The Mittag-Leffler Theorem: The Origin, Evolution, and Reception of a Mathematical Result, 1876-1884

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Abstract

The Swedish mathematician Gösta Mittag-Leffler (1846–1927) is well-known for founding *Acta Mathematica*, the first international mathematical journal. A "post-doctoral" student in Paris and Berlin (1873-76), Mittag-Leffler built on Karl Weierstrass' work by proving the Mittag-Leffler theorem, roughly: a meromorphic function is specified by its poles, their multiplicities, and the coefficients in the principal part of its Laurent expansion.

In this thesis, I explore the evolution of the Mittag-Leffler theorem, from its initial (1876) state to its final (1884) version. Aspects of the details of Mittag-Leffler's work at various stages are analyzed to demonstrate the evolution of Mittag-Leffler's technique. A key finding of the thesis is that Mittag-Leffler's research on infinite sets of singular points attracted him to Georg Cantor's set-theoretic work. The incorporation of Cantor's theory was controversial, but demonstrates Mittag-Leffler's important role in the promotion of abstract mathematics over the more concrete mathematics of the previous era.

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Chapter 1

Introduction

Magnus Gösta Mittag-Leffler was born March 16, 1846 in Stockholm, Sweden. His childhood education was greatly enriched by close contact with his father and frequent visits from family friends, and this addition to his traditional schooling proved to be fruitful in developing a multi-faceted set of talents. However, when he entered the Gymnasium in Stockholm his teachers recognized in particular his aptitude for higher mathematics, which he studied after training as an actuary. [Går94], pp. 73–74

His post-secondary mathematical pursuits began in 1865 when he became a student at the University of Uppsala, where he defended his doctoral thesis on applications of the argument principle in 1872. His subsequent appointment (1872) as a lecturer at this school did a great deal to further his mathematical career, as the salary awarded to this position was paid through an endowment with the attached condition that the holder must spend three years abroad. This gave Mittag-Leffler the opportunity to connect with and learn from many influential mathematicians of his day, and in fact his travels began in October 1873 when he left for Paris to study with Charles Hermite. He remained in Paris until the spring of 1875, when Hermite's glowing description of the German mathematical community (and Karl Weierstrass in particular) resulted in Mittag-Leffler's decision to travel to Berlin.[1] In fact, Mittag-Leffler later remarked that Hermite had encouraged the young Swede to travel

[1]Interestingly, though Mittag-Leffler later suggested that Hermite expressly advised him to make this journey, his diary entries of this time make no reference to a direct suggestion on the part of Hermite [Går94], p. 90.

to Berlin from the very outset:

> I will never forget the amazement that I experienced at the first words [with which] he [Hermite] addressed me: "You have made a mistake, Sir, I tell you; you would have to follow the courses of Weierstrass in Berlin. He is the master of all of us.[2] [Mit02], p. 131

Thus during 1875 Mittag-Leffler frequented Weierstrass' lectures and was heavily influenced by the German mathematician's work in complex function theory, particularly the material which was to become Weierstrass' 1876 paper *On the theory of single valued analytic functions*[3]. A survey of Weierstrass' relevant mathematical work has been compiled by U. Bottazzini (see [Bot86] and [Bot03]), but a summary follows in the next section.

Mittag-Leffler's contact with Weierstrass' research in complex analysis proved to be fundamental to his own mathematical research for the next eight years. In 1876 and 1877 he elaborated upon Weierstrass' 1876 factorization theorem and proved a slightly less-general version of the now-familiar theorem associated with Mittag-Leffler's name which states that given a set each of poles, corresponding multiplicities, and Laurent coefficients, it is always possible to find a function which is analytic except at those poles, with the correct multiplicities and Laurent coefficients, up to the addition of an entire function.

Prior to the first publication of the above theorem, while he was still in Berlin Mittag-Leffler heard of an opening chair of mathematics post at Helsingfors (now Helsinki), Finland and left for the University of Helsingfors in 1876. This occurred despite Weierstrass' wishes to find for him a professorial position at Berlin, a superior institution with respect to a career in mathematical research. Mittag-Leffler remained at Helsingfors for five years and then returned to Stockholm to become the first mathematics chair at the new university Stockholms Högskola. He subsequently met and married the wealthy Signe af Lindfors and upon the urging of Sophus Lie he began the organization of his *Acta Mathematica*, the first truly international journal of mathematics and the first to remain independent of all institutions [Bar02].

[2]"[J]e n'oublierai jamais la stupéfaction que j'éprouvai aux premiers mots qu'il m'adressa: "Vous avez fait erreur, Monsieur, me dit-il; vous auriez du suivre les cours de Weierstrass à Berlin. C'est notre maître á tous."

[3]"*Zur Theorie der eindeutigen analytischen Funktionen.*"

During these years, Mittag-Leffler attempted further generalizations of his 1876 theorem. The first version dealt only with the case of a function having at most one essential singularity, which could only occur at infinity. His mentors Weierstrass and Hermite were excited by this result, and each provided an alternate proof in 1880. Mittag-Leffler adopted Weierstrass' suggestions, and in 1882 had a stream of eight publications in the *Comptes Rendus de l'Académie des Sciences* of Paris between February and April. These were excerpts of letters written from Mittag-Leffler to Hermite, and during the course of these letters the reader can observe the steps taken by Mittag-Leffler to realize his goal of generalization to the case of a function having an infinite number of essential singularities. During the period of time covered by these publications, Mittag-Leffler also published a certain number of papers concerning the theory of interpolation and approximation, an interest he shared with Hermite. These publications will not be discussed in this paper; though they are, to some extent, linked to the Mittag-Leffler theorem, the two issues are sufficiently distinct to be treated separately. Furthermore, Mittag-Leffler's work on interpolation and approximation does not involve Cantor's work on infinite sets of points, and thus highlights an entirely different set of issues in the history of analysis. I hope to return to this work at a later date.

The Mittag-Leffler theorem is historically interesting for several reasons. As Mittag-Leffler was a student of Weierstrass, the theorem provides insight into key issues in analysis at that period, particularly the relationship between Weierstrass' research and that of his immediate successors. The evolution of the hypotheses of the Mittag-Leffler theorem is also worthy of study; there were several variant publications of the theorem between 1876 and 1884, the majority of which were by Mittag-Leffler himself, and there is a noticeable evolution of the ideas which is marked by changes in notation from the original Weierstrassian style and the simplification of proofs. In particular, the later versions of Mittag-Leffler's theorem differ markedly from those of his initial papers. Specifically, in 1882 we see that there is a rather abrupt appearance of Georg Cantor's recently developed theory of transfinite sets in Mittag-Leffler's theorem statements and proofs, despite the generally negative reception of Cantor's work by the prominent mathematicians of that period. I will later discuss this reception to provide the context in which Mittag-Leffler developed his ideas, as it also coloured the reaction to the Swedish mathematician's work, particularly in France.

Mittag-Leffler published the final version of his theorem in 1884, in his newly-established journal *Acta Mathematica*. This paper served as a comprehensive account of essentially all of his work on the subject. After this publication, Mittag-Leffler's focus turned to mathematical organization, though he continued to publish actively. He served as the editor in chief of *Acta* for 45 years, until his death on July 7, 1927 in Stockholm. During his life he received many honours. He was an honorary member of almost every existing scientific society, including the Cambridge Philosophical Society, the Royal Institution, the Royal Irish Academy, the London Mathematical Society, and the Institut de France. He also received honorary degrees from six different universities, and in 1886 was elected a Fellow of the Royal Society of London. [Går94]

There has been considerable historical work on various aspects of Mittag-Leffler's career and activities. A full-length biography by A. Stubhaug, intended for a broad audience, is currently in progress and expected to appear next year. L. Gårding is the author of a shorter study including both biographical and technical material [Går94]. U. Bottazzini has written about the relationship of Mittag-Leffler's work to that of Weierstrass' in [Bot86] and [Bot03]; he has also forthcoming work with J. Gray. In addition, J. Barrow-Green has investigated Mittag-Leffler's role as founder of Acta Mathematica [Bar02]. She has also described his role in creating a prize competition in Sweden, in connection with her study of Poincaré's work on the three-body problem [Bar97]. Portions of Mittag-Leffler's extensive correspondence have also been published: the letters from Hermite to Mittag-Leffler, edited by Dugac [Her84]; the correspondence with Henri Poincaré edited by Nabonnand [Nab99], and portions of correspondence with Cantor and Kovalevskaya. Mittag-Leffler has been mentioned in studies of Kovalevskaya, most notably by A. Hibner Koblitz and R. Cooke, due to his role in her career; it was due to his efforts that she received a lectureship appointment in mathematics at the university in Stockholm, and then later a life professorship [Kra73]. I. Grattan-Guiness has published various articles on Cantor's work, and J. Dauben, who has published extensively on Cantor's work and philosophy (in particular, see [Dau79]), has written about Mittag-Leffler's relationship to the German mathematician. As well, the actual material upon which this paper is based was deemed sufficiently important to be partially translated and published in the Birkhoff/Merzbach source book in classical analysis [Bir73].

However, the literature just mentioned does not provide a comprehensive account of the evolution of the Mittag-Leffler theorem, nor does it focus on Mittag-Leffler's mathematical relationship with Cantor with respect to this theorem. The intent of this thesis is then to provide a substantially fuller picture of the development of the hypotheses of the Mittag-Leffler theorem, and the subsequent reception and interpretation of the work, using Mittag-Leffler's correspondence with members of both the French and German mathematical communities (among them Hermite, Poincaré, Cantor, Weierstrass, and Kronecker), and in doing so, demonstrate Mittag-Leffler's role in the promotion of abstract objects in mathematics from the more concrete and "formula-based" mathematics of the previous era.[4]

[4]Refer to [Sø05] for a discussion of formula-based versus concept-based mathematics.

Chapter 2

Weierstrass's Influence

In order to contextualize Mittag-Leffler's achievements in the field of complex analysis, we must turn to those of his mentor Weierstrass.

Karl Weierstrass (1815–1897) was born at Ostenfelde, Germany. His early education was conducted in Münster and Paderborn, and in 1834, upon the wishes of his father, he commenced studies in *Kameralwissenschaft*[1] at Bonn University. He left after four years without even attempting to obtain a degree or even pass an examination. Returning to Münster in 1839, he studied at the local Academy to become a teacher. It was here that he attended lectures given by Gudermann on the theory of elliptic functions; Weierstrass was immediately transfixed, and commenced his own work on the subject, writing an essay about it during the summer of 1840 which would only be published in full 54 years later in his *Werke*. After receiving his teaching diploma he taught in a Münster Gymnasium, all the while continuing to work on his own mathematical research. He taught at the Progymnasium in West Prussia (this included lower classes such as gymnastics and penmanship), and then at the Gymnasium in Braunsberg in 1848. The following year he published a work on the theory of Abelian functions in the program of the Gymnasium. He further developed his theory and published two articles in Crelle's Journal (1854 and 1856) which generated a great deal of excitement within the mathematical community. As a result, he was awarded an honorary doctorate at Königsberg University, and in 1856 became an extraordinary professor at Berlin

[1] A predecessor of the modern study of public administration.

University. Shortly thereafter, he was elected a member of the Berlin Academy of Science, and in 1864 finally became a professor Ordinarius at the same institution. [Bot03], pp. 247 – 249

Weierstrass, now known as a master of rigour, did not become interested in the foundations of analysis until 1859, when he first lectured on the subject. Between the early 1860s and 1890, he developed his theories in analysis into a series of four courses of lectures which were given in cycles [Arc]. In 1863 he taught a course on the general theory of analytic functions. Interestingly, it was only at this time that Weierstrass began to think that greater rigour was necessary in analysis. His work on this subject was published in his *On the theory of single valued analytic functions* several years later, the main focus of which was complex analysis [Bot03], pp. 250–252. Regarding Weierstrass' theory of analysis, the following point should be emphasized. To Weierstrass, the way to study analysis was to work with (general) *representations* of functions. This should be kept in mind throughout the discussion of his representation theorem and Mittag-Leffler's continuation of the work.

Before turning to Weierstrass' *On the theory of single valued analytic functions*, we must introduce some of Weierstrass' terminology. The definitions presented here are actually taken from Mittag-Leffler's work [Mit76], and are based upon those of Weierstrass' lectures [Wei88].

Definition 2.1 *A function of the independent [complex] variable x has, for a finite value a, the* character of an entire function *if it can be expressed by a power series of only whole and positive powers of $(x - a)$. If the function has, for every finite value of the variable x the character of an entire function, we say that it is a function of* entire character. [Mit76], pp. 3–4

In present-day mathematics we do not differentiate between entire functions and functions of entire character, and use the term "entire" for functions which are analytic everywhere. However, when Weierstrass and Mittag-Leffler worked with these definitions, an *entire function* was strictly a polynomial, where a function of *entire character* contained infinitely many non-zero terms in its power series expansion.

Definition 2.2 *A function of the independent variable x has, for a finite value a, the* character of a rational function *if it can be expressed by a power series of whole and positive powers of $(x-a)$ and a finite number of negative powers of $(x-a)$. If the function has, for every finite value of the variable x the character of a rational function, we say that it is a function of* rational character*.* [Mit76], pp. 3–4

Hence a function is of rational character at a pole of finite order or a regular point. To be clear, to Weierstrass and Mittag-Leffler a *rational function* was a quotient of two polynomials, while a function of *rational character* was a quotient of two Taylor series, where the series in the numerator could be infinite. As one further type of function will later become important, it will be defined now in present-day terminology. The following definition is taken directly from Rudin.

Definition 2.3 *A function $f(x)$ is* meromorphic *in an open set $\mathbb{C}$ if there is a set $A \subset \mathbb{C}$ such that*

(a) A has no limit point in $\mathbb{C}$,

(b) $f(x)$ is analytic in $\mathbb{C} \setminus A$, and

(c) $f(x)$ has a pole at each point of A. [Rud74], p. 241

Thus a meromorphic function is analytic everywhere except at possibly isolated poles. Note that from (a), no compact subset of $\mathbb{C}$ has infinitely many points of A, and so A is at most countable.

It is important to understand that the term "meromorphic" was not used in mathematics during the development of the Mittag-Leffler theorem. Though the present-day statement of the theorem (refer to Appendix D) indicates that it is a *meromorphic* function that is specified by its poles, their corresponding multiplicities, and the coefficients in the principal part of its Laurent expansion, by 1882 Mittag-Leffler was actually dealing with a larger class of functions. This will be mentioned again in Chapter 4.

With that in mind, I will work with Mittag-Leffler's terminology in what follows.

With respect to the power series expansions themselves, Weierstrass refers to the coefficients of the terms $(x-a)$ with negative powers as the *constants*; of course, a is then a singularity (infinite point). It is interesting to note that in the pre-Cantor era of mathematics, the point at infinity was thought of as a very valid, though special, value for a function to take. It was thought of as actually having meaning, and not just existing as a symbol.

Definition 2.4 *A series* $\sum a_n$ *is* absolutely convergent *when the corresponding series* $\sum |a_n|$ *is convergent. Suppose a series of functions* $\sum f_n(x)$ *converges to the function* $f(x)$ *in some neighbourhood of the complex plane. We say that the convergence is uniform if, for any* $\epsilon > 0$ *there is an* N *such that* $n > N$ *makes* $|f_n(x) - f(x)| < \epsilon$ *for every* x *in the neighbourhood.*

To Weierstrass, the issue of uniform convergence was of utmost importance; that this form of convergence should be distinguished from ordinary convergence was a key feature of his lectures. Weierstrass, who essentially thought of functions as power series representations, saw the key feature of uniform convergence: if a series of analytic functions converges uniformly, the sum is then analytic. [Arc]

It is also useful to discuss briefly the concept of *monogenicity*. A function may be represented by a Taylor series that converges to that function in a neighbourhood of a point x_0 of the domain. Weierstrass referred to such series representations as *function elements* and denoted them by $P(x|x_0)$. Given two values x_0 and x_1, it may be possible to represent the function as a set of Taylor series, or function elements, on a chain of disks at points in between. These various representations are *analytic continuations* of each other (note that running into singularities imposes size limits on the disks about each point). Within a region where it is possible to conduct such analytic continuation without obstacles, the function is said to be monogenic, since all of the function elements represent the same function and have the same values.

Definition 2.5 *If a function* $f(x)$ *is single-valued and if the fact that it can be represented by a power series in a neighbourhood of* x_0 *of its domain implies that all of its elements* $P(x|x')$ *can be derived from a single element, then* $f(x)$ *is* monogenic. [Wei88], p. 114

Weierstrass created power series converging to different functions in different regions of the plane, thus showing that there is a difference between being analytic (or meromorphic) and being monogenic.

Weierstrass published his *On the theory of single valued analytic functions* in 1876, the year after Mittag-Leffler had studied with him in Berlin. This was a study regarding the foundations of analytic functions in which Weierstrass aimed to tackle the problem of representation and classification of single-valued complex functions. Enrico Betti had previously (1860) dealt with this problem to some extent, and was able to prove that entire functions (those which are analytic everywhere) can be broken down into a product of infinitely many factors of degree one and of exponential functions [Bot86], pp. 280–281. Weierstrass discovered this result independently, but was able to further generalize it. He interested himself in several general and directly related questions concerning the extent to which a function is determined by its zeros. By the fundamental theorem of algebra, it is well-known that any polynomial can be broken into linear factors, one for each zero. In this vein, Weierstrass took the idea of factorization and asked the following question: is it possible to write a representation of other entire functions as a product of factors in a way that reveals their zeros? Furthermore: given an infinite sequence of constants $\{a_n\}$ with the limit of terms approaching infinity, can one always find a function whose zeros are given by this sequence?

These questions arose in conjunction with several examples Weierstrass presented in his lectures. To clarify these ideas, some of these will be discussed now, the first being the identity

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi} \right],$$

originally found in Euler's *Introductio in analysin infinitorum* (1748). With appropriate convergence arguments we can rearrange the summand into

$$\frac{1}{z - n\pi} + \frac{1}{n\pi}$$

where we now sum over all the nonzero integers.

Integrating the left side, we have

$$\int \cot z dz = \log \sin z;$$

if we also integrate the series on the right side term by term, we find

$$\int \cot z dz = \log z + \int_0^z \sum_{n=-\infty, n\neq 0}^{\infty} \left[\frac{1}{z - n\pi} + \frac{1}{n\pi} \right] dz.$$

Equating the two,[2]

$$\log \sin z = \log z + \sum \left[\log(z - n\pi) - \log(n\pi) + \frac{z}{n\pi} \right] + c,$$

and through exponentiation

$$\sin z = kz \prod \left[1 - \frac{z}{n\pi} \right] e^{\frac{z}{n\pi}} (n \neq 0).$$

It can then be argued that $k = e^c = 1$.

Weierstrass also described a similar kind of expansion which can be found for the reciprocal of the gamma function, namely

$$\frac{1}{\Gamma(z)} = z \prod_1^\infty \left(1 + \frac{z}{n} \right) e^{-z \log \frac{n+1}{n}},$$

which had previously been studied by both Euler and Gauss. Gauss' 1812 paper on hypergeometric series, the aim of which was to develop a representation for a large class of functions as quotients of power series, was particularly influential on Weierstrass. That Gauss was able to change one of these quotients into a factorization (in other words, a product expansion) led Weierstrass to the natural question which followed: could this be done in general? [Bot03], pp. 95–96, 251

Thus product expansions of the type seen above led Weierstrass to explore the question of how to create a standard representation of this type for analytic functions. There is one important thing to notice about the above examples. The expansions of these two different entire functions have one important common feature: they display clearly the zeros (there are of course no poles) but have an additional term in the form of an exponential factor. These exponential terms are the key to making the products convergent.

Weierstrass worked with these concepts, and by defining his "prime functions"

$$E(z, 0) = 1 - z,$$
$$E(z, p) = (1 - z) \exp \left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right),$$

he was able to prove his famous *factorization theorem* which states that given the sequence $\{a_n\}$ and a sequence $\{m_n\}$ of positive integers, there exists an entire function $G(z)$ with a

[2]Note that the c in the line below serves to take care of the fact that the integral is indefinite.

zero at $x = a_n$ for each n with corresponding multiplicity m_n and no other zeros. As well, if a_n is nonzero for every n, $G(z)$ can be represented by the product

$$G(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \ldots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}},$$

where $g(z)$ is an entire function, for certain integers $\{m_n\}$. It is notable that the results were considered sufficiently important that they were translated into French by E. Picard [Wei79]. For a modern proof, one can refer to [Ahl79].

