APMA 990 — QUIZ 1

February 8, 2002

Please make sure you have received 3 sheets with 3 problems. You have 45 minutes for the exam, and you may attempt the problems in any order. All problems have equal weight. You may use the text book and all your notes during this exam. No other help is allowed.

Good Luck!

Name:	
Student number:	

Problem	Maximum	Points received
1	40	
2	40	
3	40	
Total	120	

Problem 1

(a) For 2π-periodic functions we defined the Fourier coefficients as

$$\hat{f}(n) = c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t) e^{-int} dt. \tag{1}$$

If f is square integrable on $[-\pi, +\pi]$, then (in the L^2 sense)

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{int}.$$

What is Parseval's equality, and what does that equality tell us about the relationship of f and $\hat{f}$? (A few lines are sufficient for your answer, no essay required.)

(b) Consider the function $f(t) = t/\pi$ on the interval $[-\pi, \pi]$, and extend f to be 2π-periodic. Then look at the powers of f, i.e. $g_k(t) = f(t)^k$, so $g_0 = 1$, $g_1 = t/\pi$, $g_2 = t^2/\pi^2$, etc.

Using integration by parts, write down a recursion formula for $\widehat{f_k}(n)$ in terms of $\widehat{f_{k-1}}(n)$ and some known terms. Pay attention to whether k is even or odd, and to the special case $n = 0$.

(c) Comment on the symmetry properties of the Fourier coefficients.

(d) Use (b) to write down the Fourier series for g_1, g_2, and g_3.

(e) Using Parseval's equality, calculate $\sum_{n=-\infty}^{+\infty} |\widehat{f_1}(n)|^2$. Use this to calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problem 2 Again we are working on the interval $I = [-\pi, +\pi]$.

(a) If f is a bounded function on I, is f in L^2 ? If yes, can you find a bound for the L^2-norm of f in terms of the supremum norm of f ?

$$||f||_2 = \left(\int_{-\pi}^{+\pi} |f(t)|^2 dt\right)^{1/2}, \qquad ||f||_\infty = \sup_{t\in I} |f(t)|.$$

(b) Can you find a function which is in L^2, but not bounded? Are the norms $||\cdot||_2$ and $||\cdot||_\infty$ equivalent, i.e., are there constants α and β, $0 < \alpha < \beta$, such that

$$\alpha||f||_2 \leq ||f||_\infty \leq \beta||f||_2$$

for all functions $f \in L^2$? Give as much information on α and β as you can.

(c) For a fixed value of s (a complex number) set $f_s : [-\pi, +\pi] \to \mathbf{C}, t \mapsto e^{st}$.

Find the Fourier coefficients of f_s.

(d) Use Parseval's formula to find a formula for the series

$$g(s) = \sum_{n=1}^{\infty} \frac{1}{s^2 + n^2}, \tag{2}$$

for $s \neq 0$.

Problem 3 Again we are working on the interval $I = [-\pi, +\pi]$.

(a) What are the Fourier coefficients of $f(t) = e^{it}$?

(b) Using the fact that multiplication in the time domain corresponds to convolution in Fourier domain find the Fourier coefficients $\hat{h}(k)$ of $h(t) = e^{it}g(t)$ in terms of the Fourier coefficients $\hat{g}(k)$.

(c) Suppose f is a C^2 (i.e., twice continuously differentiable) periodic function on $\mathbf{R}$ with period 2π. Write down a formula for the Fourier coefficients of

$$(Df)(t) = f''(t) + e^{it} f(t). \tag{3}$$

(d) Use (c) to write down the general form of the Fourier series for all the periodic (period 2π) solutions of the differential equation

$$(Df)(t) = f''(t) + e^{it} f(t) = 0. \tag{4}$$

You do not need to check that the Fourier series you obtain is C^2 (it is in fact C^∞).

Hint. Starting from $\hat{f}(0)$, what can you say about $\hat{f}(-1)$, $\hat{f}(-2)$, etc. ? What can you say about $\hat{f}(1)$, $\hat{f}(2)$, etc. ?