# MATHEMATICS 151



4. $x = t\sin t$ and $y = t\cos t$; $\frac{dx}{dt} = \sin t + t\cos t$ and $\frac{dy}{dt} = \cos t - t\sin t$.
At $t = \pi$, $\frac{dx}{dt} = -\pi$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = \frac{1}{\pi}$.
At the point $(0, -\pi)$ where $t = \pi$, the tangent line has equation $y - (-\pi) = \frac{1}{\pi}(x - 0)$, or $y = \frac{1}{\pi}x - \pi$.

6. $x = 5\cos t$, $y = 5\sin t$. The point $(3, 4)$ corresponds to $t = \tan^{-1}\frac{4}{3} + 2n\pi$, where $n$ can be any integer. For this value of $t$, $\cos t = \frac{3}{5}$ and $\sin t = \frac{4}{5}$.

(a) $\frac{dx}{dt} = -5\sin t = = -4$ and $\frac{dy}{dt} = 5\cos t = = 3$ at the point $(3, 4)$, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{3}{4}$. The tangent line slope is $-\frac{3}{4}$ and the tangent line has equation $y - 4 = -\frac{3}{4}(x - 3)$.

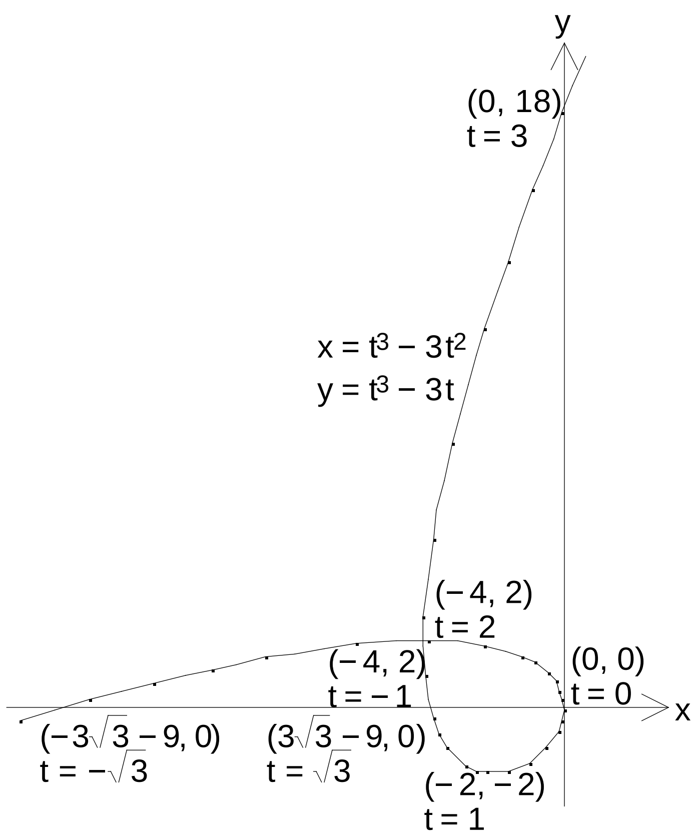
(b) Eliminating the parameter, $x^2 + y^2 = 25$ so $2x + 2y\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{x}{y}$. At the point $(3, 4)$, the tangent line slope is $-\frac{3}{4}$ and the tangent line has equation $y - 4 = -\frac{3}{4}(x - 3)$, as before.

14. $x = 1 + t^2$ and $y = t\ln t$;
$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = \ln t + 1$
so $\frac{dy}{dx} = \frac{\ln t + 1}{2t}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} =$$

$$= \frac{2t\left(\frac{1}{t}\right) - (\ln t + 1)\cdot 2}{8t^3} =$$

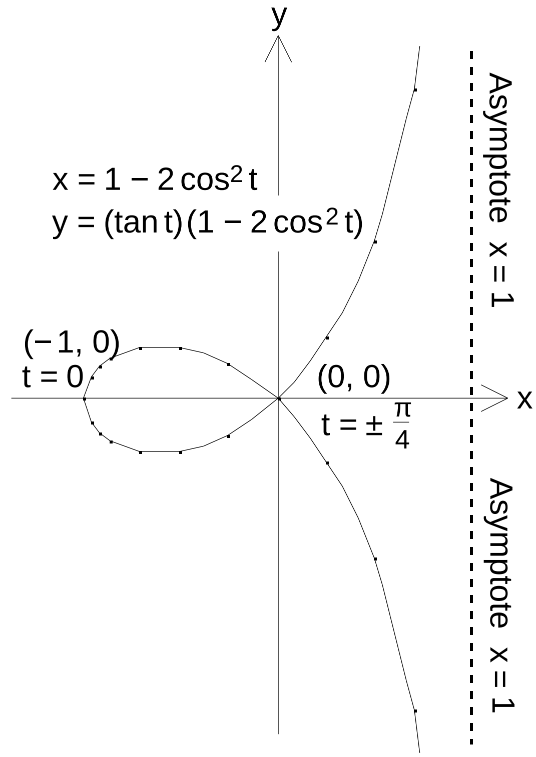
$$= \frac{-2\ln t}{8t^3} = -\frac{\ln t}{4t^3}.$$




For Exercise 16

16. See graph on the previous page.

$x = t^3 - 3t^2$ and $y = t^3 - 3t$.
If the tangent is horizontal,
$\frac{dy}{dt} = 0$ so $3t^2 - 3 = 0$ and $t = \pm 1$.
If $t = -1$, $(x, y) = (-4, 2)$.
If $t = 1$, $(x, y) = (-2, -2)$.
If the tangent is vertical, $\frac{dx}{dt} = 0$
so $3t^2 - 6t = 0$ and $t = 0$ or $t = 2$.
If $t = 0$, $(x, y) = (0, 0)$.
If $t = 2$, $(x, y) = (-4, 2)$.
The graph goes to the right and up for $-\infty < t \le -1$, to the right and down for $-1 \le t \le 0$, to the left and down for $0 \le t \le 1$, to the left and up for $1 \le t \le 2$, and to the right and up for $2 \le t < +\infty$.