These results were published while Mittag-Leffler was still in Berlin. Having attended Weierstrass' lectures during the previous year, he was very familiar with the work of the famous mathematician and chose to elaborate upon some of his representation theorems for entire functions. The next logical step was, of course, to try to apply these concepts to the class of meromorphic functions. Weierstrass himself had attempted this, but had difficulty in developing a similar generalized representation for meromorphic functions. Mittag-Leffler was aware of this difficulty, and decided to attempt the problem himself. In fact, in a passage from the first publication of his theorem Mittag-Leffler mentions the fact that he actually heard Weierstrass' lectures in Berlin and from them was led to his work on the subject:

> The author of the present work [Mittag-Leffler's [Mit76]], who had the fortune to be in Weierstrass' lecture audience at that time [Winter 1875], was led by this communication [i.e. the lectures] to pose himself a problem analogous to Weierstrass' ...[3] [Mit76], p. 15

Mittag-Leffler thus began with Weierstrass' factorization theorem, a topic which Weierstrass had also discussed in his lectures. However, where Weierstrass began with polynomials and generalized, using the idea of factorization, to the case of entire functions, Mittag-Leffler began with rational functions and attempted to generalize them to meromorphic functions using the concept of partial fraction decomposition.

[3]"Författaren till föreliggande arbete, hvilken hade den lyckan att vid denna tidpunkt tillhöra Herr Weierstrass åhörarekrets, föranleddes af detta meddelande att ställa sig det med det Weierstrassiska analoga problem ..."

His question was the following: Given arbitrary poles $\{a_n\}$, does there exist a meromorphic function $F(z)$ having arbitrarily assigned principal parts (at each pole)

$$G_n\left(\frac{1}{z-a_n}\right) = \sum_{k=1}^{\nu(n)} \frac{C^n_{\nu(n)}}{(z-a_n)^k}$$

with arbitrary poles the sequence of terms $\{a_n\}$? [Mit76], p. 4

Such a meromorphic function must have a sequence of poles that tends to infinity, and the principal part $G_n(z)$ of the Laurent series of this function about a singularity a_n must be a polynomial in $\frac{1}{z-a_n}$ without a constant term, so that the difference $F(z) - G_n(z)$ is regular in a_n. To construct such a function (with given poles and principal parts), one must ensure that the sum $\sum G_n(z)$ converges.

Thus Mittag-Leffler sought to prove the existence of such a function $F(z)$, and succeeded in doing so later that year.

Chapter 3

The first versions of Mittag-Leffler's theorem

Mittag-Leffler presented the paper containing what would later become known as the Mittag-Leffler theorem to the Royal Swedish Academy of Science on June 7, 1876. The accompanying article was entitled "A method to analytically represent a *function of rational character*, which becomes infinite exactly at certain prescribed *infinite points, whose constants* are given beforehand"[1][Mit76]. As noted above, the italicized phrases indicate terminology developed by Weierstrass. Mittag-Leffler defines these terms immediately (refer to Chapter 2) and then poses the following question: is a function of rational character completely defined by its singularities and their (respective) constants, or can we find more than one similar function when the singularities and constants are given? In other words, is a function uniquely specified by its singular points and the coefficients of its Laurent series?

Noting that if two different such functions can be found, their difference must necessarily be an entire function, and thus possesses a power series expansion that converges to the function everywhere, Mittag-Leffler proceeds to tackle the question.

He first supposes given both an infinite sequence

$$a_1, a_2, a_3, \ldots, a_p, \ldots$$

[1]"En metod att analytiskt framställa en *funktion af rationel karakter*, hvilken blir oändlig alltid och endast uti vissa föreskrifna *oändlighetspunkter, hvilkas konstanter* äro påförhand angifna."

of different singular points approaching infinity and the constants in their respective principal parts

$$\begin{array}{ccccc} c_{11} & c_{12} & c_{13} & \cdots & c_{1\lambda_1} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2\lambda_2} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3\lambda_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{p1} & c_{p2} & c_{p3} & \cdots & c_{p\lambda_p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Mittag-Leffler indicates that the singularities in the sequence $\{a_n\}$ are to be ordered by increasing distance from the origin. That is, if we fix an arbitrary positive value r, the preceding terms in the sequence will have a modulus less than r, and the following terms will have a modulus greater than r. If several terms happen to have an equal modulus (and thus they fall on a circle), say r, they are ordered by tracing the circle in a particular direction. This ordering will play an important role in Mittag-Leffler's argument for the construction of the sought function.

Now, in order to produce a function that is analytic everywhere except at the prescribed poles, it is necessary that in the neighbourhood of a pole a_p, the principal part at that point

$$\frac{c_{p1}}{(x-a_p)} + \frac{c_{p2}}{(x-a_p)^2} + \ldots + \frac{c_{p\lambda_p}}{(x-a_p)^{\lambda_p}}$$

is absolutely convergent within that neighbourhood. If this is true for every $p = 1, 2, 3, \ldots$, Mittag-Leffler notes that certainly the easiest way to create a function $f(x)$ having the poles $\{a_n\}$ is to define it as the sum of its prescribed principal parts

$$f(x) = \sum_p \left[\frac{c_{p1}}{(x-a_p)} + \frac{c_{p2}}{(x-a_p)^2} + \ldots + \frac{c_{p\lambda_p}}{(x-a_p)^{\lambda_p}} \right],$$

which can be rewritten as

$$f(x) = \frac{c_{p1}}{(x-a_p)} + \frac{c_{p2}}{(x-a_p)^2} + \ldots + \frac{c_{p\lambda_p}}{(x-a_p)^{\lambda_p}}$$

$$+ \sum_{r \neq p} \left[c_{r1} \frac{1}{(a_p - a_r)\left(\frac{x-a_p}{a_p-a_r} + 1\right)} + \ldots + c_{r\lambda_r} \frac{1}{(a_p-a_r)^{\lambda_r}\left(\frac{x-a_p}{a_p-a_r} + 1\right)^{\lambda_r}} \right].$$

However, in most cases this is unlikely to converge. It will do so, however, if the sum

$$\sum_p \left(c_{p1} + c_{p2} + \ldots + c_{p\lambda_p}\right)$$

is absolutely convergent, a requirement which is not necessarily satisfied by an arbitrary choice of constants c. This is a consequence of Abel's test[2], as adapted for uniform convergence. That is, if $\sum_p \left(c_{p1} + c_{p2} + \ldots + c_{p\lambda_p}\right)$ converges uniformly, then since

$$\left\{ \frac{1}{(a_p - a_r)\left(\frac{x-a_p}{a_p-a_r} + 1\right)} + \frac{1}{(a_p - a_r)^2\left(\frac{x-a_p}{a_p-a_r} + 1\right)^2} + \ldots + \frac{1}{(a_p - a_r)^{\lambda_r}\left(\frac{x-a_p}{a_p-a_r} + 1\right)^{\lambda_r}} \right\}$$

forms a decreasing sequence (due to Mittag-Leffler's ordering of the sequence $\{a_n\}$), by Abel's test $f(x)$ converges uniformly as well. This uniform convergence guarantees analyticity, and thus is $f(x)$ the desired representation. If the sum $\sum_p \left(c_{p1} + c_{p2} + \ldots + c_{p\lambda_p}\right)$ does not converge, it may be possible to construct functions $g_{pk}(x)$ to replace the constants c. If then

$$\sum_p \left(g_{p1}(x) + g_{p2}(x) + \ldots + g_{p\lambda_p}(x)\right)$$

is everywhere uniformly convergent, the sum of principal parts

$$f(x) = \sum_p \left[\frac{g_{p1}(x)}{(x-a_p)} + \frac{g_{p2}(x)}{(x-a_p)^2} + \ldots + \frac{g_{p\lambda_p}}{(x-a_p)^{\lambda_p}}\right],$$

plus an entire function, is the general analytic representation of the sought function. It is necessary that the series of functions $g_{pk}(x)$ converges uniformly, as this is what guarantees that the series of differentiable functions converges to another differentiable, and hence analytic, function.

To simplify the problem somewhat and determine the appropriate functions $g(x)$, Mittag-Leffler chooses to temporarily consider only sequences $\{a_n\}$ of *simple* poles. In other words, for every p, $\lambda_p = 1$, making the sum of principal parts equal to

$$f(x) = \sum_r \frac{g_r(x)}{(x-a_r)}.$$

[2]Consider the series $a_0c_0 + a_1c_1 + \ldots$. If c_n is a positive decreasing sequence and $\sum a_n$ converges, then $\sum a_nc_n$ converges [Bro08], pp. 245–247. In other words, if we multiply a convergent series by a sequence of positive and decreasing terms, the resulting series will also be convergent. Note that for uniform convergence of the sum $\sum a_nc_n$, the sequence $\sum a_n$ must be uniformly convergent as well.

He then suppose further that the origin is not a singular point. Then setting

$$g_r(x) = c_r \left(\frac{x}{a_r}\right)^{\nu_r}$$

where ν_r is finite and whole for every r produces the desired function, provided that the sum

$$\sum_r c_r \left(\frac{x}{a_r}\right)^{\nu_r}$$

is everywhere convergent. Mittag-Leffler states that this can always be accomplished through the appropriate choice of ν_r.

An analogous but more complex argument is provided for the case in which the poles are not required to be simple. This involves replacing the initially prescribed constants c with functions

$$g_{rq}(x) = k_{rq} \left(\frac{x}{a_r}\right)^{\nu_{rq}}.$$

These values k, which are defined in terms of the coefficients c by a system of linear equations, are now used to produce convergence everywhere. At this point, only one limitation remains to Mittag-Leffler: he has not yet allowed the origin to be a singular point of the sought function. However, he quickly remedies this by noting that, in that case, by adding

$$\frac{c_{01}}{x} + \frac{c_{02}}{x^2} + \ldots + \frac{c_{0\lambda_0}}{x^{\lambda_0}}$$

to the sum of the principal parts the desired function is produced.

In summary, we see that the theorem statement of his 1876 text is essentially that one can always construct a meromorphic function $f(x)$ with prescribed principal parts $G_n(z)$ of the Laurent expansion at an infinite sequence of prescribed poles $\{a_n\}$ provided that the sequence of poles approaches infinity. As well, any such function can be written as

$$\phi(z) + \sum_1^{\infty}(G_n(z) + P_n(z)),$$

where $\phi(z)$ is an entire function and $\{P_n(z)\}$ are polynomials that guarantee the convergence of the expansion.

I note for clarity, and Mittag-Leffler understood, that such a function has one essential singular point at infinity, the accumulation point of the set of poles. Such an accumulation point a is necessarily an essential singularity, as the cluster of poles around a makes it impossible to find a neighbourhood of a in which the function has a Laurent series expansion (and hence analyticity fails in every neighbourhood of a).

One interesting point follows. Toward the end of his paper, Mittag-Leffler observes that by restricting the sequence of singularities to only simple poles (i.e. letting $\lambda_0 = \lambda_1 = \ldots = \lambda_r = \ldots = 1$), and specifying that every coefficient of each principal part be equal to 1 (i.e. letting $c_{01} = c_{11} = \ldots = c_{r1} \ldots = 1$), it can be seen that the Weierstrass factorization theorem is a special case of the theorem that Mittag-Leffler has just described.

By restricting the coefficients as above, and recalling that

$$\sum_p \left[\frac{g_{p1}(x)}{(x-a_p)} + \frac{g_{p2}(x)}{(x-a_p)^2} + \ldots + \frac{g_{p\lambda_p}}{(x-a_p)^{\lambda_p}} \right]$$

is the form of the general analytic representation of a function with poles $\{a_n\}$, we see that

$$\frac{1}{x} + \sum_{r=1} \left[\frac{1}{(x-a_\nu)} \left(\frac{x}{a_r} \right)^{\nu_r} \right] + P(x)$$

is the function we seek (note that the term $\frac{1}{x}$ indicates that the origin is also a singular point). This can be easily rewritten as

$$\frac{1}{x} + \sum_{r=1} \left[\frac{1}{(x-a_\nu)} + \frac{1}{a_r} \left(1 + \frac{x}{a_r} + \left(\frac{x}{a_r} \right)^2 + \ldots + \left(\frac{x}{a_r} \right)^{\nu_r - 1} \right) \right] + P(x),$$

which is the logarithmic derivative of the product

$$x \cdot e^{P(x)} \prod_{r=1} \left[(x-a_r) \cdot e^{\frac{x}{a_r} + \frac{1}{2}\left(\frac{x}{a_r}\right)^2 + \frac{1}{3}\left(\frac{x}{a_r}\right)^3 + \ldots + \frac{1}{\nu_r}\left(\frac{x}{a_r}\right)^{\nu_r}} \right].$$

It is clear that this product is a function of entire character that disappears when $x = 0$ or any of the points $\{a_n\}$, and thus the arbitrarily specified poles have become zeros of the logarithmic antiderivative. As well, by looking at the product expansion we can see that two functions of entire character which have the same zeros and corresponding orders can differ only by a factor of the form $e^{Q(x)}$ where $Q(x)$ is a power series in x. This fact follows from Mittag-Leffler's observation that two such representations must differ by an entire function. Thus the product expansion gives the general analytic expression of a function of entire character whose zeros and their respective orders are prescribed arbitrarily (provided that in any finite neighbourhood there is at most a finite number of zeros). Mittag-Leffler was to stress the importance of this relationship between his work and the Weierstrass factorization theorem in his later publications on this subject as well.

Over the next year Mittag-Leffler produced a handful of papers in Swedish containing several variations on the theme of his 1876 paper. Each paper concerns a progressively more complicated sequence of essential singularities than the previous: [Mit76], as just mentioned, deals with the case of one essential singularity at infinity; [Mit77b] with the case of a finite number of essential singularities; and [Mit77a] with one essential singularity which has been moved from infinity to a finite value. This step-by-step progression parallels Weierstrass' procedure for generalization in his [Wei76], in which Weierstrass had asked precisely the analogous questions, namely: how is a function determined by its sequence of zeros? Then, given a sequence $\{a_n\}$ increasing in absolute value to infinity, does there exist a function that has this sequence as its zeros? [Wei79], p. 116

Under Mittag-Leffler's hypotheses, this kind of progression leads to the natural question of the representation of functions with an *infinite* set of essential singularities. Mittag-Leffler's interest in Cantor's work arose from the desire to answer this question. Cantor, whose work distinguished different kinds of infinite point sets, would turn out to provide exactly the help and information that Mittag-Leffler needed.

It may seem unusual that Mittag-Leffler published these exciting results in Swedish, and not in the French or German that would have made his work accessible to a much wider audience. However, though Mittag-Leffler had studied in both France and Germany, it is reasonable to believe that his skills in those languages would have been weaker than his abilities in Swedish. More interesting, however, is his eagerness to leave Finland, where he was lecturing, to return to the Stockholm area. On June 8, 1878, Mittag-Leffler wrote to Weierstrass:

> ...I have over 50 students and I lecture on everything alone. Besides that these damned "Fennomanen" [Finnish nationalists] do everything possible to make my life quite sour. They value only things that have a relationship to Finnish nationality. Such a cosmopolitan science as mathematics naturally can't enjoy any sympathy from them, and Weierstrassian mathematics seems to be especially

> unsympathetic to them. All this doesn't bother me very much otherwise, but the worst is that these constant struggles take up so much time.[3] [Mita]

For this reason, Mittag-Leffler's choice of Swedish may also have served to help him become noticed and gain a greater reputation amongst Stockholm's mathematicians in order to find a professorial position in the area.

The next publication of Mittag-Leffler's work was a compilation of results from 1876 and 1877. Hermite, with whom Mittag-Leffler had been corresponding since his time in Paris as a student, wrote to Mittag-Leffler on September 29, 1879, with the following request:

> The French geometers and especially Misters Briot and Bouquet will be very interested in your analytic discovery, likewise I allow myself to ask you, for all those who would be stopped by the obstacle of Swedish, to be so kind as to give, to the *Bulletin* of Mr. Darboux, a description of your work that we can read and understand.[4] [Her84], p. 58

Mittag-Leffler complied with this request, and sent an outline of his work (this did not include proofs) in a letter to Hermite, which was published in the *Bulletin des Sciences Mathématiques* in 1879, and entitled *Extract of a letter to Mr. Hermite*[5] [Mit79]. The letter appears to be the outline of the main results of a German paper[6] sent to Weierstrass. This German memoir never appeared, due to the fact that Weierstrass felt it was not wise to publish Mittag-Leffler's results in their present form; this will be discussed in Chapter 4.

The letter to Hermite contains four theorems. The first states that given sequences of values $\{x_\nu\}$ and $\{m_\nu\}$, where $\{x_\nu\}$ approaches infinity and m_ν is whole and positive for all

[3] "Ich habe über 50 Schüler und ich muss allein alles vortragen. Daneben machen diese verfluchte "Fennomanen" alles mögliches um mir das Leben recht sauer zu machen. Sie verehren nur das was auf die Finnische Nationalität eine Beziehung hat. Eine so kosmopolitanische Wissenschaft wie die Mathematik kann bei Ihnen natürlicherweise keine Sympathien geniessen und die Weierstrassische Mathematik scheint Ihnen ganz special unsympathetisch zu sein. Dies alle kümmert mich sonst sehr wenig, aber das schlimme ist dass diese ewigen Streiten so viel Zeit nehmen."

[4] "Les géomètres français et surtout MM. Briot et Bouquet s'intéresseront vivement à votre découverte analytique, aussi je me permets de vous demander pour tous ceux qu'arrêtera l'obstacle de suédois de vouloir bien donner, au *Bulletin* de Mr. Darboux, un exposé de vos travaux que nous puissions lire et comprendre."

[5] "*Extrait d'une lettre à M. Hermite.*"

[6] Apparently titled "*Arithmetische Darstellung eindeutiger analytischer Functionen einer Veränderlichen*" [Mit79].

values ν, when given the expression

$$F(x) = \sum_{\nu=1}^{\infty} (x - x_\nu)^{-m_\nu} G_{m_\nu - 1}(x - x_\nu) \left(\frac{x}{x_\nu}\right)^{\mu_\nu} \ldots,$$

it is always possible to determine the whole positive numbers $\{\mu_\nu\}$ and the entire and rational functions $\{G_{m_\nu - 1}\}$ in such a way that the function $F(x)$ is analytic in x, has exactly the poles $\{x_\nu\}$ and one essential singularity at infinity. Adding an entire function $H(x)$ to the above representation of $F(x)$ thus gives the general representation for such a function. Furthermore, when given the coefficients of the principal part, within the neighbourhood of the pole x_ν $F(x)$ can be expressed as a Laurent series. Notice that the above equation can thus be written as

$$F(x) = \sum_{\nu=1}^{\infty} (x - x_\nu)^{-m_\nu} G_{m_\nu - 1}(x - x_\nu) \left(\frac{x}{x_\nu}\right)^{\mu_\nu} \ldots + \Pi(x)H(x),$$

where $H(x)$ is an arbitrary entire function and $\Pi(x) = 1$ for all x.

This is identical to Mittag-Leffler's 1876 result, though it is phrased in a slightly different manner. Once again, the terms $\left(\frac{x}{x_\nu}\right)^{\mu_\nu}$ are used to ensure convergence.

The second theorem of [Mit79] states that given sequences of values $\{x_\nu\}$, $\{m_\nu\}$, and $\{n_\nu\}$, where $\{x_\nu\}$ approaches infinity and m_ν and n_ν are whole and positive for all values ν, when given the expression

$$F(x) = \sum_{\nu=1}^{\infty} (x - x_\nu)^{-(m_\nu n_\nu)} G_{m_\nu + n_\nu - 1}(x - x_\nu) \left(\frac{x}{x_\nu}\right)^{\mu_\nu} \ldots + \Pi(x)H(x),$$

where $H(x)$ is a known but arbitrary entire function, it is always possible to determine the entire function $\Pi(x)$, the whole and positive numbers $\{\mu_\nu\}$, and the entire and rational expressions $\{G_{m_\nu + n_\nu - 1}\}$ (of degree $m_\nu + n_\nu - 1$) such that $F(x)$ is entire in the variable x, and has exactly the poles $\{x_\nu\}$ and one essential singularity at infinity. Furthermore, when given the coefficients of the principal part, within the neighbourhood of the pole x_ν $F(x)$ can be expressed as a Laurent series.

