed by other approaches, which were more susceptible of generalization of the basic theory, so it seems that the ultimate influence of his work is small, maybe mostly expressed through the work of his student S. Beatty.

All this said, it is worth considering the following. Fields' research legitimized him as a scientific authority. His authority surely must have played a key role in his push to get governmental support for scientific research. Fields and others were to see this goal come to fruition, first with provincial support of research at the University of Toronto, and then later, with the establishment of the National Research Council (the main funding avenue for research mathematicians in Canada until the Natural Sciences and Engineering Research Council was created). So Fields' research helped nurture the growing community of research mathematicians in Canada. According to S. Beatty, Fields "by his insistence on the value of research as well as by the importance of his published papers, has, perhaps, done most of all Canadians to advance the cause of mathematics in Canada" [Bea39, 109]. The Canadian mathematical community is now a thriving one and Fields' legacy surely plays some part in this.

Appendices

A A Note on Fields' Berlin Notebooks

As was noted in the main text, Fields spent a period of time studying in Berlin, Germany, in the 1890s after finishing his PhD at Johns Hopkins University in Baltimore, Maryland. There is a rich archival resource that remains from this part of Fields' life, namely his many notebooks, now housed at the University of Toronto Archives from this period. The notebooks are all notes of lecture courses Fields either attended or copies of other people's notes (for example, the notebooks that record lectures by Weierstrass are transcriptions). Those interested in looking at Fields' notebooks should visit the University of Toronto Archives, located in Toronto, Ontario, Canada. The accession number for Fields' notebooks is B1972-0024 and consists of two boxes.

B The Fields Medal

The Canadian sculptor R. Tait McKenzie was commissioned by the organizing committee of the Toronto ICM to design the medal. The final design has on one side a head that represents Archimedes and an inscription in Greek that translates into "To transcend one's spirit and to take hold of the world," and on the other an inscription on a background with a Laurel branch and a sphere being inscribed in a cylinder (like

Figure 6.1: The Fields Medal given to Maxim Kontsevich in 1998 in Berlin. ©International Mathematical Union. Used with permission

that of the engraving believed to have been on Archimede's tomb), that reads in Latin "The mathematicians having congregated from the whole world awarded (this medal) because of outstanding writings" [IMU07]. The name of the recipient is engraved on the rim of the medal. The medal is struck every four years at the Royal Canadian Mint.

At the Zürich Congress, Synge brought forward Fields' proposal for the medals. The idea was accepted and the first committee in charge of choosing awardees was formed. This committee consisted of G. D. Birkhoff, C. Carathéodory, E. Cartan, F. Severi and T. Takagi. The first medals went to Lars Ahlfors of Harvard and Jesse Douglas of MIT at the 1936 Congress in Oslo. And thus begun the tradition of awarding the Fields medals at successive Congress to mathematicians of the highest calibre. For a short survey of the work of the Fields medallists, see [Mon97]. The permanent trustee of the Fields Medal prize fund is the National Trust Company, Ltd., located in Toronto. Even though the Fields Medal is often referred to as the "Nobel Prize in mathematics," the monetary value of the Fields Medal is currently only about $15,000 Canadian in 2002 [Rie, 781].

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