APMA 990 Wavelets — Solutions to Problem Set 5

Page 157, Chapter 5.3

3. The modulation matrix for the 4-tap Daubechies filter is

$$H_m(z) = \begin{pmatrix} C(z) & C(-z) \\ D(z) & D(-z) \end{pmatrix}$$

$$= \frac{1}{4\sqrt{2}} \begin{pmatrix} 1+\sqrt{3}+(3+\sqrt{3})z^{-1}+(3-\sqrt{3})z^{-2}+(1-\sqrt{3})z^{-3} & \cdots \\ 1-\sqrt{3}-(3-\sqrt{3})z^{-1}+(3+\sqrt{3})z^{-2}-(1+\sqrt{3})z^{-3} & \cdots \end{pmatrix}$$

$$\cdots \begin{matrix} 1+\sqrt{3}-(3+\sqrt{3})z^{-1}+(3-\sqrt{3})z^{-2}-(1-\sqrt{3})z^{-3} \\ 1-\sqrt{3}+(3-\sqrt{3})z^{-1}+(3+\sqrt{3})z^{-2}+(1+\sqrt{3})z^{-3} \end{matrix} \Bigg).$$

Page 164, Chapter 5.4

1. Suppose $|a| < 1$ and $|b| < 1$. Then the minimum phase spectral factor is

$$C(z) = (1 - az^{-1})(1 - bz^{-1})(1 - z^{-1})^3.$$

4. Roots of the polynomial $C(z)$ computed in Matlab are tabled below. The number in paranthesis indicates multiplicity. The polynomial $P(z)$ has roots at $z = z_k$ and $z = z_k^{-1}$, where z_k denote the tabulated roots.

N	Roots of $C(z)$	mult
3	$z=-1$	(2)
	.26794919243112	
5	$z=-1$	(3)
	$.28725137804402 \pm i.15289233388220$	
7	$z=-1$	(4)
	.32887591778515	
	$.28409629819104 \pm i.24322822591084$	
9	$z=-1$	(5)
	$.27705081179133 \pm i.30653754184488$	
	$.33718958204674 \pm i.09134056597451$	
11	$z=-1$	(6)
	.35254298476311	
	$.26937873604023 \pm i.35469814824096$	
	$.33711562223288 \pm i.15638426950199$	
13	$z=-1$	(7)
	$.26192301501371 \pm i.39318454417261$	
	$.33406281883084 \pm i.20673317774853$	
	$.35726492400363 \pm i.06523214805684$	

Page 172, Chapter 5.5

1.

$$P(\omega) = 1 + \sum_{n \text{ odd}} p(n) e^{-in\omega}$$

satisfies $P(\pi) = 0$.

Except for $p(0)$ we only have nonzero **odd** coefficients. Hence, we obtain

$$P(\pi) = 1 + \sum p(n) e^{-in\pi} = 1 + \sum p(n)(-1)^n = 1 - \sum p(n) \Rightarrow \sum p(n) = 1,$$

and

$$P(0) = 1 + \sum p(n) = 1 + 1 = 2.$$

Page 186, Chapter 6.1

1. The Fourier transform of $f(2t)$ is $\frac{1}{2}\hat{f}\left(\frac{\omega}{2}\right)$:

$$\int_{-\infty}^{+\infty} f(2t) e^{-i\omega t} dt = \frac{1}{2} \int_{-\infty}^{+\infty} f(s) e^{-i\omega s/2} ds = \frac{1}{2}\hat{f}\left(\frac{\omega}{2}\right).$$

Therefore, $\frac{1}{2}\hat{f}\left(\frac{\omega}{2}\right)$ is in $\hat{V}_{j+1}$ (the space containing the transforms of functions in v_{j+1} when $\hat{f}(\omega)$ is in $\hat{V}_j$. So, $\hat{f}(\omega)$ is in $\hat{V}_j$ when $\hat{f}(2\omega)$ is in $\hat{V}_{j-1}$.

5. The problem had a typo – it should say W_0 instead of W_1. As posed, the answer is obviously no. With the "correction" the answer is still no. Since $V_1 = V_0 \oplus W_0$, we can write every g in V_1 as $g = f + h$, $f \in V_0$, $h \in W_0$, with f and h uniquely determined by g. So we cannot choose an arbitrary f in V_0.

Page 193, Chapter 6.2

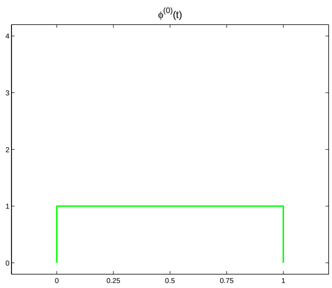

Figure 5.1: $\phi^{(0)}(t)$

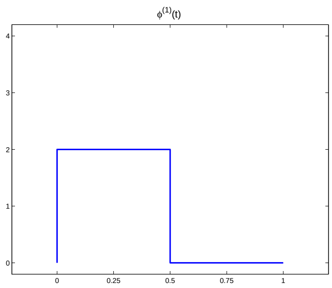

Figure 5.2: $\phi^{(1)}(t)$

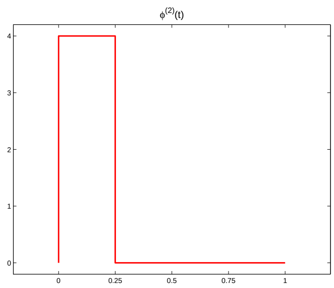

Figure 5.3: $\phi^{(2)}(t)$

3. By the cascade algorithm,

$$\phi^{(i+1)}(t) = \sum_{k=0}^{N} 2h(k)\phi^{(i)}(2t-k).$$

If $\phi^{(0)}(t)$ is the stretched box on $[0, 2N]$, then

$$\begin{array}{rll} \phi^{(0)}(2t) & \text{is supported on} & [0, N] \\ \phi^{(0)}(2t-k) & \text{is supported on} & [\frac{k}{2}, N + \frac{k}{2}]. \end{array}$$

The index k runs from 0 to N, therefore

$$\begin{array}{rll} \phi^{(1)}(t) & \text{is supported on} & [0, \frac{3N}{2}]. \\ \text{Likewise,} & & \\ \phi^{(2)}(t) & \text{is supported on} & [0, \frac{5N}{4}]. \\ \phi^{(3)}(t) & \text{is supported on} & [0, \frac{9N}{8}]. \\ \cdots & & \cdots \\ \phi^{(i)}(t) & \text{is supported on} & [0, \frac{(2^i+1)N}{2^i}]. \end{array}$$

Clearly, as $i \to \infty$, the interval of support of $\phi^{(i)}(t)$ approaches $[0, N]$.

4. Yes, for example the Daubechies 4-tap filter D_4 has a negative coefficient and $\sum h(k) = 1$.

6. See Figures 5.1–5.3. The cascade algorithm converges weakly to the *delta* function. The dilation equation

$$\delta(t) = 2\delta(2t)$$

is verified by integrating $\delta(t)$ times a smooth function f in L_2:

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0) = \int_{-\infty}^{+\infty} f(t)\delta(2t)2dt.$$