Thus the second theorem is very similar to the first, but is slightly more general as $\Pi(x)$

represents an arbitrary entire function. Mittag-Leffler notes that, according to the Weierstrass Factorization Theorem, the function $\Pi(x)$ is determined according to the equation

$$\Pi(x) = \prod_{\nu=1}^{\infty} \left[\left(1 - \frac{x}{x_\nu}\right) e^{\frac{x}{x_\nu} + \frac{1}{2}\left(\frac{x}{x_\nu}\right)^2 + \ldots + \frac{1}{\lambda_\nu}\left(\frac{x}{x_\nu}\right)^{\lambda_\nu}} \right]^{n_\nu + 1}.$$

The third theorem is also similar: if given an infinite sequence of quantities $\{x_n\}$ approaching infinity, and sequences $\{m_\nu\}$ and $\{n_\nu\}$ of whole and positive numbers, it is possible to represent arithmetically a function of x that is uniform and entire, and whose zeros are exactly the terms of the sequence $\{x_n\}$. No explicit representation is given, but Mittag-Leffler comments that such a function can be written as a power series of the form

$$c_{\nu,p_\nu}(x - x_\nu)^{p_\nu} + c_{\nu,p_\nu+1}(x - x_\nu)^{p_\nu+1} + \ldots + c_{\nu,p_\nu+n_\nu}(x - x_\nu)^{p_\nu+n_\nu+1} P(x - x_\nu)$$

in the neighbourhood of any x_n, provided the appropriate coefficients are also given. Thus in the case of the third theorem, Mittag-Leffler is looking to specify the zeros of a function and not its poles.

The fourth theorem in [Mit79] is somewhat different from the others. It states that when given sequences of values $\{x_\nu\}$ which approaches infinity, whole and positive numbers $\{p_\nu\}$ and $\{q_\nu\}$ such that for each ν at least one of the values p_ν, q_ν will always equal zero, and whole and positive numbers $\{n_\nu\}$, it is always possible to determine two entire functions which will never equal zero at the same time and such that their quotient is a function of x. This quotient will be uniform (single-valued), and its poles and/or zeros will be determined exactly by the sequence $\{x_\nu\}$ with the only essential singularity at infinity. Furthermore, when given the coefficients of the principal part, within the neighbourhood of the pole x_ν the quotient can be expressed as having a function of entire character in both the numerator and the denominator. Note that in today's language this quotient is a meromorphic function.

Mittag-Leffler concludes this paper with a passage that is key in foreshadowing his further work:

> I want to give a general arithmetic representation of uniform functions, which have a multiple infinity of essential singular points.[7] [Mit79], p. 275

[7] "Je veux donner une représentation arithmétique générale de fonctions uniformes, qui aient une infinité multiple de points singuliers essentiels."

Mittag-Leffler's use of the phrase "multiple infinity" is somewhat vague, but it is clear that he is referring to an infinite collection of points; the very fact that this is initially ill-defined demonstrates how much clarity Cantor's language would later bring to his theorem. Despite the words, however, the meaning of the statement is unproblematic, and thus by 1879 Mittag-Leffler aimed to generalize his theorem in such a way that the function $f(x)$ may possess an infinite number of essential singularities. It would take three years for Mittag-Leffler to finally express in print that he had further generalized his theorem, though he would still fall short of his goal of generalization to the case of a function having infinitely many essential singularities. It was during this period that Mittag-Leffler made serious contact with Cantor's ideas.

Chapter 4

Contact with Cantor's ideas

The next publication of a version of Mittag-Leffler's theorem occurred in 1880, when Weierstrass produced a version which improved both its notation and proof [Wei80a]. He explained to Mittag-Leffler in a letter dated June 7, 1880, his reason for these revisions:

> I am so impressed by the importance of your theorem, that on a closer examination I can't conceal that *the establishment of the theorem must be made significantly shorter and simpler if it is to find its place in the elements of analysis, where it belongs* [emphasis mine]. The large apparatus of formulas that you apply would, I fear, scare off many readers; in any case it makes it harder to penetrate into the essence of the matter. I therefore believed that it did not lie in your interest for the memoir to be published in the present form, and wanted to suggest to you that you give me permission to present the theorems in question in a free treatment to our Academy, the more so since they are already published and your property rights to them are in any case secured.[1] [Wei]

[1]"So sehr ich aber von der Wichtigkeit Ihres Satzes durchdrungen bin, so konnte ich mir doch bei genauere Durchsicht Ihrer Abhandlung nicht verkehlen, dass die Begründung des Satzes erheblich kürzer und einfacher gestaltet werden müsse, wenn derselbe seinen Platz in den Elementen der Analysis, wohin er gehört, finden soll. Der grosse Apparat von Formeln, den Sie anwenden, würde manchen Leser, wie ich fürchte, abschrecken; jedenfalls erschwert er das Eindringen in das Wesen der Sache. Ich habe deshalb geglaubt, dass es nicht in Ihrem Interesse liege, wenn die Abhandlung in der vorliegenden Form veröffentlicht würde, und wollte Ihnen vorschlagen, dass Sie mir die Erlaubniss geben möchten, die in Rede stehenden Sätze in freien Bearbeitung unserer Akademie vorzulegen, zumal da sie bereits publicirt sind und Ihr Eigenthumsrecht daran jedenfalls gesichert ist."

Weierstrass published the new and concise version of this under the title *On a function-theoretic theorem of Mr. Mittag-Leffler*[2] which appeared in the *Monatsbericht der Königlichen Akademie der Wissenschaften zu Berlin* [Wei80a]. Weierstrass' proof method was immediately adopted by Mittag-Leffler and actually remains the basis of the treatment in Ahlfors [Ahl79].

The news of Weierstrass' treatment of Mittag-Leffler's work spread, and on December 24, 1880, Hermite mentioned in a letter to Mittag-Leffler the following:

> It is from Mr. Picard that I learned, before having the translation of the most recent work of Mr. Weierstrass (which we are still awaiting), in how profound and simple a way the great geometer proves the important theorem that will henceforth bear your name: that the sum of rational functions which correspond to the infinitely many poles of a single-valued function, forms a convergent series.[3] [Her84], p. 86

In his next letter to Mittag-Leffler, dated the following day, Hermite provided a variant proof of the theorem and presented an application to finding series expansions for the beta-function, an important tool in applied mathematics. However, Mittag-Leffler continued to use Weierstrass's method,[4] which will be outlined in Section 4.1 as it was used in Mittag-Leffler's subsequent publications of his theorem, the first of which occurred two years later in the *Comptes Rendus de l'Académie des Sciences* of Paris.

At around this point in time, Mittag-Leffler appears to have made serious contact with Cantor's work for the first time. This is indicated by a letter from Hermite written on May 9, 1881, which closes with the following request:

> Could you satisfy my great curiosity on the subject of infinites of a new kind

[2]"*Über einen functionentheoretischen Satz des Herrn Mittag-Leffler*."

[3]"C'est par Mr Picard que j'ai appris, avant d'avoir la traduction du dernier travail de Mr Weierstrass, qui se fait attendre, de quelle manière si profonde et si simple le grand géomètre démontre l'importante proposition, qui désormais portera votre nom, que la somme des fractions rationelles, correspondant aux pôles en nombre infini d'une fonction uniforme, forme une série convergente."

[4]In fact, several years later, in a letter written on September 2, 1884, Hermite himself even admitted that Weierstrass' argument was preferable to his version: "...the principle of Weierstrass, for the demonstration of your theorem, is infinitely superior to my process ...". ("... le principe de Weierstrass, pour le démonstration de votre théorème, est infiniment supérieur à mon procédé ...") [Her85], p. 93.

> considered by Mr. Cantor in the theory of functions?[5] [Her84], p. 122

Mittag-Leffler answered on June 22 of that year that as soon as he had further studied Cantor's work he would report on it to Hermite [Her84], p. 250. Thus it seems as though Cantor's work on this subject was initially met with curiosity and interest from Hermite. This also indicates that by 1881, Mittag-Leffler had become aware of Cantor's work, and if he had not yet begun to study it he intended to do so.

Hermite's letter of May 9 concluded with a note stating that he [Hermite] had already begun to teach Mittag-Leffler's theorem in the Sorbonne, and that it would begin to appear in examinations. This is a clear demonstration of the importance of Mittag-Leffler's work, and the excitement with which it was greeted by his peers and mentors.

4.1 Mittag-Leffler's "pre-Cantorian" *Comptes Rendus* publications

Between February and April of 1882, Hermite submitted to the *Comptes Rendus de l'Académie des Sciences* eight excerpts of letters written to him by Mittag-Leffler, each one entitled *On the theory of uniform functions of one variable. Extract of a letter addressed to Mr. Hermite*[6] [Mit82b]. This method of announcing and detailing mathematical or scientific results in a letter written to a member (Hermite was one) was the standard way for non-members of the *Académie* to publish announcements of their research in the *Comptes Rendus* [Arc02], p. 135.

The first of these letters was published on February 13, 1882, and it serves to recapitulate the breakdown of Mittag-Leffler's work between 1876 and 1877.

4.1.1 First Letter: Theorem statement and proof

The proof sketched in this article differs from that found in Mittag-Leffler's earlier work, as Weierstrass' letter of 1880 provided a more elegant version which Mittag-Leffler adopted

[5]"...pourriez vous satisfaire ma vive curiosité au sujet des infinis d'un genre nouveau considérés par Mr Cantor dans la théorie des fonctions?"

[6]"*Sur la théorie des fonctions uniformes d'une variable. Extrait d'une lettre adressée à M. Hermite.*"

thereafter and began to use in his lessons at the University of Helsingfors. Curiously, however, he claims to have begun his teachings of "that same method" ("*cette même méthode*") at the beginning of 1879 [Mit82b], p. 415. This is puzzling, as he could not have learned of Weierstrass' results prior to June 7, 1880. It seems unlikely that Mittag-Leffler would intentionally make an erroneous statement in a letter he knew would be widely read, and thus it is not entirely clear which method he claims to have been teaching.

The statement of the theorem itself, however, is unproblematic, and follows:

If given an infinite sequence of distinct complex numbers $a_1, a_2, a_3, \ldots$ approaching infinity and an infinite sequence of entire rational or transcendental functions

$$\begin{aligned} G_1(\gamma) &= c_1^{(1)}\gamma + c_2^{(1)}\gamma^2 + c_3^{(1)}\gamma^3 + \ldots, \\ G_2(\gamma) &= c_1^{(2)}\gamma + c_2^{(2)}\gamma^2 + c_3^{(2)}\gamma^3 + \ldots, \\ &\vdots \\ G_\nu(\gamma) &= c_1^{(\nu)}\gamma + c_2^{(\nu)}\gamma^2 + c_3^{(\nu)}\gamma^3 + \ldots \end{aligned}$$

that vanish when $\gamma = 0$, it is always possible to find an analytic function $F(x)$ such that:

1. its only singular points are the terms $a_1, a_2, a_3, \ldots$, and

2. for every fixed value of ν, the difference

$$F(x) - G_\nu\left(\frac{1}{x - a_\nu}\right)$$

   is finite at $x = a_\nu$ such that in the neighbourhood of $x = a_\nu$ $F(x)$ may be expressed under the form

$$G_\nu\left(\frac{1}{x - a_\nu}\right) + P(x - a_\nu).$$

To be clear, Mittag-Leffler intends the sequence $\{a_\nu\}$ to represent the singularities (infinitely many poles and one essential singularity at infinity) of such a function $F(x)$, making the function

$$G_\nu\left(\frac{1}{x - a_\nu}\right)$$

the principal part of F at a_ν. The function $P(x - a_\nu)$ is then a power series in $(x - a_\nu)$; this, combined with the function G_ν, completes the Laurent series at the point a_ν.

Mittag-Leffler outlines the construction of such a function in the following way. To begin, we take arbitrarily an infinite sequence of positive numbers $\{\epsilon_i\}$ having a finite sum. This series $\{\epsilon_i\}$ will actually be used as a comparison series in what we would now call the Weierstrass M-test for uniform convergence. This was taught by Weierstrass in his lectures, which is doubtless where Mittag-Leffler learned of the technique. We take also another number $\epsilon < 1$. For brevity, we consider here only the case $a_\nu \neq 0$ for all indices ν. In this case, since

$$G_\nu\left(\frac{1}{x - a_\nu}\right)$$

is analytic except at $x = a_\nu$ it can be expanded in a Taylor series of the form

$$\sum_{\mu=0}^{\infty} A_\mu^{(\nu)} x^\mu$$

about the origin (in fact, it's already written as a Taylor series) inside its radius of convergence $|x| < |a_\nu|$. Because G_ν is analytic everywhere except at the specified poles, its Taylor series about $x = 0$ for

$$G_\nu\left(\frac{1}{x - a_\nu}\right)$$

converges for all $|x| < |a_\nu|$, and thus it is possible to find a whole number m_ν large enough that

$$\left|\sum_{\mu=m_\nu}^{\infty} A_\mu^{(\nu)} x^\mu\right| < \epsilon_\nu$$

when $|x| < \epsilon|a_\nu|$. We know that this is possible, because this convergence indicates that the sum of the tail end of the series must approach zero, and so we can make it as small as we want. This is the argument provided by Mittag-Leffler. He later received, however, a communication from H. A. Schwarz in Göttingen indicating that the proof needed to be modified because the assumption that the same ϵ would always work was incorrect. Mittag-Leffler then altered his original proof to deal with this small problem. Mittag-Leffler published Schwarz' correction in the August 14, 1882 *Comptes Rendus* [Mit82c].

After finding such a number m_ν, we set

$$F_\nu(x) = G_\nu\left(\frac{1}{x - a_\nu}\right) - \sum_{\mu=0}^{m_\nu - 1} A_\mu^{(\nu)} x^\mu,$$

thus making $F_\nu(x)$ equal to the tail end of the Taylor series, which we know to have a finite sum.Then

$$F(x) = \sum_{\nu=1}^{\infty} F_\nu(x),$$

which converges uniformly because it is less than the sum of all ϵ_ν (a finite value), gives us a function having the poles in the sequence $\{a_\nu\}$. It is because of this uniform convergence that the functions F_ν are analytic, and thus $F(x)$ is also analytic except at the specified singular points. The result is that in the neighbourhood of each a_ν, $F(x)$ has a Laurent expansion allowing it to be written in the form

$$G_\nu\left(\frac{1}{x - a_\nu}\right) + P(x - a_\nu)$$

(the principal part plus an entire function expressed as a Taylor series).

This first publication in the Parisian *Comptes Rendus* indicates that it is possible to construct a function with a specified set of (infinitely many) poles and one essential singularity at infinity, having principal parts (rational or transcendental entire functions) with specified coefficients. The content itself differs little from the first theorem of Mittag-Leffler's 1879 *Bulletin des Sciences Mathématiques* publication, though the newer article contains the sketch of a proof and is somewhat more generalized than the earlier writings as the terms in the infinite sequence of entire functions G may now be transcendental as well as rational; Mittag-Leffler himself comments that if one restricts the functions G_ν to only those which are rational and entire, the two results are identical. However, the proof of this 1882 version is noticeably different from Mittag-Leffler's earlier versions; it was thanks to Weierstrass that by the time the Mittag-Leffler theorem was published in the *Comptes Rendus* its proof had been pared down, making it into a clear and concise result. It should also be noted that Mittag-Leffler has not yet dealt with the construction of a function having poles with more than one accumulation point, or one shifted from infinity; in fact, there was just not room to do so in the *Comptes Rendus*, which had a three-page article limit. These would have to wait until later articles which deal with exactly those cases.

4.1.2 Second Letter

The second letter in the series of eight was published the following week (February 20, 1882), and addresses the converse problem: given a known function, is it always possible to express it as a series of the type Mittag-Leffler's theorem prescribed? This is in parallel with what Weierstrass did in 1876, as mentioned earlier. Hermite found this result particularly interesting, and wrote to Mittag-Leffler on February 14, 1882, to say:

> I am extremely excited by your announcement that you can, in a large number of cases, obtain the fractions [principal parts] $G(x)$, and deduce the proposed function $f(x)$ ...[7] [Her84], p. 143

The reader will soon see that obtaining the actual representation of a function is in general more difficult than actually proving theorem of its existence. I will provide some detail in order to demonstrate the flavour of Mittag-Leffler's methods, but I will not delve into specific applications.

To obtain a representation for a given function, Mittag-Leffler begins by supposing the existence of a function of the type considered in his last letter, namely one of the form

$$F(x) = \sum_{\nu=1}^{\infty} F_\nu(x) + G(x),$$

where $G(x)$ is entire and rational or transcendental.[8] Notice that the function $F(x)$ considered in the previous letter was simply of the form $F(x) = \sum_{\nu=1}^{\infty} F_\nu(x)$; however, as an entire function will have no poles except possibly one at infinity, this function $F(x)$ (with the added entire function $G(x)$) will possess exactly the same poles as that of the previous letter.

Now, when $F(x)$ is known, the problem of expressing it as a series of type $\sum_{\nu=1}^{\infty} F_\nu(x) + G(x)$ amounts to finding the natural numbers m_ν (one for each a_ν) such that the series $\sum_{\nu=1}^{\infty} F_\nu(x)$ is uniformly convergent (or at the very least, to show that these m_ν exist), as well as determining $G(x)$. Mittag-Leffler aims to develop a method for solving this problem

[7] "Vous m'avez extrèmement intéressé en m'annonçant que vous pouvez dans un grand nombre de cas obtenir les fractions $G(x)$, en les déduisant de la fonction proposée $f(x)$..."

[8] To be clear, when Mittag-Leffler uses the term *entire rational function*, he is referring strictly to a polynomial. When he uses the term *entire transcendental function*, he is indicating that the Taylor series expansion of said function has an infinite number of terms, such as that of the function e^x.

in a very general case, though it will become clear that this method will not work for every such function $F(x)$.

To begin, Mittag-Leffler considers the (positively-oriented) simply connected contour S, which contains the origin, the singular points a_1, a_2, $\ldots$, a_n of the known function F, and the nonzero complex number x which is not a singular point. His aim, at this point, is to construct a function having the same poles as $F(x)$, but also having poles at the origin and the specified point x considered above, all of which are included in the contour S. This function is

$$\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m.$$

Mittag-Leffler now applies a standard method for the calculation of contour integrals in regions containing several poles to the function he has just created, in order to split the integral into contours which each contain exactly one of the singular points a_1, a_2, $\ldots$, a_n, 0, and x. We now have

$$\int_S \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz = \int_{(x)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz + \int_{(0)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz + \sum_{\nu=1}^{n}\int_{(a_\nu)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz,$$

where the notation (a) indicates the simply connected contour containing only the singular point a. If the origin happens to belong to the set of poles a_1, a_2, $\ldots$, a_n of $F(z)$ it is to be considered separately and not taken under the summation sign. By multiplying each term by $\frac{1}{2\pi i}$, the above can be rewritten as

$$\frac{1}{2\pi i}\int_{(x)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz =$$

$$-\frac{1}{2\pi i}\int_{(0)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz - \sum_{\nu=1}^{n}\frac{1}{2\pi i}\int_{(a_\nu)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz + \frac{1}{2\pi i}\int_S \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz.$$

Mittag-Leffler's next step is to apply the Cauchy integral formula[9] to the function

$$\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m,$$

[9]Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}.$$

[Bro96], p. 123

since x is obviously a point interior to the curve around x. We then have, when we evaluate the function at the point $z = x$,

$$\begin{aligned} F(x) &= F(x)\left(\frac{x}{x}\right)^m \\ &= \frac{1}{2\pi i}\int_{(x)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz \\ &= \underbrace{-\frac{1}{2\pi i}\int_{(0)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz}_{(A)} \underbrace{-\sum_{\nu=1}^{n}\frac{1}{2\pi i}\int_{(a_\nu)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz}_{(B)} + \frac{1}{2\pi i}\int_{S} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz. \end{aligned}$$

We have now isolated the function $F(x)$. Determining the Mittag-Leffler series for $F(x)$ then amounts to calculating the parts labeled (A) and (B). Notice now that the contour in (A) contains exactly one singular point of the function $\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m$, and thus by the Cauchy residue theorem[10] calculating (A) is equivalent to determining the residue at the point $z = 0$. Similarly, calculating (B) is equivalent to determining the sum of residues at the points a_1, a_2, ..., a_n.