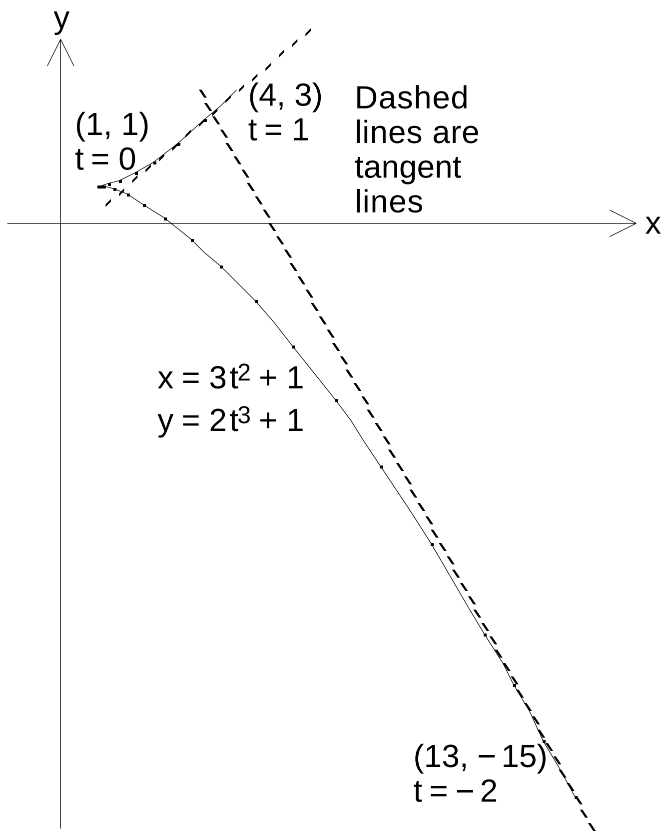

For Exercise 24

24. $x = 1 - 2\cos^2 t$; $y = (\tan t)(1 - 2\cos^2 t)$.
The curve meets itself (if $x \ne 0$) for values of $t_1$ and $t_2$ where $1 - 2\cos^2 t_1 = 1 - 2\cos^2 t_2 \ne 0$ and $\tan t_1 = \tan t_2$.
Then $t_2 = t_1 + n\pi$ for some integer $n$.
Conversely if $t_2 = t_1 + n\pi$ then $(\tan t_1)(1 - 2\cos^2 t_1) = (\tan t_2)(1 - 2\cos^2 t_2)$. The whole curve is periodic with period $\pi$.
We won't get different tangents just by retracing the same curve over and over in the same way, so we look for other ways the curve can meet itself.
If $x = 1 - 2\cos^2 t = 0$, then $x = y = 0$, $\cos^2 t = \frac{1}{2}$, and $\cos t = \pm \frac{1}{\sqrt{2}}$, so $t = \pm \frac{\pi}{4} + n\pi$.

$\frac{dy}{dt} = (\sec^2 t)(1 - 2\cos^2 t) + (\tan t)(4\cos t \sin t) = \sec^2 t - 2 + 4\sin^2 t = 2$ and
$\frac{dx}{dt} = 4\cos t \sin t = 2$ if $t = \frac{4n+1}{4}\pi$, so $\frac{dy}{dx} = 1$. The tangent line is $y = x$ in this case.
$\frac{dy}{dt} = (\sec^2 t)(1 - 2\cos^2 t) + (\tan t)(4\cos t \sin t) = \sec^2 t - 2 + 4\sin^2 t = 2$ but
$\frac{dx}{dt} = 4\cos t \sin t = -2$ if $t = \frac{4n-1}{4}\pi$, so $\frac{dy}{dx} = -1$. The tangent line is then $y = -x$.
See graph above and to the right.

28. If $x = 3t^2 + 1$ and $y = 2t^3 + 1$, $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$ so $\frac{dy}{dx} = t$.
For any particular value of $t$, the tangent to the curve through $(3t^2 + 1, 2t^3 + 1)$ has equation $y - (2t^3 + 1) = t\{x - (3t^2 + 1)\}$, or $y = tx - t^3 - t + 1$.
This passes through $(4, 3)$ when $3 = 4t - t^3 - t + 1$, so $t^3 - 3t + 2 = 0$.
Factoring, $(t + 2)(t - 1)^2 = 0$, and thus either $t = -2$ or $t = 1$.
So the tangent lines in question are $y = -2x + 11$ and $y = x - 1$.
They are tangent to the curve at $(13, -15)$ and at $(4, 3)$ respectively.

See graph on the next page.




For Exercise 28