APMA 990 Wavelets — Solutions to Problem Set 4

Page 121, Chapter 4.2

1. $X_{even}(z) = 1 + z^{-5}$, $X_{odd}(z) = 2z^{-2}$. We have

$$\begin{aligned} \frac{1}{2}(X(z) + X(-z)) &= 1 + z^{-10} = X_{even}(z^2) \\ \frac{1}{2}z(X(z) - X(-z)) &= 2z^{-4} = X_{odd}(z^2) \end{aligned}$$

3. Denote by S the delay. On the left side, the delay is in the second channel

$$H_p^{left}(z) = \begin{pmatrix} 1 & \\ & S \end{pmatrix} \begin{pmatrix} C_0 & C_1 \\ D_0 & D_1 \end{pmatrix} = \begin{pmatrix} C_0 & C_1 \\ SD_0 & SD_1 \end{pmatrix},$$

whereas on the right side the delay is in the odd phase

$$H_p^{right}(z) = \begin{pmatrix} C_0 & C_1 \\ D_0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & S \end{pmatrix} = \begin{pmatrix} C_0 & SC_1 \\ D_0 & SD_1 \end{pmatrix}.$$

In z-domain, the polyphase representations are

$$\begin{aligned} H_0^{left}(z) &= C_0(z^2) + z^{-1}C_1(z^2) \quad ; \quad & H_1^{left}(z) &= z^{-2}D_0(z^2) + z^{-3}D_1(z^2), \\ H_0^{right}(z) &= C_0(z^2) + z^{-3}C_1(z^2) \quad ; \quad & H_1^{right}(z) &= D_0(z^2) + z^{-3}D_1(z^2). \end{aligned}$$

The transform functions are not equal, hence the polyphase matrix does not commute with the delay matrix.

Page 126, Chapter 4.3

1. We replace z by $-z$ in the polyphase decomposition:

$$\begin{aligned} H(z) &= H_{even}(z^2) + z^{-1}H_{odd}(z^2) \\ H(-z) &= H_{even}(z^2) - z^{-1}H_{odd}(z^2). \end{aligned}$$

In matrix form:

$$\begin{pmatrix} H(z) & H(-z) \end{pmatrix} = \begin{pmatrix} H_{even}(z^2) & H_{odd}(z^2) \end{pmatrix} \begin{pmatrix} 1 & \\ & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For a two-channel analysis filter bank we obtain

$$\begin{pmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(z) \end{pmatrix} = \begin{pmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{pmatrix} \begin{pmatrix} 1 & \\ & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1},$$

we have, in fact, inverted equation (4.48) in the text.

The relation between H_m and H_p is therefore

$$H_m(z) = H_p(z^2) \begin{pmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{pmatrix}.$$

7. $H_0(z) = 1$, $F_0(z) = \frac{1}{2}(1 + z^{-1})^2$, and $H_1(z) = F_0(-z) = \frac{1}{2}(1 - z^{-1})^2$, $F_1(z) = -H_0(-z) = -1$. The PR condition is verified by

$$F_0(z)H_0(z) + F_1(z)H_1(z) = F_0(z) - F_0(-z) = 2z^{-1}.$$

The exponent $\ell = 1$ indicates a one-step delay. The product filter P_0 and centered product filter P are

$$\begin{aligned} P_0(z) = F_0(z)H_0(z) &= \frac{1}{2}(1 + z^{-1})^2 = \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2} \\ P(z) &= zP_0(z) = \frac{1}{2}z^{-1} + 1 + \frac{1}{2}z^1. \end{aligned}$$

We have $P(z) + P(-z) = 2$ — the halfband property leads to perfect reconstruction.

Page 133, Chapter 4.4

1. In time domain the polyphase decomposition is

$$x = (\uparrow 2)x_0 + S(\uparrow 2)x_1,$$

where $x_0(n) = x(2n)$ is x_{even}, and where $x_1(n) = x(2n+1)$ is x_{odd}.