The calculations of each residue are not particularly complicated. To determine (A), the residue

$$-\frac{1}{2\pi i}\int_{(0)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz,$$

in the neighbourhood of zero, Mittag-Leffler writes $F(z)$ as its Taylor series or Laurent series, depending on whether or not zero is actually an element of the set a_1, a_2, ..., a_n. For instance, if we suppose that $z = 0$ is not a singular point of the function $F(z)$, then when z

[10] Let C be a positively oriented simple closed contour. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z).$$

[Bro96], p. 183

is close to zero, by using the Taylor series for $F(z)$ we can expand our function $\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m$ as

$$\begin{aligned}
-\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m &= \frac{1}{x}\left[\frac{1}{1-\frac{z}{x}}\right]\left(\frac{x}{z}\right)^m F(z) \\
&= \frac{1}{x}\left[1+\left(\frac{z}{x}\right)+\left(\frac{z}{x}\right)^2+\ldots\right]\left(\frac{x}{z}\right)^m F(z) \\
&= \frac{1}{x}\left[1+\left(\frac{z}{x}\right)+\left(\frac{z}{x}\right)^2+\ldots\right]\left(\frac{x}{z}\right)^m \left[F(0)+F'(0)z+\frac{F''(0)z^2}{2!}+\ldots\right].
\end{aligned}$$

It is then easy to find the coefficient of the term $\frac{1}{z-0}$. By expanding the right side of the above equation, we note that the terms containing $\frac{1}{z}$ are

$$\frac{1}{x}\left(\frac{x}{z}\right)F(0)+\frac{1}{x}\left(\frac{x}{z}\right)^2 F'(0)z+\frac{1}{x}\left(\frac{x}{z}\right)^3 F''(0)\frac{z^2}{2!}+\ldots+\frac{1}{x}\left(\frac{x}{z}\right)^m F^{(m-1)}(0)\frac{z^{m-1}}{(m-1)!}$$

and thus

$$-\frac{1}{2\pi i}\int_{(0)}\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz = F(0)+F'(0)x+F''(0)\frac{x^2}{2!}+\ldots+F^{(m-1)}(0)\frac{x^{m-1}}{(m-1)!}$$

which we will abbreviate by the function $G_1(x)$.

On the other hand, if $z=0$ is contained in $a_1, a_2, \ldots, a_n$, the steps to calculating the residue are almost identical, with the Taylor series being replaced by a Laurent expansion.

Mittag-Leffler now needs to determine each residue of (B). He begins by considering the general residue at the point $z=a_\nu$, namely

$$-\frac{1}{2\pi i}\int_{(a_\nu)}\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz.$$

He rewrites it as

$$-\frac{1}{2\pi i}\int_{(a_\nu)}\frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz =$$

$$\underbrace{-\frac{1}{2\pi i}\int_{(a_\nu)}\frac{F(z)}{z-x}dz}_{(C)}+\underbrace{\sum_{\mu=0}^{m-1}\frac{1}{2\pi i}\frac{1}{a_\nu}\left(\frac{x}{a_\nu}\right)^\mu\int_{(a_\nu)}\left(\frac{z}{a_\nu}\right)^{-(\mu+1)}F(z)dz}_{(D)}$$

by replacing the geometric series

$$\sum_{\mu=0}^{m-1}\left(\frac{x}{z}\right)^\mu$$

in (D) by its sum

$$\frac{1-\left(\frac{x}{z}\right)^m}{1-\left(\frac{x}{z}\right)},$$

and in this way splits (B) into two parts, (C) and (D), which he then proceeds to calculate.

To calculate (C), Mittag-Leffler rewrites $\frac{F(z)}{z-x}$ as

$$\frac{F(z)}{z-x}=\left(-\frac{1}{x-a_\nu}\right)\frac{F(z)}{1-\frac{z-a_\nu}{x-a_\nu}}.$$

Then, close to the pole a_ν it is possible to replace $F(z)$ by its Laurent series

$$F(z)=G_\nu\left(\frac{1}{z-a_\nu}\right)+P_\nu(z-a_\nu)$$

and to replace the sum $\frac{1}{1-\frac{z-a_\nu}{x-a_\nu}}$ by its geometric series $\sum_{n=0}^{\infty}\left(\frac{z-a_\nu}{x-a_\nu}\right)^n$. As $P_\nu(z-a_\nu)$ is a Taylor series in $z-a_\nu$, the coefficient of the term $\frac{1}{z-a_\nu}$ indicates that the residue $-\frac{1}{2\pi i}\int_{(a_\nu)}\frac{F(z)}{z-x}dz$ is thus equal to the function $G_\nu\left(\frac{1}{x-a_\nu}\right)$, which is the principal part at a_ν.

The calculation of the residues

$$\sum_{\mu=0}^{m-1}\frac{1}{2\pi i}\frac{1}{a_\nu}\left(\frac{x}{a_\nu}\right)^\mu\int_{(a_\nu)}\left(\frac{z}{a_\nu}\right)^{-(\mu+1)}F(z)dz$$

of (D) still remains. Two steps are required to accomplish this. For the first, Mittag-Leffler uses the rather clever trick of translating a binomial series. That is, letting $x=u-1$ the series $(1+x)^k$ becomes $(u)^k=1+k(u-1)+\frac{k(k-1)}{2!}+\ldots$. Mittag-Leffler uses this to write

$$\left(\frac{z}{a_\nu}\right)^{-(\mu+1)}F(z)=\left[1+\left(\frac{z}{a_\nu}-1\right)\right]^{-(\mu+1)}$$

using its binomial expansion. Then

$$\begin{aligned}-\frac{1}{a_\nu}\left(\frac{z}{a_\nu}\right)^{-(\mu+1)} &= -\frac{1}{a_\nu}\left[1-\left(\frac{\mu+1}{1!}\right)\left(\frac{z-a_\nu}{a_\nu}\right)+\left(\frac{(\mu+1)(\mu+2)}{2!}\right)\left(\frac{z-a_\nu}{a_\nu}\right)^2-\ldots\right]\\ &\times\left[G_\nu\left(\frac{1}{z-a_\nu}\right)+P_\nu(z-a_\nu)\right],\end{aligned}$$

and thus

$$-\frac{1}{2\pi i}\int_{(a_\nu)}\left(\frac{z}{a_\nu}\right)^{-(\mu+1)}F(z)dz=-\frac{c_1}{a_\nu}+\frac{c_2(\mu+1)}{1!a_\nu^2}-\frac{c_{-3}(\mu+1)(\mu+2)}{2!a_\nu^3}+\ldots.$$

Mittag-Leffler then sums over all the points a_μ to get

$$\begin{aligned}\sum_{\mu=0}^{m-1} \frac{1}{2\pi i}\frac{1}{a_\nu}\left(\frac{x}{a_\nu}\right)^\mu \int_{(a_\nu)} \left(\frac{z}{a_\nu}\right)^{-(\mu+1)} F(z)dz &= \sum_{\mu=0}^{m-1}\left(\frac{x}{a_\nu}\right)^\mu \\ &\times \left[-\frac{c_1}{a_\nu} + \frac{c_2(\mu+1)}{1!a_\nu^2} - \frac{c_{-3}(\mu+1)(\mu+2)}{2!a_\nu^3} + \dots\right] \\ &= -\sum_{\mu=0}^{m-1} A_\mu^{(\nu)}\left(\frac{x}{a_\nu}\right)^\mu,\end{aligned}$$

and thus $(D) = -\sum_{\mu=0}^{m-1} A_\mu^{(\nu)}\left(\frac{x}{a_\nu}\right)^\mu$. Putting this together with (C), we see that a general residue of (B) is equal to

$$G_\nu\left(\frac{1}{x-a_\nu}\right) - \sum_{\mu=0}^{m-1} A_\mu^{(\nu)}\left(\frac{x}{a_\nu}\right)^\mu.$$

Letting then $m_\nu = m$, and summing all residues of (B), it is clear that

$$(B) = \sum_{\nu=1}^{n} F_\nu(x).$$

Mittag-Leffler was thus able to show that

$$F(x) = -\frac{1}{2\pi i}\int_{(0)} \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz - \sum_{\nu=1}^{n}\frac{1}{2\pi i}\int_{(a_\nu)} + \frac{1}{2\pi i}\int_S \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz$$

can be expressed in the form

$$F(x) = G(x) + \sum_{\nu=1}^{n} F_\nu(x) + \frac{1}{2\pi i}\int_S \frac{F(z)}{z-x}\left(\frac{x}{z}\right)^m dz$$

where $G(x)$ is is entire, and rational or transcendent, which is the result Mittag-Leffler sought.[11] Evidently Mittag-Leffler's letter to Hermite containing these results exceeded the page limit of *Comptes Rendus* publications. Hermite wrote to him on February 23, 1882, to say that he was forced to suppress 22 lines [Her84], p. 149.

One important thing to note at this point is that using this argument Mittag-Leffler has prevented himself from considering functions $F(x)$ having infinitely many singular points due to his use of the Cauchy residue theorem, which allows only a finite number of singular

[11] It is interesting to note that in his Swedish publications, Mittag-Leffler used the modern notation $x!$ to represent $x(x-1)(x-2)\cdots(2)(1)$. However, in a letter of February 23, 1882, Hermite advised a switch to the notation familiar to the French, namely $|\underline{x}$, for works to be published in the French *Comptes Rendus*. This is true also for the notation $|x|$; the French represented this by mod.x. [Her84], p. 149

points a_1, a_2, ..., a_n within a simply connected contour S. Mittag-Leffler has also only dealt with one essential singularity, at infinity. However, we know that he certainly wants to deal with more; he had explicitly mentioned this in his 1879 publication in the *Bulletin des Sciences Mathématiques*.

As well, it can be seen that the method provided here will not work for every function $F(x)$, as it may not be possible to evaluate the integral in the last term of the above expression. However, the fourth letter in this series presents a very nice example of an appropriate function, namely $f(x) = \pi \cot(\pi x)$, a known expansion mentioned earlier. The third letter also deals with applications of Mittag-Leffler's theorem to interpolation and approximation, a shared interest of Mittag-Leffler and Hermite [Mit82b]. This subject will not be dealt with here.

To this point, Mittag-Leffler's theorem has been entirely based upon Weierstrass' foundations and concepts of analysis — Mittag-Leffler is even working with a proof developed by Weierstrass himself. However, Mittag-Leffler's next publications make use of the concepts of another important mathematician of that time.

4.2 Georg Cantor

Georg Cantor (1845–1918) was born St. Petersburg, Russia, but emigrated to Germany with his family when he was still a child. He attended the Gymnasium in Wiesbaden, but did not develop an interest in mathematics until studying at the Grossherzoglich-Hessische Realschule in Darmstadt. His university education commenced in 1862 in Zurich, but his father's death in the following year led him back to Germany. Cantor then began university studies in Berlin under the illustrious Weierstrass, from whom he received a strong and precise foundation of analysis. [Mes71], p. 53

As a student, Cantor became a member of the Berlin Mathematical Society, and achieved the status of president from 1864 until 1865. In 1869 he began to teach at the University of Halle. Shortly thereafter, he was awarded a position as associate professor, and became a full professor in 1879, a position which he held until his death. Cantor married Vally Guttmann 1874, and they had five children together. With such a large family, it was fortunate that

Cantor had inherited money from his late father; on the sole income from his professorial position at Halle, where the faculty was poorly paid, he would not have been able to build a house and would have been rather dire financial situation. The circumstances were different in Berlin, where Cantor had dreamed of finding a more financially-sound and ambitious position, but his professional clashes with the powerful mathematician Kronecker (a staunch opponent of Cantor's theory of transfinite sets) prevented this goal from becoming a reality. Kronecker was not Cantor's only opponent, and by the end of his mathematical career Cantor had severed ties with almost every one of his former colleagues and correspondents, including his confidant Mittag-Leffler. Cantor suffered from depression from 1884 on, and on several occasions spent time in a sanatorium. According to Schoenfliess, this was perhaps due to his exhausting mathematical efforts and, perhaps more significantly, to the fact that his avant-garde work was rejected by virtually all of his contemporaries. [Mes71], p. 53

Though he is primarily remembered for single-handedly founding the theory of sets, Cantor also made significant contributions to classical analysis, such as his work on real numbers and on Fourier series. He was also the first president (1890 – 1893) of the Association of German Mathematicians, an organization which he succeeded in founding in 1890, and worked for the organization of the first ever international congress of mathematicians, held in Zurich in 1897. [Mes71], p. 53

Today we associate Cantor with set theory as a whole. However, this did not become a general framework for mathematics until significantly after Cantor began to write on the subject. His work initially inspired a great deal of controversy in both France and Germany due to traditional and longstanding opposition to the concept of the *actual* infinite [Dau79], p. 120. To be specific, the body of work for which Cantor faced the greatest disapproval was his *Foundation of the theory of manifolds*[12]. Published in 1883, it laid the foundation for the theory of transfinite sets, the culmination of his work of the previous decade. This built upon the concepts published in his *On infinite and linear sets of points*[13], a series of papers crucial to the development of the Mittag-Leffler theorem which will be discussed in Chapter 7.

[12]"*Grundlagen einer allgemeinen Mannigfaltigkeitslehre*"
[13]"*Ueber unendliche lineare Punktmannigfaltigkeiten*"

This opposition to Cantor's work was rooted in the great controversy about the nature of the infinite itself. Aristotle had classically opposed the idea of completed infinities, and arguments to make analysis rigorous had focused on potential infinities. Some Christians felt that the concept of an actual infinity challenged the "unique and absolute infinite nature" of God [Dau79], p. 120. The majority of mathematicians simply refused to apply the concept of the actual infinite, and instead dealt only with the idea of a potential infinite. Cantor, however, faced this dispute head-on in his writings. Mathematics and philosophy played equal roles in his *Foundation*, as he felt the two disciplines to be deeply connected. Moreover, Cantor actually believed that his *Foundation* served as not only a mathematical treatment of this new theory of transfinite sets, but as a defense of the actual infinite [Dau79], p. 120.

In light of the popular sentiment of Cantor's time, it is not unreasonable to believe that, with respect to Cantor's desire to shake the basic foundation of mathematics and philosophy, he was doomed from the beginning. Indeed, the majority of his contemporaries[14] denounced his new theory, believing it to be an empty abstraction. This was to change over time. There were already some exceptions amongst the naysayers: in 1882 Mittag-Leffler found the mathematical concepts outlined by Cantor of direct use in his own work, and by 1926, Hilbert even felt that Cantor had created a new "paradise" for mathematicians [Dau79], p. 1. In general, however, this opposition led Cantor to feel increasingly isolated amongst his contemporaries.

4.3 Mittag-Leffler's *Comptes Rendus* publications using Cantorian terminology

It can be observed that in the published portions of the first four *Comptes Rendus* letters there is no obvious usage of Cantor's theory of transfinite sets. However, a letter sent from Mittag-Leffler to Hermite on February 13 of 1882 opens with the following paragraph:

> You ask me of which type are the new theorems that I announced to you in my letter of 19 June 1879, published by you in the third volume of the second series of the "Bulletin des Sciences Mathématiques". The theorems are essentially the

[14]Kronecker, Hermite, and Poincaré are some notable examples.

> same as the four theorems that I describe to you there but they include, to use Mr. Cantor's terminology, all the uniform, monogenic functions whose singularities are a collection of values (mass of values? Werthmenge) of the first type. To give you an idea of my research, I will describe to you a theorem that can serve as the foundation of the entire theory.[15] [Mitb]

It is interesting to note Mittag-Leffler's difficulty in expressing the concept of a set. Cantor used the term *Werthmenge* or simply *Menge*, which imply the concept of an *amount*. The uncertainty with this concept will surface again when Mittag-Leffler incorporates Cantor's terminology and theory into his work in the following letter to Hermite. In any case, it is clear that prior to the publication of the first *Comptes Rendus* letter Mittag-Leffler had begun to consider Cantor's ideas, specifically that of a function of the first type (*premier genre*), though he does not noticeably take advantage of this definition until the fifth letter to appear in the *Comptes Rendus*.

This letter does nothing more than rephrase the contents of the theorem presented in the first letter in terms of Cantor's terminology. This structural change took some work on the part of Mittag-Leffler; approximately two of the paper's two and a half printed pages were devoted to stating the theorem in terms of sets of singular points, and no indication of a proof was given (presumably it would follow steps similar to those of the former letter). It is interesting to observe Mittag-Leffler's exchange of *sequence* for *set*. The awkwardness with which he discusses the concept of a set seems surprising now, as the set has since become a very standard mathematical entity.

This was not, however, Mittag-Leffler's first publication in which his theorem was phrased in Cantorian notation. In this fifth *Comptes Rendus* letter he comments that this new formulation has already appeared in the February 8 publication of the *Översigt* of the Royal Swedish Academy of Science, and thus the intention of this later French letter was to provide

[15]"Vous me demandez dans quel genre sont les théorèmes nouveaux que je vous annonce dans ma lettre du 29 juin 1879 publiée par vous dans le tome III de la deuxième série de 'Bulletin des Sciences mathématiques'. Les théorèmes sont au fond les mêmes que les quatre théorèmes que je vous y expose mais ils embrassent pour employer la terminologie de M. Cantor toutes les fonctions uniformes et monogènes dont les singularités présentent un nombre de valeurs (une masse de valeurs? *Werthmenge*) de la première éspèce. Pour vous donner une idée de mes recherches je vous exposerai un théorème qui peut servir comme fondement de toute la théorie."

an excerpt of the more detailed Swedish text. As the *Comptes Rendus* publication was the first such appearance of these ideas in a milieu generally accessible to the French and German mathematicians, it will be discussed in detail here.

Mittag-Leffler's fifth publication in the French *Comptes Rendus* is dated April 3, 1882. It begins with a paragraph describing his work on the theorem to that date. In it, Mittag-Leffler indicates that he has been able to generalize a part of his 1877 results which appeared in the *Översigt* of the Royal Swedish Academy of Science and that he will now present a portion of this work to Hermite [Mit82d], pp. 938–939.

The use of Cantor's ideas is immediately obvious. Instead of beginning the theorem statement with an infinite sequence of distinct singularities (poles) $\{a_\nu\}$, these distinct singular points of a uniform and monogenic function $F(x)$ are defined to be the values which comprise the set P. The following definitions are then given.

Definition 4.1 *If Q is then a collection of terms belonging to P, a* limit point *of Q is a point which has, in all of its neighbourhoods, other values also belonging to Q.* [Mit82d], p. 939

With this in mind, Mittag-Leffler then defines P' (the *derived set*).

Definition 4.2 *Let P be a set. Its* derived set*, P', is then the set of limit points of P. This process can be continued: P'' is the set of limit points of P', ..., and in the same way, $P^{(r)}$ is the set of limit points of $P^{(r-1)}$.* [Mit82d], p. 939

Note that these sets P are nested so that $P \supseteq P' \supseteq \ldots \supseteq P^{(r-1)} \supseteq P^{(r)} \supseteq \ldots$. The fact that $P \supseteq P'$ applies because of the way in which Mittag-Leffler is assuming the singularities are located; it is not true in general.

Mittag-Leffler then states Cantor's distinction between sets of the "first type" ("*premier genre*") and "second type", and his definition for sets of the "rth kind" ("*rth espèce*").

Definition 4.3 *A set P of the* first type *is one such that successive derivations of the set lead eventually to the case where $P^{(r)}$ is the empty set (= 0 in Cantor's words) for some r. If no such r exists, then P is a set of the* second type*.* [Mit82d], p. 939

If P is a set of the first type, then a further classification exists.

Definition 4.4 *If the $P^{(r-1)}$ is nonempty but $P^{(r)} = \emptyset$, we say that P is of the first* type *and the* $(r-1)^{th}$ kind. [Mit82d], p. 939

All of the above definitions can be found in Cantor's [Can83], a French translation of Cantor's 1879 article [Can79]. One will notice that the translations into English of Mittag-Leffler's Cantorian terminology are not literal. The *Comptes Rendus* publication contains the phrases *premier genre* and r^{th} *espèce*, of which the literal translations would be, respectively, *genus* and *species*. I have chosen not work with the literal translations, and instead to use more general terms.

Mittag-Leffler's theorem is then stated in the following manner. Suppose given

1. A set of distinct elements P, such that P is of the first type and the n^{th} kind and that $P^n = \{a_1, a_2, \ldots, a_m\}$. The other elements of P should be partitioned into sets so that $P^{(n)} \setminus P^{(n+1)} = \{a_{\mu\nu}\}$ where $\mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m$; and

$$\lim_{\mu\to\infty} a_{\mu\nu} = a_\nu.$$

   Similarly, $P^{(n-2)} \setminus P^{(n-1)} = \{a_{\lambda\mu\nu}\}$ where $\lambda = 1, 2, \ldots; \mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m$; and

$$\lim_{\lambda\to\infty} a_{\lambda\mu\nu} = a_{\mu\nu}$$

   and so on such that $P \setminus P' = \{a_{\alpha\beta\gamma\ldots\lambda\mu\nu}\}$ where $\alpha = 1, 2, \ldots; \beta = 1, 2, \ldots; \gamma = 1, 2, \ldots; \ldots; \lambda = 1, 2, \ldots; \mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m$; and

$$\lim_{\alpha\to\infty} a_{\beta\gamma\ldots\lambda\mu\nu} = a_{\mu\nu},$$

   and

2. A series of entire rational or transcendental functions with no constant term (so that each function equals zero for $y = 0$)

$$G_{\alpha\beta\gamma\ldots\lambda\mu\nu}(y) = c_{-1}^{\alpha\beta\gamma\ldots\lambda\mu\nu} y + c_{-2}^{\alpha\beta\gamma\ldots\lambda\mu\nu} y^2 + \ldots$$

   where $\alpha = 1, 2, \ldots; \beta = 1, 2, \ldots; \gamma = 1, 2, \ldots; \ldots; \lambda = 1, 2, \ldots; \mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m$.

Then it is always possible to construct an analytic function $F(x; a_{\alpha\beta\gamma\ldots\lambda\mu\nu})$ where $\alpha = 1, 2, \ldots; \beta = 1, 2, \ldots; \gamma = 1, 2, \ldots; \ldots; \lambda = 1, 2, \ldots; \mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m$ with exactly the singular points contained in P and such that for each value of $(\alpha\beta\gamma\ldots\lambda\mu\nu)$ the difference

$$F(x) - G_{\alpha\beta\gamma\ldots\lambda\mu\nu}\left(\frac{1}{x - a_{\alpha\beta\gamma\ldots\lambda\mu\nu}}\right)$$

is finite when $x = a_{\alpha\beta\gamma\ldots\lambda\mu\nu}$ and can be expressed under the form

$$G_{\alpha\beta\gamma\ldots\lambda\mu\nu}\left(\frac{1}{x - a_{\alpha\beta\gamma\ldots\lambda\mu\nu}}\right) + P_{\alpha\beta\gamma\ldots\lambda\mu\nu}(x - a_{\alpha\beta\gamma\ldots\lambda\mu\nu})$$

in the neighbourhood of $x = a_{\alpha\beta\gamma\ldots\lambda\mu\nu}$. [Mit82d] The argument for the proof of this theorem is virtually identical to that of [Mit76].

Notice as well that this theorem statement requires that the set of poles P of a function be of the first type, and not the second. Mittag-Leffler claims that his proof allows only for the former case. However, the proof is not included, likely due to the limited page count as mentioned earlier.

Mittag-Leffler also claims that this is by no means the most general theorem of its type. He indicates that there are others which embrace a whole class of functions of the second type, and comments:

> To be able to state those, I am initially forced to account for the classification which Mr. Cantor has introduced for a number of real points located between finite limits and which belong to the second class[16] [Mit82d], p. 941

which refers to Cantor's distinction between countable and uncountable sets.

A few comments must be made at this point.

First, Mittag-Leffler is clearly dealing only with countable sets of poles, as his use of subscripts running through the positive integers can provide at most a countably infinite set of essential singularities, though it is certainly possible to construct a function with uncountably many singular points which form a set of the first type (for example, the Cantor

[16] "Pour pouvoir énoncer ceux-ci, je suis d'abord forcé de rendre compte de la classification qu'a introduite M. Cantor pour un nombre de points réels et situés entre des limites finies et qui appartient à la seconde classe."

set). It may seem as though Mittag-Leffler has overlooked such cases, but it will later become clear that his proof technique only applies for countable sets of poles.

Second, Mittag-Leffler has made a rather large step in his process of generalization: this use of derived sets now allows Mittag-Leffler to deal not only with sets of poles having more than one limit point, but with sets of poles having even infinitely many of them, which would occur if the derived set P' is infinite. The significance of this accomplishment should not be overlooked — an infinite set P' indicates that the corresponding function must have infinitely many essential singularities. Mittag-Leffler does not mention this in the publication, which is strange unless he himself was not yet aware of it.

By the Spring of 1882, Mittag-Leffler was thus on his way to the generalization of his theorem to the case of infinitely many essential singularities (in fact, as just mentioned he had essentially found the key to dealing with an arbitrary number of essential singularities, though he either did not realize it or did not make it clear to the reader), but he had not yet dealt with functions having sets of singular points of the second type. However, this new statement of the Mittag-Leffler theorem in a widely-read journal did spark a correspondence between Mittag-Leffler and Cantor which would prove instrumental in the theorem's generalization. This correspondence will be discussed in the following section, as it didn't appear to affect the contents of the *Comptes Rendus* publications.

The sixth *Comptes Rendus* letter was published one week later, on April 10, 1882. It presents a modification of the theorem given in [Mit82b].[17]

In this sixth publication, Mittag-Leffler still supposes given a sequence of entire rational or transcendental functions $G(y)$ without a constant term, but instead of supposing given an infinite sequence of values $\{a_\nu\}$ increasing to infinity, these terms are subject to the condition

$$\lim_{\nu\to\infty} |a_\nu| = R,$$

where R is an arbitrary positive number. Thus the sequence of poles still has a single accumulation point, but it can now occur at a positive finite number. An example of a set

[17]Mittag-Leffler actually referred to the version of his theorem which appeared in the February 18 edition of the *Comptes Rendus*. However, this weekly publication was printed on February 13 and February 20 of 1882. The close relation of this theorem to work printed on February 13 (and the fact that the work on February 20 concerns the converse statement of the theorem) seems to indicate that *18 février* should have been *13 février*.

of singularities which possesses accumulation points at a finite distance from the origin R is

$$\{a_{nk}\} = \left(1 + \frac{(-1)^{n+1}}{n+1}\right) e^{\frac{2\pi k i}{n+1}},$$

where $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots, \infty$. This set will be discussed further in Chapter 6.

It appears that the construction of the function $F(x)$ is unchanged from the first letter; the sketch of the procedure is identical. However, at the end of this letter Mittag-Leffler comments that it can happen that every portion of the circle $|x| = R$ contains an infinite number of singular points. In that case, it is not possible to continue to create power series expansions for such a function (one for each radius of convergence), and so such a function must exist only in the domain $|x| < R$. (If this is not the case, however, the function can be extended beyond this domain by analytic continuation.) He also notes that though there is not necessarily a simple and direct method for the general case, it is easy to extend the procedure to $R = \infty$ for a wide class of functions.

Curiously, Mittag-Leffler has phrased this modification in terms of a sequence of poles $\{a_\nu\}$ and not a set. It is unclear as to why he has dropped Cantor's terminology after having used it in the previous letter to Hermite.

The seventh and eighth *Comptes Rendus* letters were inadvertently switched in their order of publication, and in a letter to Mittag-Leffler dated May 8, 1882, Hermite acknowledges the error and is apologetic:

> I regret extremely the unfortunate inversion of your last notes in the publication; I do not doubt that the reader will recognize it himself, but I will ask, to answer your intention, if the inversion can be indicated in an *errata* at the end of the volume.[18] [Her84], p. 159

These last two letters serve to inductively prove that the Mittag-Leffler theorem holds under the Cantorian notation for an infinite set of poles and a finite set of essential singularities. The pair is doubtless easier to understand when the letters are viewed in their intended progression, as the first letter demonstrates the basis case and the second provides the inductive argument, and so the error is, in fact, obvious to the reader.

[18]"Je regrette extrêmement l'inversion malheureuse dans la publication de vos deux dernières notes; le lecteur je n'en doute point la reconnaîtra de lui-même, mais je demanderai, pour répondre à votre intention, si l'inversion peut être indiquée dans un *errata* à la fin du volume."

Mittag-Leffler first supposes that the set of singular points P is of the first kind and first type, such that the only limit value (and hence essential singularity) occurs at a, and that $P \setminus P' = \{a_1,\ a_2,\ \ldots\}$. He notes that if this one singular point occurs at $a = \infty$, this is reduced to the theorem of his first letter; if the essential singularity occurs at a finite value a, he notes that the argument is almost identical, but that one must develop the function

$$G_\nu \left(\frac{1}{x - a_\nu} \right)$$

in a series

$$\sum_{\rho=0}^{\infty} A_\rho^{(\nu)} \left(\frac{a_\nu - a}{x - a} \right)^\rho$$

in order to develop a series expansion about the point $x = a$ when a is an essential singularity. Note that because a is an essential singularity, infinitely many of the coefficients are not zero.

This establishes the result for a set P of the first kind and type, if there is exactly one essential singularity.

Mittag-Leffler then sketches the extension to a finite number of essential singularities, which requires the construction of functions $F_\nu(x; a_{\mu\nu}; \mu = 1, 2, \ldots); \nu = 1, 2, \ldots, m$ so that the sum

$$\sum_{\nu=1}^{m} F_\nu(x; a_{\mu\nu}; \mu = 1, 2, \ldots)$$

is a uniform and monogenic function having poles $a_{\mu\nu}$ and essential singularities a_ν.

Mittag-Leffler then describes the case in which P is still of the first kind and type but now contains a set of m limit points. The construction of such a function F is similar to the case in which there exists only one essential singularity. In this case, $P \setminus P'$ is ordered in a "double series" $\{a_{\mu\nu}\}$ such that $\mu = 1,\ 2,\ \ldots$ and $\nu = 1,\ \ldots,\ m$ such that the series $a_{\mu\nu}$ ranging through μ has only the limit point a_ν. For a visual representation, refer to Figure 4.1, in which the singular point a_{mn} is represented by $a_{m,n}$. For each pole $a_{\mu\nu}$, a function

$$F_{\mu\nu}(x; a_{\mu\nu}; \mu = 1, 2, \ldots; \nu = 1, 2, \ldots, m) = G_{\mu\nu} \left(\frac{1}{x - a_{\mu\nu}} \right) - \sum_{\rho=0}^{m_{\mu\nu}-1} A_{\rho^{\mu\nu}} \left(\frac{a_{\mu\nu} - a\nu}{x - a\nu} \right)^\rho$$

is created. These are used to form the m functions

$$F_\nu(x; a_{\mu\nu}; \mu = 1, 2, \ldots) = \sum_{\mu=1}^{\infty} F_{\mu\nu}(x); \nu = 1, 2, \ldots, m$$

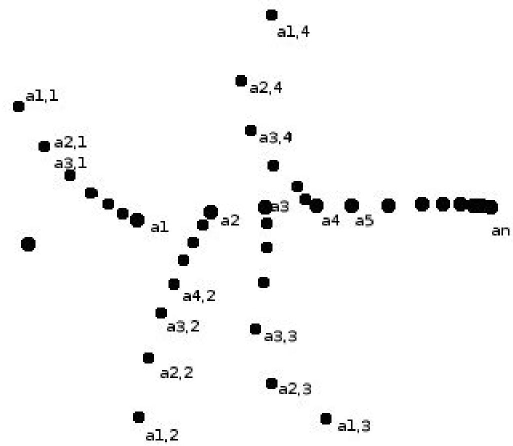

Figure 4.1: An example of singular points arranged in a "double series".

and thus the function

$$F(x) = \sum_{\nu=1}^{m} \sum_{\mu=1}^{\infty} F_{\mu\nu}(x)$$

is a uniform and monogenic function of the first type and first kind having exactly the singular points P.

The inductive step occurs in what should have been Mittag-Leffler's eighth publication in the *Comptes Rendus*. Supposing that the theorem holds in the case where P is of the first type and the $(n-2)^{th}$ or $(n-1)^{th}$ kind, Mittag-Leffler demonstrates that it must also be true when P is of the n^{th} kind. Supposing that $P^{(n)} = \{a_1\}$, and $P^{(n-1)} = \{a_{\mu 1}\}$ for $\mu = 1, 2, \ldots$, it is possible to form a sequence of positive values ρ_μ for $\mu = 1, 2, \ldots$ such that $a_{\mu 1}$ is the only value of x belonging to $P^{(n-1)}$ that satisfies the conditions $|x - a_{\mu 1}| \leq \rho_\mu$ and there is no value of x which satisfies the inequality for two different values of μ. The singularities are then divided into two groups, P_1 and P_{11} such that P_1 contains all elements of P which do not satisfy any of the conditions $|x - a_{\mu 1}| \leq \rho_\mu$. P_1 is clearly a subset of P, and thus $P_1^{(n-1)} \subset P^{(n-1)}$. However, since P_1 is a set of singularities it contains all of its limit points. Thus it is also true that $P_1^{(n-1)} \subset P_1$. Since $P_1 \cap P^{(n-1)} = \emptyset$, $P_1^{(n-1)}$ is empty and thus P_1 is a set of the $(n-2)^{th}$ kind. Under the original supposition, it is then possible to construct an entire monogenic function f having singular points $a_{\alpha\beta\gamma\ldots\lambda\mu 1}$ in P_1, which has a Laurent expansion

$$G_{\alpha\beta\gamma\ldots\lambda\mu 1}\left(\frac{1}{x-a_{\alpha\beta\gamma\ldots\lambda\mu 1}}\right)+P_{\alpha\beta\gamma\ldots\lambda\mu 1}(x-a_{\alpha\beta\gamma\ldots\lambda\mu 1}).$$

The remaining singular points belong to the set P_{11} which contains all elements of P satisfying at least one of the inequalities. Since $P_{11}\subset P$ it must be true that $P_{11}^{(n-1)}\subset P^{(n-1)}$. Inside the circle of radius ρ_μ, however, exists exactly one value $a_{\mu 1}\in P^{(n-1)}$, and thus $P_{11}^{(n-1)}=a_{\mu 1}$ or is empty, and thus P_{11} is of the $(n-1)^{th}$ or $(n-2)^{th}$ type. In this way, under the supposition it is possible to form a function F_μ having the singular points contained in the circle of radius ρ_μ. The desired function F, which is of the $(n+1)^{th}$ kind and has exactly the singular points of $P=P_1+P_{11}$ is thus of the form

$$F(x)=f(x)+\sum_{\mu=1}^{\infty}F_\mu(x).$$

This completes the inductive proof, and thus any function with a set of singularities of the first kind can be expressed in a series of the type prescribed by the Mittag-Leffler theorem.

It is important to observe now that Mittag-Leffler has actually constructed a function F having infinitely many essential singularities. This arose rather naturally through the use of the Cantorian notation; any derived set of a set of singularities is itself a set of essential singularities. This is so because a limit point of a set of singularities contains a singularity in any neighbourhood, thus preventing it from being a pole.

Therefore, if P is a set of the n^{th} type, each of the sets P', P'', $\ldots$, $P^{(n-1)}$ contains infinitely many essential singularities of F. This of course cannot be the case for a set of the first type, as dealt with in the basis case of the inductive proof. In that situation, all essential singularities must be contained in P' as it is only possible to derive P one time before arriving at the empty set. That said, P' must then contain a finite number of elements and thus a set of singular points P of the first kind and first type can contain only finitely many essential singularities.

Though this introduction of infinitely many essential singular points arose almost without notice, it should be emphasized as a significant moment in the history of this theorem — this had been Mittag-Leffler's goal at least as early as 1879 when he expressed this desire in his 1879 *Bulletin des Sciences Mathématiques* publication. Of course, a function with more than one essential singularity (or simply one with an essential singularity anywhere other than infinity) is not meromorphic. It is clear that by 1882 the Mittag-Leffler theorem encompassed a much larger class of functions than indicated in modern textbooks. Furthermore, as mentioned in Chapter 2, the term "meromorphic" was not used during the course of Mittag-Leffler's work. The fact that most modern statements treat only meromorphic functions is due to the use of Runge's theorem (refer to Appendix D) in the accompanying proof.

Meanwhile, Georg Cantor was about to contact Mittag-Leffler with his own thoughts pertaining to Mittag-Leffler's work.

Chapter 5

The Cantor/Mittag-Leffler correspondence

It is well-known that Mittag-Leffler did a great deal to promote Cantor's work in (transfinite) set theory through his journal *Acta Mathematica* and translations of the material into French (see [Dau79]); Dauben notes that, as we have just seen, this interest in Cantor's work was due in part to its importance to the formulation of the Mittag-Leffler theorem [Dau79]. One valuable source from which it is possible to glean information pertaining to Mittag-Leffler's use of Cantor's set theory is a collection of Cantor's letters as edited by Meschkowski and Nilson. Though the text is not complete and contains only excerpts of the letters from Mittag-Leffler to Cantor, it has been very useful in creating a fuller picture of this history [Can91], p. 80. For this chapter, the work by Meschkowski and Nilson is supplemented with archival material.

As mentioned above, prior to February 8, 1882, Mittag-Leffler became aware of Cantor's work and began to use the Cantorian idea of a derived set. In fact, on that date a short paper appeared in the Stockholm Academy Proceedings that shows his awareness of an 1879 paper of Cantor that deals with exactly this issue. In this 1882 paper, Mittag-Leffler mentions that Cantor has introduced a terminology for "infinite linear point sets"[1] and footnotes two papers by Cantor. The first paper was published in 1871 under the title *On the expansion of*

[1]"Oändliga lineera punktmångfalder."

a theorem from the theory of the trigonometric series[2], and the second was published in two parts, one each in 1879 and 1880, entitled *On infinite and linear sets of points*[3]. [Mit82a]

While it is not clear exactly when Mittag-Leffler learned of Cantor's work, it seems certain that it was after Mittag-Leffler's 1879 paper in the *Bulletin des Sciences Mathématiques*. As well, as Cantor's 1871 paper was published several years prior to the initial version of the Mittag-Leffler theorem, it seems more likely that Mittag-Leffler's knowledge of Cantor's theory of infinite sets came from Cantor's 1879 and 1880 publications. In any event, the ideas about derived sets and nowhere dense sets that Mittag-Leffler cites in 1882 are all to be found in Cantor's 1879 paper [Can79].

A letter from Hermite of December 24, 1881, also seems to indicate that Mittag-Leffler had mentioned Cantor in an earlier letter, as it states that Hermite, at this time, was very eager to learn from Mittag-Leffler about Cantor's new research[4] [Her84], p. 138.

Cantor noticed Mittag-Leffler's *Comptes Rendus* paper of April 3 and responded to Mittag-Leffler. The first such response is dated April 21, 1882.

In this letter, Cantor comments upon the content of Mittag-Leffler's April 3, 1882 *Comptes Rendus* article. It appears that he believed Mittag-Leffler to have misunderstood some of the theory. In particular, Cantor believed Mittag-Leffler to be under the incorrect impression that the derived set must always be contained in the set from which it derived, and he wrote to Mittag-Leffler with the response that this is only true after the first such derivation. He also commented that he had extended his research on point sets and planned to send Mittag-Leffler a portion of his new works, to be printed in the *Mathematische Annalen* in the near future. [Can91], p. 68

Mittag-Leffler's response, as summarized by Meschkowski and Nilson, indicated that he was not in error, as it was true for the sets with which he was concerned, namely those consisting of singularities of a single-valued function of a complex variable. Evidently, Mittag-Leffler then complimented Cantor on his work and indicated that he would like to visit Halle. [Can91], p. 68

In return, on May 1, 1882, Cantor acknowledged that he, and not Mittag-Leffler, had

[2]"*Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*" (see [Can71]).
[3]"*Ueber unendliche lineare Punkt mannichfaltigkeiten*" (see [Can79] and [Can80]).
[4]"Vous me ferez le plus grand plaisir en me donnant l'idée des nouvelles recherches de Mr Cantor ..."

made the oversight. He was enthusiastic about Mittag-Leffler's work in this area and commented that it was very close to his own. He also shared with Mittag-Leffler his thoughts pertaining to further work on the subject:

> As far as the real (most comprehensive) generalization of your theorem and that of Weierstrass is concerned, I fear that you will not get there by the path you have started out on. One must, I believe, have in view the common distinguishing feature of all point sets P which are singular points of a single valued monogenic function. These distinguishing features are, I believe, the following:
>
> 1. *countability* so that all singular points can be placed in the form of a series $\pi_1, \pi_2, \ldots$
> 2. P is in not everywhere dense[5] in any continuous one or two dimensional domain of the plane.
>
> Perhaps these two characteristics suffice so that based on them one can create a function with the given singular points $\pi_1, \pi_2, \ldots$ and can specify the general form of it.[6] [Can91], p. 69

Cantor gave no explanation as to why he believed these features to be significant, and concluded the letter in a friendly tone by saying

> I hope with joy that you will visit me this summer as you say in your letter you intend; then we can discuss thoroughly [these matters] ...[7] [Can91], p. 69

Thus as Mittag-Leffler attempted to use the theory of infinite sets to extend his theorem, Cantor gave him a place at which he could begin. This enthusiasm for Mittag-Leffler's work

[5]If the union of P and its limit points is equal to a set R, then P is everywhere dense in R.