9. The analysis bank has

$$H_0(z) = z^{-2}, \qquad H_1(z) = z^{-1}.$$

This leads to the Type 1 polyphase matrix

$$H_p(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

For the synthesis bank we have $F_0(z) = 1$ and $F_1(z) = z^{-1}$ with the Type 2 polyphase matrix

$$F_p(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $H_p F_p$ does not equal the identity matrix. We have, however,

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2z^{-2}$$

indicating perfect reconstruction with a two-step time delay. In time domain, the output from the lowpass channel is the even components (delayed by 2):

$$w_0(2k) = x(2k-2) \qquad \text{and} \qquad w_0(2k+1) = 0.$$

The high-pass channel keeps only the odd components and delays the signal by 2:

$$w_1(2k) = 0 \qquad \text{and} \qquad w_1(2k+1) = x(2k-1).$$

Therefore, the sum of w_0 and w_1 reconstructs $\hat{x}(n) = x(n-2)$ — a PR system.

Page 142, Chapter 4.5

1. From the polyphase matrix we can read off the filter coefficients of the two filters:

$$\begin{aligned}
H_0(z) &= \frac{1}{2}(\cos\theta - z^{-1}\sin\theta + z^{-2}\sin\theta + z^{-3}\cos\theta) \\
H_1(z) &= \frac{1}{2}(\cos\theta - z^{-1}\sin\theta - z^{-2}\sin\theta - z^{-3}\cos\theta) \\
h_0 &= \frac{1}{2}(+\cos\theta, -\sin\theta, +\sin\theta, +\cos\theta) \\
h_1 &= \frac{1}{2}(+\cos\theta, -\sin\theta, -\sin\theta, -\cos\theta)
\end{aligned}$$

h_0 is orthogonal to its double shift and to h_1. Hence we have an orthogonal filter bank, but not linear phase.

The polyphase matrix of degree 1 has lattice form

$$\begin{aligned} H_p(z) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & z^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a + cz^{-1} & b + dz^{-1} \\ a - cz^{-1} & b - dz^{-1} \end{pmatrix}. \end{aligned}$$

Comparing with the given matrix we find a, b, c, d to be the filter coefficients of h_0:

$$(a, b, c, d) = \frac{1}{2}(+\cos\theta, -\sin\theta, +\sin\theta, +\cos\theta)$$

The inverse of $H_p(z)$ is

$$\begin{aligned} H_p^{-1}(z) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & \\ & z^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= 2\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} 1 & \\ & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta + z\sin\theta & \cos\theta - z\sin\theta \\ -\sin\theta + z\cos\theta & -\sin\theta - z\cos\theta \end{pmatrix}. \end{aligned}$$

12. We find the product

$$H_p(z) = \begin{pmatrix} 1 - 6z^{-1} + 6z^{-2} - 4z^{-3} & -2 + 12z^{-1} - 3z^{-2} + 2z^{-3} \\ 2 - 3z^{-1} + 12z^{-2} - 2z^{-3} & -4 + 6z^{-1} - 6z^{-2} + z^{-3} \end{pmatrix}.$$

The analysis filters are obtained from H_p:

$$\begin{aligned} H_0(z) &= 1 - 2z^{-1} - 6z^{-2} + 12z^{-3} + 6z^{-4} - 3z^{-5} - 4z^{-6} + 2z^{-7} \\ H_1(z) &= 2 - 4z^{-1} - 3z^{-2} + 6z^{-3} + 12z^{-4} - 6z^{-5} - 2z^{-6} + z^{-7}. \end{aligned}$$

Note that $H_0(z) = z^{-7}H_1(z^{-1})$, the order flip.

Page 152, Chapter 5.2

4. We have

$$\begin{aligned} c &= (c(0), c(1), c(2), c(3), c(4), c(5)), \\ d &= (c(5), -c(4), c(3), -c(2), c(1), -c(0)). \end{aligned}$$

Double shift orthogonality:

$$\begin{aligned} \sum c(k)d(k) &= c(0)c(5) - c(1)c(4) + c(2)c(3) \\ &\quad -c(2)c(3) + c(1)c(4) - c(0)c(5) = 0 \\ \sum c(k)d(k-2) &= c(0)c(3) - c(1)c(2) + c(2)c(1) - c(3)c(0) = 0 \\ \sum c(k)d(k-4) &= c(0)c(1) - c(1)c(0) = 0. \end{aligned}$$