[6]"Was die eigentliche (umfassendste) Verallgemeinerung Ihrer und der Weierstraßschen Sätze anbetrifft, so fürchte ich. dafl Sie auf den eingeschlagenen Wegen nicht dazu kommen werden. Man muß, wie ich glaube, die gemeinschaftlichen Merkmale oller Punctmengen P. die von singulären Stellen einer eindeutigen monogenen Function gebildet werden. ins Auge fassen; diere Merkmale sind, wie ich glaube, folgende:

1. *Abzählbarkeit*, so daß alle singulären Puncte sich in der Form einer Reihe: $\pi_1, \pi_2, \ldots$ setzen lassen.
2. P ist in keinem stetigen ein-oder zweidimensionalen Gebiete der Ebene *überalldicht*.

vielleicht reichen diese beiden Merkmale aus, um auf Grund derselben eine Function herzustellen, die die gegebenen Singularitätsstellen $\pi_1, \pi_2, \ldots$ besitzt und die allgemeine Form derselben zu bestimmen."

[7]"Ich hoffe mit Freuden, dafl Sie in diesem Sommer, wie in Ihrem Briefe in Ausicht genommen, mich besuchen werden; ..."

was likely quite encouraging, as Cantor was already an established mathematician, and it seems reasonable that this alone might encourage a less-established individual to run with the suggestion. Mittag-Leffler wisely did so, as it would later become clear that Cantor's theory and particularly the notion of a derived set would be significant in not only generalizing his theorem, but also in tidying up the theorem statement itself.

Mittag-Leffler did follow Cantor's advice, as he apparently questioned Cantor about everywhere dense sets in a letter dated June 21, 1882. Cantor responded in a letter of June 25, 1882. They continued to correspond about this topic, and Mittag-Leffler visited Cantor at Halle in July, 1882. [Can91]

Later that year, Cantor provided Mittag-Leffler with the following theorem:

Theorem 5.1 *If P is a countably infinite set of singular points of a single-valued analytic function of a complex variable, then there always exists an α such that $P^{\alpha} = 0$.*

Cantor encouraged Mittag-Leffler to use this theorem to extend his own and Weierstrass' theorems on the existence of single-valued analytic functions with a restricted set of singularities to the existence of those having an arbitrary countable set of singularities P. He noted that more work (which he was already considering) was necessary to extend the result to an uncountable set P. [Can91], p. 88

Mittag-Leffler, in a letter of October 22, 1882, was very pleased and wished to see the proof. He wrote that Weierstrass and Poincaré did not believe that such a theorem existed, and he commented that his theorems could be generalized to countable sets of singularities such that $P^r = 0$ for some r, but that he had felt that this was of little interest until one knew exactly which countable sets had this property. [Can91], pp. 88–89

It should be emphasized that Mittag-Leffler's interest in Cantor's ideas was quite exceptional at this time. The general opinion of Cantor's set theory was negative in both Germany and France. Cantor, in a letter to Mittag-Leffler dated September 9, 1883, quotes Kronecker, who visited him during July, as saying that Cantor's theory of infinite sets and infinite numbers was "Humbug" (referring to the concept of different orders of infinity). [Can91], p. 127

The French side of the story was not much better. Hermite, who may or may not have been influenced by Kronecker (as their correspondence is lost it is difficult to be sure), told Mittag-Leffler in a letter dated April 13, 1883, that:

> The impression that Mr. Cantor's papers produce in us is pitiful. Reading them seems to all of us a real chore. [Her84], pp. 209–210

Yet Mittag-Leffler continued to work in the direction suggested by Cantor. Mittag-Leffler felt very close to generalizing his theorem to the fullest extent through the use of Cantor's language. It seems likely that by this point Mittag-Leffler felt a personal loyalty to Cantor, with whom he had formed a friendship, and it is clear that he was happy about Cantor's praise for his work. Indeed, his result appeared to show a concrete use for Cantor's work that might be seen to justify it to mathematicians such as Kronecker and Hermite. I will discuss other aspects of Mittag-Leffler's relationship with Cantor and his work in Chapter 7.

Chapter 6

The final version of Mittag-Leffler's theorem

The final version of the Mittag-Leffler theorem appeared in his journal *Acta Mathematica*, under the title *On the analytical representation of uniform monogenic functions of an independent variable*[1] [Mit84].

The main feature of this text, which is the final generalization of Mittag-Leffler's theorem and includes complete proofs, is a slightly altered use of Cantor's set theory. The concept of a "derived set", or collection of limit points of another set, remains fundamental. In the 1884 version, however, Cantor's idea use nowhere dense sets has given way to the notion that the set of singularities should be an *isolated* set. This definition appears in a lengthy introduction which sets the stage for the newest and last version of his theorem.

Definition 6.1 *A point is* isolated *in a given set Q if it has a neighbourhood that contains no other point of the set. A set is isolated when all of its points are isolated.* [Mit84], p. 4

To see what this means in the context of singularities, suppose P is an arbitrary set and $P' \subset P$. It can happen that P contains elements not included in P'; we denote this set $P \setminus P'$ by Q. As the elements of Q are not limit points of P, there belongs, to each point in Q, a neighbourhood in which no other point of Q can be found. That is, the elements of

[1]"*Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*"

Q form an *isolated set.* Of course, as $Q \subset P$, it follows that $Q' \subset P'$; when P is the set of singular points of a function $F(x)$, then $Q' \subset P$ as well.

Another new addition to the theorem is the concept of a *continuum*, a term which, at that point in time, had different meanings amongst mathematicians. To Weierstrass, a continuum was a connected region in the complex plane. If we let infinitely many rational functions of the (complex) variable x be given in a specific sequence $f_0(x), f_1(x), f_2(x), \ldots$, we can define the *domain of convergence* to be the collection of all values of x for which the series

$$\sum_{\nu=0}^{\infty} f_\nu(x)$$

is finite. We denote the domain of convergence by A. If then a is contained in this domain, it is possible to find a (two-dimensional) neighbourhood of a in the complex plane within which the series above converges uniformly for all x. Of course, the domain of convergence of the series may consist of several (two-dimensional) areas of the plane which are separated from each other. With this in mind, we can now define Weierstrass' concept of a continuum.

Definition 6.2 *(Weierstrass) Take arbitrarily a point in A, in a neighbourhood of which there is an arbitrary second point, in a neighbourhood of which there is an arbitrary third, and so on in this manner. A* continuum (A_1) *is then the collection of points of A which one can get to by this means. In this way, (A_1) is a connected piece of the complex plane, and has a boundary which consists of either points, lines, or a combination of the two.* [Wei80b], pp. 201–203

Of course, by this definition there may exist points of A outside of (A_1); this could be caused by a set of singularities that blocks analytic continuation. If this is the case, it indicates that there exists in this domain at least one other continuum (A_2). (A_2) has thus no points in common with A_1 except possibly on their boundaries which may coincide partially or completely. In the same way, if there are further points of A which are contained neither in (A_1) nor (A_2) there must exist at least a third continuum (A_3) ...[2]

[2] Continua are directly connected to the concept of monogenicity. Under Weierstrass, a function is essentially a collection of power series representations at different points within its domain. The domain is then divided into continua depending on whether or not the function is monogenic. If it is monogenic, there is only one continuum. More generally, inside each continuum, a function is monogenic.

It is significant that to Weierstrass, neighbourhoods and continua should be defined in terms of the domains within which power series converge. The idea that series are key to learning about the functions they represent is central to his entire theory of the foundations of analysis, which can certainly be seen in, for example, his famous factorization theorem.

Not all mathematicians echoed Weierstrass' thoughts. Some, such as Georg Cantor, had entirely different agendas. Cantor, though his studies had begun in function theory, was now concerned with general sets of points. He therefore sought concepts that were independent of ideas about functions and that went beyond the complex plane. In particular, he was interested in learning about countability and dimensionality of point sets. The following remark on continua is taken from a letter addressed to Mittag-Leffler written on February 10, 1883:

> ...the system of all real numbers $\geq a$ and $\leq b$ is a continuum in my sense and conversely I can prove that every linear continuum *is* a complete interval $(a \ldots b)$ and can *not* be anything else.[3] [Can91], p. 114

Thus in Cantor's sense, linear continua are one-dimensional intervals on the real number line, which can be generalized to higher dimensions. Cantor's ideas were more general than those of Weierstrass. In fact, Cantor felt that Weierstrass' strong connection of continua to the convergence of power series was preventing him from achieving full generality. In a letter dated February 7, 1883, Mittag-Leffler asked Cantor the following question:

> What do you have against the following definition of the straight line between zero and one: this line is the set of points that contains *all* rational and irrational numbers that are greater than 0 and less than one. One could in an analogous way give the definition of any continuum.[4] [Can91], p. 115

Cantor's response was the following:

[3]"Allerdings ist das System aller rellen Zahlen $\geq a$ und $\leq b$ ein Continuum in meinem Sinne und auch umgekehrt kann ich beweisen jede *lineare* (d.h. in einer geraden Linie enthaltene) Punctmenge, die meiner Definition eines Continuums genügt, *ist* ein vollständiges Intervall $(a \ldots b)$ und kann *nichts* anders sein."

[4]"Was haben Sie gegen die folgende Definition der geraden Linie zwischen Null und Eins einzuwenden: diese Linie ist die Punkt-Menge, welche *alle* rationalen oder irrationalen Zahlen, die grösser als Null und kleiner als Eins sind, umfasst. Auf analoge Weise könnte man dann die Definition irgend welches Continuum erhalten."

> *If however you go to* point sets that belong to a space G_n, where $n > 1$, the usual definition will not work if you want to have the most general concept of continuum; and only with this, as I will describe later, do you get to the most general definition of *lines, surfaces, bodies.*
>
> In your way [as proposed in Mittag-Leffler's question above] and also on the path that Weierstrass follows in his lectures, you don't get to generally valid concepts, because you are dependent on analytic representations, which never provide the conviction that you have not overlooked something. Likewise I hold everything that we have from Riemann on this to be completely insufficient. [5] [Can91], p. 114

It is not particularly clear what Cantor saw as problematic in Mittag-Leffler's definition. However, Weierstrass' definition requires the existence of a function before the set, a sharp contrast to Cantor's belief that the set was the most general object and the key concept. Thus Cantor felt that the emphasis on the representation of functions interfered with the generality of this concept, and in fact the generality of all concepts in this vein. Cantor advocated a general shift from dealing with functions, and in particular the representations of functions, to dealing with set of points (of their domain and range) and this discussion clearly illustrates this fact. Mittag-Leffler seems to have agreed with Cantor about this issue, but maintained that Weierstrass' definitions were still important. On February 27 of the same month he wrote to the Cantor:

> I very much agree with your definition of continuum, and would however like to refer to what Weierstrass calls a continuum as a "completely connected point set". I am thinking of using this terminology in my work and I think this will serve the purpose very well. It will follow sufficiently from my work that such completely

[5]" *Wenn* Sie aber übergehen zu Punctmengen, welche einem Raum G_n angehören, wo $n > 1$, so reichen Sie meit der gewöhnlichen Definition nicht aus, wenn Sie den allgemeinsten Begriff des Continuums haben wollen; und erst fon diesem aus komment Sie, wie ich später ausf|"uhren werde, zu den *allgemeinsten* Definitionen von *Linien, Flächen, Körpern.*

" Auf Ihrem Wege und auch auf dem Wege, welchen Weierstrass in seinen Vorlesungen befolgt, kommt man nicht zu allgemeingültigen Begriffen, weil man von analytischen Darstellungen abhängig ist, die einem niemals die Ueberzeugung liefern, dass man nichts vergessen hat. Ebenso halte ich Alles was man von Riemann darüber hat für gar night ausreichend."

> connected point sets have their necessary place in the theory of analytic functions and can't be replaced by your continua.[6] [Mitc]

With this discussion in mind, we now prepare to state the Mittag-Leffler theorem as it appeared in *Acta Mathematica*.

We begin as in [Mit82d] by letting P be an infinite set of elements in the complex plane, and then constructing the derived set P'. However, at this point in [Mit84] Mittag-Leffler makes use of the concept of an isolated set.

We are now ready to look at the new formulation of the Mittag-Leffler theorem.

Let $Q = \{a_1, a_2, \ldots, a_\nu, \ldots\}$ be an isolated set belonging to the domain of a variable x, and let

$$G_1\left(\frac{1}{x-a_1}\right), G_2\left(\frac{1}{x-a_2}\right), \ldots, G_\nu\left(\frac{1}{x-a_\nu}\right), \ldots$$

be a series of uniform, monogenic functions, entire and rational or transcendental in $\frac{1}{x-a_\nu}$, disappearing when $\frac{1}{x-a_\nu} = 0$. It is always possible to form an analytic expression which behaves regularly in the vicinity of the points belonging to $Q + Q'$ (under Cantor's notation, this refers to the union of the sets Q and Q') and which, for each value of ν, can be developed under the form

$$G_\nu\left(\frac{1}{x-a_\nu}\right) + P(x - a_\nu)$$

where P is a power series in $(x - a_\nu)$.

That is, the newest version of Mittag-Leffler's theorem states that if the collection of poles of a function $F(x)$ form an isolated set Q whose limit points form the set of essential singularities Q' of $F(x)$, then the function specified by the Mittag-Leffler theorem can be formulated in the usual way, analytic everywhere except at the points of $Q + Q'$.

Assuming that $Q = \{a_1, a_2, \ldots, a_\nu, \ldots\}$ is an infinite set of points, Mittag-Leffler states that it is always possible to adjoin to each value $a_\nu \in Q$ not equal to zero or infinity another value $b_\nu \in Q'$, where b_ν is chosen so that

$$\lim_{\nu\to\infty} |a_\nu - b_\nu| = 0.$$

[6]"Ich bin mit Ihrem Definition von Continuum sehr einverstanden, möchte aber dann das was Weierstrass Continuum nennt, eine "vollkomen zusammenhängende Punktmenge" nennen. Ich denke diese Terminologie in meiner Arbeit zu brauchen und glaube dass dies sehr zweckmässig wird. Dass solche volkommen zusammenhängende Punktmenge in der Theorie der analytischen Funktionen ihre nothwendige Stellung haben und nicht durch ihre Continua ersetzt werden können wird aus meiner Arbeit genügend hervorgehen."

A paragraph follows on how these points b_ν are to be selected, the details of which are omitted for brevity. We then choose arbitrarily a positive quantity $\epsilon < 1$, and a series of positive terms $\epsilon_1, \epsilon_2, \ldots$ such that $\sum \epsilon_n$ is finite. We see that this is exactly the technique developed by Weierstrass in 1880 that Mittag-Leffler used in his earlier work, with the exception that it has been adjusted to deal with new terminology and collections of singularities. Thus to form the analytic expression sought, I will present only the case in which a_ν and b_ν are finite. In this situation, it is necessary to find a whole number m_ν such that

$$\sum_{\mu=m_\nu+1}^{\infty} A_\mu^{(\nu)} \left(\frac{a_\nu - b_\nu}{x - b_\nu}\right)^\mu < \epsilon_\nu$$

when

$$\left|\frac{a_\nu - b_\nu}{x - a_\nu}\right| \leq \epsilon.$$

Once m_ν is found, as usual we let

$$F_\nu(x) = G_\nu\left(\frac{1}{x - a_\nu}\right) - \sum_{\mu=0}^{m_\nu} A_\mu^{(\nu)} \left(\frac{a_\nu - b_\nu}{x - b_\nu}\right)^\mu,$$

and thus

$$F(x) = \sum_{\nu=1}^{\infty} F_\nu(x)$$

is the desired function.

Mittag-Leffler then indicates to the reader that to find m_ν it is first necessary to determine the coefficients A_ν in terms of the given coefficients $c_{-1}^{(\nu)}, c_{-2}^{(\nu)}, \ldots$. The necessary steps in accomplishing this can be found in Appendix C.

Mittag-Leffler notes that if one restricts the set Q' to include only the point at infinity, and restricts the functions $G_\nu\left(\frac{1}{x-a_\nu}\right)$ to polynomials (thus entire and rational functions), then this result is identical to that found in his 1876 publication in the *Öfversigt* of the Royal Swedish Academy; if the functions $G_\nu\left(\frac{1}{x-a_\nu}\right)$ are allowed to be transcendental, the result is identical to that of his 1877 Swedish publication, and the April 13 *Comptes Rendus* letter.

Mittag-Leffler then changes his tactics and rephrases his theorem using the notion of a *continuum* in order to demonstrate that his techniques apply to situations in which the singularities of the desired function impose a natural boundary upon its domain which prevents analytic extension to the remainder of the complex plane. In this case, Q must form an isolated set in the complex plane such that $Q + Q'$ is the *complete limit* (i.e. the set of all limit points of Q, and *not* the boundary of Q as one would expect in the present day) of a *continuum* A. This follows Weierstrass' notion of a continuum; we would think of A as a connected subset of the complex plane representing the domain of Mittag-Leffler's function $F(x)$. The functions $G_\nu\left(\frac{1}{x-a_\nu}\right)$ may then be entire rational or transcendental functions of $\frac{1}{x-a_\nu}$ without constant terms.

As an example of singularities which establish a natural boundary, Mittag-Leffler considers the set

$$\{a_{nk}\} = \left(1 + \frac{(-1)^{n+1}}{n+1}\right) e^{\frac{2\pi k i}{n+1}},$$

where $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots, \infty$, and the functions $G_{nk}\left(\frac{1}{x-a_{nk}}\right)$. The term $e^{\frac{2\pi k i}{n+1}}$ consists of points (these turn out to be essential singularities) which fall on the unit circle centered at the origin; however, the negative sign in the term $1 + \frac{(-1)^{n+1}}{n+1}$ causes the singularities to fall alternately inside and outside of the circle. In this situation, we see that $Q = \{a_{nk}\}$ and $Q' = \{b_{nk}\} = e^{\frac{2\pi k i}{n+1}}$. Refer to Figure 6.1 to see the plot of the first 3600 points of $\{a_{nk}\}$.

However, at this point something must be noted — this set of singularities is not a set of type one, but a set of type two; as its singularities have all the points of the unit circle as limit points, further derivations will produce exactly the unit circle as values. This raises the question as to how deeply Cantor's theory of infinite sets of points is actually being used — at this point, it is only to order and organize the singular points of a function. The fact that Mittag-Leffler's work was thought to be verging on the dubious, even with this almost insignificant use of Cantor's work, is something that will be discussed in Chapter 7.

The last main theorem of [Mit84] concerns the Weierstrass factorization theorem. Mittag-Leffler outlines this theorem in the following way:

Let Q be an infinite isolated set of points in the domain of the variable x, such that the set $Q + Q'$ forms the complete limit of a continuum A under the Weierstrassian definition. Let now $a_1, a_2, \ldots, a_\nu, \ldots$ form the different points (i.e. poles) of Q, and let $n_1, n_2, \ldots, n_\nu, \ldots$

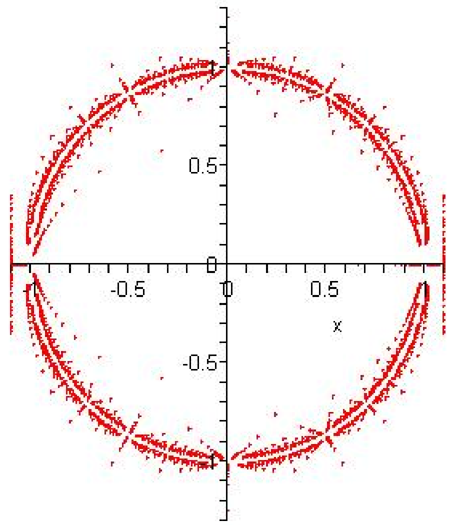

Figure 6.1: A set of singular points for a non-monogenic function.

exist as a series of integers. It is then always possible to form an analytic expression representing a uniform monogenic[7] function which behaves regularly inside the continuum A and which does not become zero inside that domain. Moreover, in the neighbourhood of each point a_ν, the function can be represented under the form

$$(x - a_\nu)^\nu e^{P(x-a_\nu)}.$$

Each point of Q' is then an essential singularity of that function.

The steps taken by Mittag-Leffler to prove this theorem are essentially identical to those given for the theorem presented at the beginning of [Mit84]. That is, to each singularity a_ν of Q, we "adjoin" a point b_ν of Q', and choose a sequence $\{\epsilon_\nu\}$ having finite sum. As before we consider the case where a_ν is neither zero nor infinity.[8] If this is the case, provided

$$\left|\frac{a_\nu - b_\nu}{x - b_\nu}\right| < 1$$

[7] Inside this continuum.

[8] The steps are only slightly different for the case where a_ν equals zero or infinity; for brevity, these cases will be omitted here.

it is possible to construct the equations

$$\frac{n_\nu}{x-a_\nu} = \frac{n_\nu}{x-b_\nu} + \frac{n_\nu}{x-b_\nu}\sum_{\mu-1}^{\infty}\left(\frac{a_\nu - b_\nu}{x-b_\nu}\right)^{\mu}.$$

It is not difficult to show that the equation above, for finite[9] b is the logarithmic derivative of

$$\left(1-\frac{a_\nu - b_\nu}{x-b_\nu}\right)^{n_\nu} = e^{-n_\nu \sum_{\mu=1}^{\infty}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}},$$

which can be rewritten as

$$\left(1-\frac{a_\nu - b_\nu}{x-b_\nu}\right)^{n_\nu} e^{n_\nu \sum_{\mu=1}^{m_\nu}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}} e^{-\left[-n_\nu \sum_{\mu=m_\nu+1}^{\infty}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}\right]} = 1,$$

or

$$\left(1-\frac{a_\nu - b_\nu}{x-b_\nu}\right)^{n_\nu} e^{n_\nu \sum_{\mu=1}^{m_\nu}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}} = e^{-n_\nu \sum_{\mu=m_\nu+1}^{\infty}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}}.$$

Mittag-Leffler claims that it is always possible to select a number m_ν such that the absolute value of the exponent on the right side is less than the previously specified value ϵ_ν; it is this m_ν that should be used.

Then, if we set

$$E_\nu(x) = \left(1-\frac{a_\nu - b_\nu}{x-b_\nu}\right)^{n_\nu} e^{n_\nu \sum_{\mu=1}^{m_\nu}\frac{1}{\mu}\left(\frac{a_\nu-b_\nu}{x-b_\nu}\right)^{\mu}},$$

the product

$$\prod_{\nu=1}^{\infty} E_\nu(x)$$

has the properties specified at the beginning of the theorem. Specifically, inside the continuum A, it is monogenic and uniform, behaves regularly, and in the neighbourhood of each pole a_ν the function can be put into the appropriate form.

According to Mittag-Leffler, this result was presented by Picard to the members of the *Académie des Sciences* on March 21, 1881, for the case in which Q contains the *zeros* of the function considered, and in which the continuum $A + Q$ represents the domain of the variable x contained entirely either within the interior of a circle centered at the origin, or outside of this circle [Mit84], p. 38.

[9]If b is infinite, $\frac{a_\nu - b_\nu}{x-b_\nu}$ is replaced by $\frac{x}{a_\nu}$.

He then comments that if the domain $A+Q$ is contained entirely within the circle given above, and $b_\nu = \infty$ for all ν, the above representation can be rewritten as

$$\prod_{\nu=1}^{\infty} E_\nu(x) = \left(1 - \frac{x}{a_\nu}\right)^{n_\nu} e^{n_\nu \sum_{\mu=1}^{m_\nu} \frac{1}{\mu}\left(\frac{x}{a_\nu}\right)^\mu},$$

as given by the Weierstrass Factorization Theorem [Mit84]. This indicates that even though Mittag-Leffler is giving serious consideration to, and making use of, Cantor's theory of infinite set of points, he is still as tightly bound to Weierstrass' research program as he was in 1876.

Chapter 7

Reception and Concluding Remarks

7.1 Reception

The reception of the Mittag-Leffler theorem is somewhat complicated. On one hand, his results were perceived by many readers as being important and exciting, especially while they adhered strictly to Weierstrass' theory of functions and had not yet incorporated Cantor's new ideas. Hermite summarizes this sentiment in a letter written to Mittag-Leffler on April 19, 1881, with the statement:

> What you say to me of the general formulas that you have, and which give the general analytic expression of uniform functions having "an infinity of singular points", either on a line, or in a surface, interests me extremely, and I wait impatiently to be able to study your results on such an important and so difficult question.[1] [Her84], p. 119

On the other hand, when Mittag-Leffler began to make use of Cantor's theory of infinite sets of points, the controversy surrounding Cantor's work settled on Mittag-Leffler's as well.

We'll begin to study the reception of the Mittag-Leffler theorem by looking at the reactions of his mentors, particularly Hermite, to Mittag-Leffler's work on the subject prior to

[1] "Ce que vous me dites des fonctions générales que vous possédez, et qui donnent l'expression analytique des fonctions uniformes ayant une 'infinité des points essentiels', soit sur une ligne, soit dans une aire, m'intéresse extrêmement, et j'attends avec impatience de pouvoir étudier vos résultats sur une question si importante et si difficile."

the inclusion of Cantor's notation.

7.1.1 The reactions of Hermite and Weierstrass prior to the introduction of Cantor's set theory

As mentioned earlier in this paper, after Mittag-Leffler published his work in Swedish in 1876 and 1877 he sent a German version of his results to Weierstrass entitled *Arithmetic representation of single-valued analytic functions of one variable.*[2] When Hermite learned of this, he wrote to Mittag-Leffler to say that he had asked the mathematician and journal editor Carl Borchardt[3] (1817–1880), who would be meeting with Weierstrass and his sisters in Berchtesgaden shortly, to get Weierstrass' opinion of Mittag-Leffler's memoir. Reporting on Borchardt's response, Hermite noted that "[t]he judgement of the illustrious analyst on your [Mittag-Leffler's] research will be very interesting for me to know"[4], as he (Hermite) was not able to penetrate the Swedish in order to determine Mittag-Leffler's method. Hermite then requested that a French exposition of the work be presented in Darboux's *Bulletin.* [Her84], p. 58

Hermite wrote back to Mittag-Leffler on November 10, 1879, with Borchardt's report that Weierstrass' opinion was quite favourable. Weierstrass felt that result obtained by the young Swedish mathematician was extremely interesting and demonstrated a remarkable talent, but that the work was far too long and especially too charged with calculations [Her84], p. 61. Weierstrass' response is certainly not surprising, as he wrote as much to Mittag-Leffler himself. However, Hermite's desire to learn of Weierstrass' opinion demonstrates his opinion about the significance of these results.

Thus it is clear that Mittag-Leffler's work attracted a great deal of attention from several prominent mathematicians of that time; this fact is further emphasized when we consider that Weierstrass and Hermite each published an alternate proof in 1880.

Weierstrass and Hermite were interested in different aspects of Mittag-Leffler's work. Weierstrass, who had already been working on theorems for the representation of functions,

[2]"*Arithmetische Darstellung eindeutiger analytischer Functionen einer Veränderlichen*".

[3]Bortchardt (1817–1880) was the second editor of the journal founded by Crelle, the journal *Für die reine und angewandte Mathematik*, and a professor at the university in Berlin.

[4]"Le judgement de l'illustre analyste sur vos recherches me sera d'autant plus intéressant à connaître . . . "

felt that Mittag-Leffler's theorem was an important part of the foundations of analysis. Hermite, on the other hand, had little interest in abstract mathematics, of the kind referred to by Sørensen as concept-based, and was more interested in the theorem's potential for application to his areas of interest. In 1882 Hermite published a paper entitled *Extract of a letter addressed to Mr Mittag-Leffler, of Stockholm, by Mr Ch Hermite, of Paris, on an application of [the] Theorem of Mr Mittag-Leffler in the Theory of Functions*[5] in Crelle's Journal [Her82]. This letter focused on finding a representation for the beta function

$$F(x) = \frac{\Gamma(x)\Gamma(\alpha)}{\Gamma(x+\alpha)},$$

which has poles at negative integers and residues

$$R_n = \frac{(-1)^n(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{n!},$$

and for other related expressions. Hermite was particularly interested in the idea of forcing a divergent series to converge by adding a constant or another function, a key aspect of Mittag-Leffler's proof. [Her82]

Even more indicative of the esteem for Mittag-Leffler's work, however, is the fact that his contemporaries felt it was important enough to teach. In particular, Hermite lectured on Mittag-Leffler's theorem in his course on analysis at the Faculté des Sciences in Paris (Sorbonne). This is fairly significant, as this was an introductory course, typically for first and second year students [Her91]. The class included the *crème de la crème* of French undergraduates, since the students at the Ecole Normale Supérieure attended lectures at the Faculté des Sciences of the Sorbonne due to the close proximity of the two schools. [Arc02]

Hermite was not the only French mathematician teaching this material. On June 29, 1882, Hermite wrote a letter to Mittag-Leffler containing the passage:

> My dear friend,
>
> I hasten to offer you my warmest and sincere congratulations. A good providence certainly watches over your destiny and gives you, at the moment of your marriage, a position [the mathematics chair at the newly-founded University of

[5] "*Extrait d'une lettre adressée a M. Mittag-Leffler, de Stockholm, par M. Ch Hermite, de Paris, sur une Application du Théorème de M. Mittag-Leffler dans la Théorie des Fonctions.*"

> Stockholm] that is not inferior to your talent and your work. Instead of others, and long before I did, you will have a suitable chair, and students who will reap the fruits of your efforts and follow your path. You will keep me abreast, I hope, of all the circumstances of your teaching which will not be without some connection to that of the Sorbonne. The students of *Ecole Normale* have studied your theorem with great zeal, as Mr Darboux tells me, and I may say without exaggeration that it has now become classic among us. [6] [Her84], p. 123

This text makes us aware of several points. First, it indicates that Mittag-Leffler's work had also entered the classrooms of *Ecole Normale* by the Spring of 1881, and that the students there were eager to learn the material.[7] More importantly, we learn that the Mittag-Leffler theorem was already considered a standard result in analysis among the next generation of mathematicians. Finally, Hermite's warm remarks regarding Mittag-Leffler's professional capabilities emphasize the value Mittag-Leffler was felt to have as a lecturer and researcher within the European mathematical community.

It is clear, then, that with its Weierstrassian proof and notation the Mittag-Leffler theorem was seen as an exciting and important result to be studied by great mathematicians and beginners alike. We will now look at the role of the theorem upon Mittag-Leffler's introduction of Cantor's notation.

7.1.2 Reception of the Cantorian version

Interestingly, as mentioned in Chapter 4, when Mittag-Leffler mentioned to Hermite Cantor's new treatment of the theory of infinite point sets, the idea was not greeted with suspicion, but rather with interest and curiosity. Hermite was not the only French mathematician who

[6] "Mon cher ami,
Je m'empresse de vous offrir mes sincères, mes plus vives félicitations. Il y a certainement une bonne providence qui veille sur votre destinée en vous donnant au moment de votre mariage une position non inférieure à votre talent et à vos travaux. Plutôt que d'autres et bien avant moi vous aurez eu une chaire à cotre concenance, et des élèves qui recueilleront le fruit de vos efforts et suivront votre trace. Vous me tiendrez j'espère au courant de toutes les circonstances de votre enseignement qui ne sera point sans quelque connexion avec celui de la Sorbonne. Les élèves de l'Ecole Normale ont mis un très grand zèle, dont Mr Darboux m'a fait part, à etudier votre théorème, devenu maintenant classique parmi nous, je puis le dire sans exagération."

[7] At this time, Gaston Darboux (1842–1917) held a position akin to an assistant professor or tutor at the Ecole. He held tutorials for the students there, which included lectures on the Mittag-Leffler theorem.

wished to learn of this new work. As he wrote to Mittag-Leffler on December 24, 1881:

> You produce in me the greatest pleasure by giving me the idea of new research of Mr. Cantor, and as for what you announce to me on the uniform functions having an "infinity of essential [singular] points" you cannot imagine how impatient Emile Picard and I are to know it.[8] [Her84], p. 138

Later in the same letter, Hermite commented that "the name[s] of you and Mr. Weierstrass have been repeated from mouth in mouth in all the Ecole [Normale Supérieure], where you are now as well-known as in the Sorbonne."[9]

Clearly, then, Mittag-Leffler's fame connected to his theorem was not in decline. Indeed, his work remained popular and useful even after Mittag-Leffler published his fifth letter in the *Comptes Rendus* and begun to make use of Cantor's set-theoretic language. In fact, many French mathematicians seem to have been open to the possibility of adopting Cantor's concepts. In the April 24, 1882 publication of the *Comptes Rendus* Poincaré published a paper in which he cited an example of a function of the *deuxième genre* under the classification presented to the *Académie* by Mittag-Leffler on April 3. [Poi82], p. 1167

Thus it seems as though Mittag-Leffler's work continued to be widely read and even influential throughout the course of his publications in the French *Comptes Rendus*. However, as the French acquaintance with Cantor's work deepened, Hermite and others began to show concern over the reliance Mittag-Leffler was placing on Cantor's methods.

In the winter of 1882 Hermite told Mittag-Leffler that the French mathematicians would help Mittag-Leffler in translating into French the German work deemed by him to be the most important. In particular, Hermite mentions in a letter of January 26, 1882, that Paul Appell had indicated he would be working to translate the first memoir of Cantor [Her84], p. 192.

The work to which Appell was referring was originally published in 1879, and is the first of the six-part series entitled *On infinite and linear sets of points*[10]. This work was

[8]"Vous me ferez le plus grand plaisir en me donnant l'idée des nouvelles recherches de M Cantor, et quant à ce que vous m'annoncez sur les fonctions uniformes possédant une infinité de points essentiels vous ne pouvez vous imaginer à quel point Emile Picard et moi nous sommes impatients de le connaître."

[9]"...le nom de M Weierstrass et le vôtre ont été répétés de bouche en bouche dans toute l'Ecole, où vous êtes maintenant aussi connu qu'à la Sorbonne." [Her84], p. 138

[10]"*Ueber unendliche lineare Punktmannigfaltigkeiten.*"

mentioned in Chapter 4 for its significance to Mittag-Leffler, who used it, along with [Can80], in developing his early knowledge of Cantor's theory of infinite point sets as used in [Mit82d] and [Mit82a]. The French had agreed to translate this work of Cantor's on Mittag-Leffler's suggestion, so that it may be made available to a greater audience. When the translation began, however, the work was greeted with uncertainty due to its extensive philosophical content.

Hermite reported to Mittag-Leffler in a letter dated March 5, 1883, that:

> Mr. Poincaré judges that the French readers will almost all be completely repelled by the both philosophical and mathematical research of Mr. Cantor, where the arbitrary has too much of a role, and I do not believe that he is mistaken.[11] [Her84], p. 199

Mittag-Leffler seemed sympathetic to this concern, though he still believed Cantor's work to have merit. His response to Hermite three days later requested that Cantor's last large report not be translated. Mittag-Leffler indicated that he would ask that the memoir be written in another manner, one which excluded philosophy [Her84], p. 276. He then commented:

> I am persuaded for my part that this mathematical piece is of great importance and I believe that Mr. Poincaré himself would draw at once from them from it considerable advantages. But we will see![12] [Her84], p. 276

Cantor evidently agreed to modify the work accordingly, when Mittag-Leffler told him that it would be "more easily appreciated in the mathematical world" [Her84], p. 276.

Nonetheless, the French mathematicians generally resisted Cantor's concepts. There were a few exceptions to this statement, however. Picard seems to have felt that Cantor's new theory could be useful, but had some reservations. On May 15, 1884, Mittag-Leffler wrote to Cantor quoting a letter he had received from Picard:

> I acknowledge that in the beginning Cantor's conjectures seemed uninteresting to me, if only from the philosophical point of view, I am now beginning to believe

[11]"Mr Poincaré juge que les lecteurs français seront à peu près tous absolument réfracteurs aux recherches à la fois philosophiques et mathématiques de Mr Cantor, où l'arbitraire a trop de part, et je ne crois pas qu'il se trompe."

[12]"Je suis persuadé pour ma part que cette partie mathématique est d'une grand importance et je crois que M. Poincaré lui-même en tirerait une fois des advantages considérables. Mais nous verrons!"

> that all of this will be able to have applications in analysis: some of his theorems on trigonometric series, where it is a question of points of the first type, really struck me. Aren't you going to publish something on these questions soon? That will complete my conversion to point sets.[13] [Mitd]

When Mittag-Leffler published his *On the analytical representation of uniform monogenic functions of an independent variable* in 1884, the reader will recall that he had maintained his use of Cantor's concepts in dealing with the singular points of the function he wished to specify or represent. From today's standpoint, the role of Cantor's theory in Mittag-Leffler's argument (the steps of which still adhered closely to Weierstrass' initial revision) is almost insignificant. The continued protest from the French mathematicians, then, indicates that to them it really wasn't insignificant.

Perhaps this is explained by the following passages, taken from letters written by Hermite to Mittag-Leffler in August of 1884.

> ...I will make the critical study of your recent report of which Picard and Appell already tell me the best things, their only reservations concerning the same concepts which are due to Mr. Cantor; we all wonder if the definitions of the new singularities would not have something a bit artificial and factitious, instead of resulting from the nature of things, as we would like it.[14] [Her85], p. 91

Then, in October of 1884 Hermite sent Mittag-Leffler some critical remarks regarding the substance and style of his work:

> Two things are to be distinguished in your work: the propositions, the results at which you arrived, then the exposition that you have made, that is to say, the substance and the style. For the substance, I am certainly the echo of all analysts by recognizing that it constitutes, with the famous theorems of Weierstrass,

[13]"Je vous avoue qu'au début les spéculations de Cantor m'avaient paru sans intérêt, si ce n'est au point de vue philosophique, je commence a croire maintenant que tout cela pourra avoir des applications en analyse: quelques uns de ses théorèmes sur les séries trigonométriques, où il est question de points du premier genre, m'ont extrêmement frappé. N'allez-vous pas aussi publier bientôt quelque chose sur ces questions, cela va achever de me convertir aux ensembles de points. "

[14]"C'est là que je ferai l'étude critique de votre récent mémoire dont Picard et Appell m'ont déjà dit le plus grand bien, leur seule réserve concernant les notions mêmes qui sont dues à M Cantor; nous nous demandons tous si les définitions des nouvelles singularitiés n'auraient point quelque chose d'artificiel et de factice, au lieu de résulter de la nature des choses, comme nous le voudrions."

> the very base of the theory of single-valued functions, and Picard with whom I discussed it shares my opinion entirely . . . [but regarding the style,] we both are agreed that, in proceeding by that way which consists of starting from entirely new abstract concepts to arrive through a sequence of deductions at realities of analysis, you obeyed the German tendency, with a mentality that is not ours. What is an imperative need for the French mind is to proceed in the opposite sense, by taking all possible care to show how a new concept results from the former concepts, and to make the reader at the outset, at the birth of the more general propositions, as a continuation of known particular cases, never leaving, if I may say, objective reality.[15] [Her84], p. 94

This necessity for new concepts to be born of previously-existing and accepted ideas stems from the French belief that mathematics was tightly bound to the natural laws of the universe. To Hermite and his school, mathematics was not a social or human construct but existed independently. Several years earlier, Hermite had summarized exactly this sentiment in the context of the recent interest in discontinuous functions. In a letter to Paul du Bois-Reymond dated February 1, 1881, he wrote:

> . . . the study of the functions of analysis is also the study of the laws of nature; I believe for my part that all the facts of analysis exist outside of us, and are imposed on us just as necessarily as the properties of matter and the phenomena of the real world. Consequently I see in the study of functions a study of objective reality, and in the laws relative to functions [I see] a reflection of physical laws.[16]

[15]"Deux choses sont à distinguer dans votre travail: les propositions, les résultats auxquels vous êtes parvenu, puis l'exposition que vous en avez faite, c'est-à-dire le fond et la forme. Pour le fond, je suis certainement l'écho de tous les analystes en reconnaissant qu'il constitue, avec les théorèmes célèbres de Weierstrass, le fondement même de la théorie des fonctions uniformes, et Picard avec qui je m'en suis entretenu partage entièrement mon opinion. Il n'est pas non plus d'un avis opposé au mien pour ce qui concerne la forme sous laquelle vous les présentez et les exposez. Tous deux nous sommes convenus que, en procédant par cette voie qui consiste à partir des notions abstraites entièrement nouvelles pour arriver par un enchaînement de déductions aux réalités de l'analyse, vous avez obéi à la tendance allemande, à une nature d'esprit qui n'est point la nôtre. Ce qui est un besoin impérieux pour l'esprit francais, c'est de procéder en sense inverse, en mettant tout le soin possible à montrer de quelle manière une nouvelle notion résulte des notions antérieures, et à faire assister le lecteur à l'origine, à la naissance des propositions plus générales, comme suite des cas particuliers connus, sans jamais quitter, si je puis dire, la réalité objective."

[16]". . . l'étude des fonctions de l'analyse est aussi l'étude des lois de nature; je crois pour mon compte que tous les faits analytiques existent en dehors de nous, et s'imposent à nous tout aussi nécessairement que les

[Her16], p. 203

It seems that to the French mathematicians of the time, Mittag-Leffler's use of Cantor's transfinite sets (clearly of the "German tendency") went against their very nature. Mittag-Leffler was acutely aware of this discrepancy, though he continued to believe that Cantor's work was of significant worth. To remedy this situation he attempted (as mentioned earlier) to make Cantor's publications in *Acta Mathematica* more appealing to the general mathematical community by suggesting reductions in the amount of philosophical content and more indications of the utility of Cantor's work in science [Dau79], p. 291. However, Cantor grew tired of such compromises. By 1887 Cantor had completely severed his professional relationship with Mittag-Leffler and refused to publish his work in *Acta*, after Mittag-Leffler suggested that it was unwise to print a premature version of Cantor's general theory of order types [Dau79], p. 3. This caused a serious rift in their friendship which never quite recovered.

Despite the concerns about Mittag-Leffler's use of Cantorian ideas, the Mittag-Leffler theorem itself continued to be taught, notably by Hermite. Picard was later to recall an event that illustrates the impact of the theorem among students in the 1880s. In 1937, when Picard was awarded the Prix Mittag-Leffler at the Institut de France, he made an acceptance speech at the award ceremony which recounted the fame of both Weierstrass and Mittag-Leffler:

> It even happened that in the ceremony called the Shadows, where the Polytechniciens make innocent fun of their professors, someone announced the discovery of a new verse of Genesis, where it was written: "God created Weierstrass, then, not finding it good that Weierstrass should be alone, He created Mittag-Leffler.[17]
>
> [Pic38], pp. xxii–xxiv

propriétés de la matière et les phénomènes du monde réel. Par conséquent, je vois, dans l'étude des fonctions, une étude de la réalité objective, et dans les lois relatives aux fonctions, un reflet des lois physiques."

[17]"Il arriva même que dans une de ces cérémonies, dites les Ombres, oú les Polytechniciens font d'innocentes plaisanteries sur les professeurs, on annonça la d'ecouverte d'un nouveau verset de la Genèse, où il était écrit: "Dieu créa Weierstrass, puis, ne trouvant pas bon que Weierstrass fût seul, il créa Mittag-Leffler."

7.2 Concluding Remarks

The Mittag-Leffler theorem began as an extension of Weierstrass' 1876 factorization theorem. As it was a direct continuation of Weierstrass' work and was undertaken with exactly the tools acquired from Weierstrass' 1875–1876 lectures, it is clear that Mittag-Leffler was an important contributor to Weierstrass' research program concerning the foundations of analysis. In addition, the rather complicated publication history of the Mittag-Leffler theorem, spanning eight years and with many publications in Swedish, German, and French, illustrates the relative complexity of scientific communication at the time and underlines why an international journal was a good idea.

The evolution of the Mittag-Leffler theorem was somewhat complicated. Several contributions were instrumental to its development, notably the alternate proof provided by Weierstrass in 1880 which Mittag-Leffler subsequently adopted. This version was far more concise than Mittag-Leffler's proofs of the 1870s, and Mittag-Leffler's later versions remained tightly-linked to Weierstrass' work in analysis throughout the entire course of its development. The introduction of Cantor's controversial work in the theory of infinite sets of points marks another pivotal moment in the evolution of the Mittag-Leffler theorem. Through his correspondence with Cantor, Mittag-Leffler gained the necessary tools to bring his work to its final and most-generalized version, allowing for the representation of functions having an infinite number of essential singularities. Around this time, Mittag-Leffler was one of very few advocates of Cantor's transfinite set theory. The majority of the European mathematical community favoured concrete mathematics based on formulas, and Mittag-Leffler's promotion of Cantor's unorthodox work through his own research and through his journal *Acta Mathematica* indicates the important role played by Mittag-Leffler in promoting the kind of abstract mathematics that would later be praised as a major achievement. One of the most significant findings of this thesis concerns Mittag-Leffler's relationship to Cantor's set theory: his desire to deal with infinite sets of singular points attracted him to Cantor's work.

Despite the uncertain reception of Cantor's work, the Mittag-Leffler theorem was seen as a classical element of the theory of analysis. It remained popular and widely-studied within both French and German mathematical circles even in its latest version. Hermite played a significant role in distributing Mittag-Leffler's work to the French mathematical community. That Mittag-Leffler's theorem generated research in the following generation of mathematicians (including the well-known figures Picard, Appell, and Poincaré) is due, in large part, to Hermite's promotion of Mittag-Leffler's work in French classrooms. This teaching of the Mittag-Leffler theorem to the undergraduate elite of France is perhaps the most important indication of its status at that time. Furthermore, though the theorem statements and accompanying proofs taught today tend to differ slightly from Mittag-Leffler's work of the late nineteenth-century, this work remains a fundamental part of the repertoire in complex analysis. The theorem appears in current standard textbooks such as [Ahl79] and [Rud74], and even in [Spi99] of the Schaum's Outlines series.

Future work on this subject falls into several categories. I would like to return to Mittag-Leffler's publications on interpolation and approximation between 1876 and 1884 in order to more fully understand their connection to the Mittag-Leffler theorem. Regarding Mittag-Leffler's role as a promoter of Cantor's work, in [Can79] Cantor claims that both Ulisse Dini and Giulio Ascoli found applications for his theory of infinite and linear sets of points. I would like to investigate these applications.

I would also like to determine how the Mittag-Leffler theorem was used in analysis in the late 1880s. In particular, Mittag-Leffler's work generated some research by a handful of Hermite's students, notably Appell, Poincaré, and Picard.

Finally, I would like to examine more extensively Mittag-Leffler's promotion of mathematical work (such as Cantor's set theory) by investigating the following questions: what other mathematical work did Mittag-Leffler actively support, as a mathematician, but more specifically as a journal editor? What sorts of mathematical research did Mittag-Leffler support through his journal? What subjects did he consider to be important and interesting, and how and how much did his position as the editor of *Acta Mathematica* influence the sorts of mathematics studied in Europe around that time? More generally, what was the overall influence of Mittag-Leffler's editorial policy and related activities on the evolution and development of mathematical research in Europe?

C Note for "The Final Version of Mittag-Leffler's Theorem"

Determining m_ν

In Chapter 6 we discussed the fact that Mittag-Leffler thought it necessary to give his readers a precise account of how to explicitly calculate the exponents m_ν such that

$$\sum_{\mu=m_\nu+1}^{\infty} A_\mu^{(\nu)} \left(\frac{a_\nu - b_\nu}{x - b_\nu}\right)^\mu < \epsilon_\nu$$

when

$$\left|\frac{a_\nu - b_\nu}{x - a_\nu}\right| \leq \epsilon.$$

To determine an appropriate value for m_ν, Mittag-Leffler begins by writing the series $G\left(\frac{1}{x-a}\right)$ in the form $(x \neq b)$

$$\begin{aligned} G\left(\frac{1}{x-a}\right) &= \frac{c_{-1}}{(x-b)-(a-b)} + \frac{c_{-2}}{[(x-b)-(a-b)]^2} + \dots \\ &= \frac{1}{a-b}\frac{c_{-1}}{1-\frac{a-b}{x-b}}\frac{a-b}{x-b} + \frac{1}{(a-b)^2}\frac{c_{-2}}{\left[1-\frac{a-b}{x-b}\right]^2}\frac{(a-b)^2}{(x-b)^2} + \dots \end{aligned}$$

Recognizing that $\frac{1}{1-\frac{a-b}{x-b}}$ is the sum of a geometric series, we can rewrite the above as

$$G\left(\frac{1}{x-a}\right) = \frac{c_{-1}}{a-b}\frac{a-b}{x-b}\sum_{n=0}^{\infty}\left(\frac{a-b}{x-b}\right)^n + \frac{c_{-2}}{(a-b)^2}\frac{(a-b)^2}{(x-b)^2}\left(\sum_{n=0}^{\infty}\left(\frac{a-b}{x-b}\right)^n\right)^2 + \dots$$

By expanding and rearranging,

$$G\left(\frac{1}{x-a}\right) = \left[\frac{c_{-1}}{a-b}\right]\left(\frac{a-b}{x-b}\right) + \left[\frac{c_{-1}}{a-b} + \frac{c_{-2}}{(a-b)^2}\right]\left(\frac{a-b}{x-b}\right)^2 + \dots$$

and thus the coefficients A_ν can be found to be

$$\begin{aligned} A_\mu &= \frac{c_{-1}}{a-b} + \frac{\mu-1}{1!}\frac{c_{-2}}{(a-b)^2} + \frac{(\mu-1)(\mu-2)}{2!}\frac{c_{-3}}{(a-b)^3} + \dots \\ &+ \frac{(\mu-1)(\mu-2)\cdots(\mu-[r-1])}{(r-1)!}\frac{c_{-r}}{(a-b)^r} + \dots \end{aligned}$$

Because the function $G_\nu\left(\frac{1}{x-a_\nu}\right)$ is entire and absolutely convergent,

$$\sum_{k=1}^{\infty}\frac{|c_{-k}|}{(x-a)^k} < \infty;$$

letting $x - a = \xi$, we write the above as

$$\frac{|c_{-1}|}{\xi} + \frac{|c_{-2}|}{\xi^2} + \frac{|c_{-3}|}{\xi^3} + \ldots = g_\xi < \infty.$$

Noticing that the above implies $|c_{-r}| \leq g_\xi \xi^r$ for every value of r, we can write, using the triangle inequality,

$$\begin{aligned} A_\mu &\leq \frac{g_\xi \xi}{|a-b|} + \frac{\mu-1}{1!}\frac{g_\xi \xi^2}{|a-b|^2} + \ldots + \frac{(\mu-1)(\mu-2)\cdots(\mu-[r-1])}{(r-1)!}\frac{g_\xi \xi^r}{|a-b|^r} + \ldots \\ &= \frac{g_\xi \xi}{|a-b|}\left(1 + \frac{\mu-1}{1!}\frac{\xi}{|a-b|} + \ldots + \frac{(\mu-1)(\mu-2)\cdots(\mu-[r-1])}{(r-1)!}\frac{\xi^{r-1}}{|a-b|^{r-1}} + \ldots\right) \\ &= \frac{g_\xi \xi}{|a-b|}\left(1 + \frac{\xi}{|a-b|}\right)^{\mu-1}. \end{aligned}$$

We define now two positive values α and β such that $\beta < 1$, $(1+\alpha)\epsilon < 1$, and $\frac{\epsilon}{1-\beta} < 1$. It is then always possible to find a positive quantity ϵ' such that $(1+\alpha)\epsilon < \epsilon' < 1$ and $\frac{\epsilon}{|1-\beta|} < \epsilon' < 1$. We can then choose ξ for the function G_ν such that $\frac{\xi}{|a-b|} < \alpha$. Then when b is finite and such that $\left|\frac{a-b}{x-b}\right| < \epsilon$,

$$\begin{aligned} \sum_{\mu=m_\nu+1}^{\infty} A_\mu \left|\frac{a-b}{x-b}\right|^\mu &\leq \sum_{\mu=m_\nu+1}^{\infty} \frac{g_\xi \xi}{|a-b|}\left(1 + \frac{\xi}{|a-b|}\right)^{\mu-1} \\ &\leq \sum_{\mu=m_\nu+1}^{\infty} g_\xi \alpha (1+\alpha)^{\mu-1}\epsilon^\mu \\ &\leq \sum_{\mu=m_\nu+1}^{\infty} g_\xi \alpha (1+\alpha)^{\mu-1}\left(\frac{\epsilon'}{1+\alpha}\right)^\mu \\ &= g_\xi \frac{\alpha}{1+\alpha}\sum_{\mu=m_\nu+1}^{\infty} \epsilon'^\mu. \end{aligned}$$

By factoring out the common term $\epsilon'^{m_\nu+1}$, this can be written as

$$g_\xi \frac{\alpha \epsilon'^{m_\nu+1}}{1+\alpha}\sum_{\mu=0}^{\infty} \epsilon'^\mu,$$

and thus

$$\sum_{\mu=m_\nu+1}^{\infty} A_\mu \left|\frac{a-b}{x-b}\right|^\mu \leq g_\xi \frac{\alpha \epsilon'^{m_\nu+1}}{1+\alpha}\frac{1}{1-\epsilon'}.$$

One can then choose m_ν such that

$$g_\xi \frac{\alpha \epsilon'^{m_\nu+1}}{1+\alpha}\frac{1}{1-\epsilon'} < \epsilon_\nu$$

and then set, for a finite and nonzero,

$$F_\nu(x) = G_\nu\left(\frac{1}{x-a_\nu}\right) - \sum_{\mu=1}^{m_\nu} A_\mu^{(\nu)}\left(\frac{a_\nu - b_\nu}{x-b_\nu}\right)^\mu,$$

or, for a_ν zero or infinite,

$$F_\nu(x) = G_\nu\left(\frac{1}{x-a_\nu}\right)$$

making the desired function

$$F(x) = \sum_{\nu=1}^{\infty} F_\nu(x).$$

The general form of such a function is of course

$$\overline{F(x)} = F(x) + G(x),$$

where $G(x)$ represents an entire function. To expound these calculations of such a value m_ν is certainly unnecessary by today's standards, as the concern is simply to ensure that such a value exists. In fact, Mittag-Leffler's detailed instructions appear to serve no purpose whatsoever, as there is no constructive use of the actual values of m_ν in the text which follows. This sort of analytic exercise likely served solely to demonstrate to the reader that it could actually be done explicitly, and to show Mittag-Leffler's mastery of the technique.

D Modern Statement and Proof

In modern language, the Mittag-Leffler Theorem may be expressed as:

Theorem. Suppose Ω is an open set in the plane, $A \subset \Omega$, A has no limit point in Ω, and to each $\alpha \in A$ there are associated a positive integer $m(\alpha)$ and a rational function

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} c_{j,\alpha}(z-\alpha)^{-j}.$$

Then there exists a meromorphic function $f \in \Omega$, whose principal part at each $\alpha \in A$ is P_α and which has no other poles in Ω. [Rud74]

The standard proofs of this theorem used today rely heavily upon *Runge's theorem*, which states that if K is a compact set in the plane and $\{a_j\}$ is a set containing one point in each component of $S^2 - K$ (where S^2 represents the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$), then if Ω is open, $K \subset \Omega$, $f \in H(\Omega)$ (where $H(\Omega)$ represents the class of all holomorphic functions in Ω), and $\epsilon > 0$, there exists a rational function $R(z)$, all of whose poles lie in $\{a_j\}$, such that $|f(z) - R(z)| < \epsilon$ for all $z \in K$. [Rud74] Using Runge's theorem, we proceed as follows.

Proof. Let $\{K_n\}$ be a sequence of compact sets in Ω such that:

1. $K_n \subset K_{n+1}$ for $n = 1, 2 = 3, \ldots$ (so the K_ns are nested)
2. Every compact subset of Ω lies in some K_n (so as $n \to \infty$, $K_n \to \Omega$)
3. Every component of $S^2 - K_n$ contains a component of $S^2 - \Omega$, for $n = 1, 2, 3, \ldots$ (in other words, K_n must have no holes except those which are forced upon it by holes in Ω, where there is no requirement for connectivity of Ω).

Now, let $A_1 = A \cap K_1$ and $A_n = A \cap (K_n - K_{n-1})$ for $n = 2, 3, \ldots$. Since $A_n \subset K_n$ and A has no limit point in $K_n \subset \Omega$ (that is, there is no value ω in Ω such that a sequence in A converges to ω) it follows that each A_n is a finite set. Now set

$$Q_n(z) = \sum_{\alpha \in A_n} P_\alpha(z)$$

for $n = 1, 2, 3, \ldots$. Since A_n is finite, it is clear that each Q_n is a rational function, with poles of Q_n lying in $K_n - K_{n-1}$ for $n \geq 2$ (since $\alpha \in A_n \subset (K_n - K_{n-1})$). In particular, Q_n is holomorphic in an open set containing K_{n-1}, and so by *Runge's theorem* there exist rational functions R_n with all poles in $S^2 - \Omega$ such that $|R_n(z) - Q_n(z)| < 2^{-n}$ for $z \in K_{n-1}$.

Claim:

$$f(z) = Q_1(z) + \sum_{n=2}^{\infty} (Q_n(z) - R_n(z))$$

for $z \in \Omega$ has the desired properties (namely that F is meromorphic in Ω with principle part P_α at each α and no other poles in Ω). To show this, we fix N. On K_N we have

$$f = Q_1 + \sum_{n=2}^{N} (Q_n - R_n) + \sum_{N+1}^{\infty} (Q_n - R_n).$$

By Runge's theorem, each term in the second sum of the above equation for F is less than 2^{-n} on K_N and so the last series converges uniformly on K_N to a function which is holomorphic inside K_N. As well, since we have assumed that the poles of each function R_n lie outside Ω, we know that $f - (Q_1 + \cdots + Q_N)$ is holomorphic inside K_N, and so f has the assigned principal parts inside of K_N (and hence inside Ω), since N was arbitrarily selected. ∎

E Table of important publications

Table 7.1: Important publications pertaining to the Mittag-Leffler theorem, 1876 – 1877

Year	Author	Title	Journal
1876	K. Weierstrass	*Zur Theorie der eindeutigen analytischen Funktionen*	*Abhandlungen der Koeniglich Akademie der Wissenschaften*; Also in *Werke*, vol?
1876	G. Mittag-Leffler	*En metod att analytiskt framställa en funktion af rational karakter, hvilken blir oändlig alltid och endast uti vissa föreskrifna oändlighetspunkter, hvilkas konstanter äro på förhand angifna*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*
1877	G. Mittag-Leffler	*Ytterligare om den analytiska framställningen af functioner utaf rationel karakter. Pars I.*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*
1877	G. Mittag-Leffler	*Om den analytiska framställningen af en funktion af rationel karakter med en godtyckligt vald gränspunkt*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*
1877	G. Mittag-Leffler	*Till frågan om den analytiska framställningen af en funktion af rationel karakter genom qvoten af två beständigt konvergerande potens-serier*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*
1877	G. Mittag-Leffler	*Om den analytiska framställningen af funktioner af rationel karakter utaf flere oberoende variabler. Pars 1 und 2.*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*

Table 7.2: Important publications pertaining to the Mittag-Leffler theorem, 1878 – 1884

Year	Author	Title	Journal
1879	G. Mittag-Leffler	*Extrait d'une lettre à M. Hermite*	*Bulletin des Sciences Mathématiques*
1879	G. Cantor	*Ueber unendliche lineare Punkt mannichfaltigkeiten*, Part 1	*Mathematische Annalen*
1880	G. Cantor	*Ueber unendliche lineare Punkt mannichfaltigkeiten*, Part 2	*Mathematische Annalen*
1880	K. Weierstrass	*Über einen Funktionentheoretischen Satz des Herrn G. Mittag-Leffler*	*Monatsbericht der K. Akad. der Wiss. Berlin*
1882	C. Hermite	*Extrait d'une lettre adressée a M. Mittag-Leffler, de Stockholm, par M. Ch Hermite, de Paris, sur une Application du Théorème de M. Mittag-Leffler dans la Théorie des Fonctions*	*Crelle's Journal*
1882	G. Mittag-Leffler	*Sur la théorie des fonctions uniformes d'une variable. Extrait d'une lettre adressée à M. Hermite*, Parts 1 – 8	*Comptes Rendus de l'Académie des Sciences, Paris*
1882	G. Mittag-Leffler	*Fullständig analytisk framställning af hvarje entydig monogen funktion, hvars singulära ställen utgöra en värdemängd af* första *slaget*	*Öfversigt af Kongl. Vetenskaps-Akad. Förhandlingar Stockholm*
1884	G. Mittag-Leffler	*Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*	*Acta Mathematica*

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