MATH 155 — FINAL EXAM

April 18, 1995

Please make sure you have received 12 pages (including this cover page) with 6 problems. You have three hours for the exam, and you may attempt the problems in any order. You may use a handwritten three-page summary of your notes during this exam. No other help is allowed. Write your answers in the space provided. If you need more space, attach additional pages.

Good Luck!

Name:	
Student number:	

Problem	Maximum	Points received
1	15	
2	15	
3	16	
4	15	
5	24	
6	15	
Total	100	

Problem 1 *Differential equations.* A bursary fund starts with an initial endowment of \$50,000. It is expected that the fund will raise \$4,000 monthly. Each month the entire interest income plus 2% of the total fund is paid out in bursaries.

1. Denoting by $A(t)$ the amount of money in the fund, where time t is measured in months, find a differential equation for $A(t)$.

2. Find the solution of this differential equation with the appropriate initial condition.

3. How much money is in the bursary fund in the long run ($t \to \infty$)?

4. If the initial endowment is raised to \$120,000.–, how much money is in the fund in the long run ($t \to \infty$)?

Solution of Problem 1.

Solution of Problem 1 – continued.

Problem 2 *Binomial and normal distribution.* Assume a fair coin is tossed 400 times.

1. Write down the formula for the exact probability of getting at most 230 heads.

2. Approximate the binomial distribution by the normal distribution. Without using continuity correction, what are the approximate probabilities of

(i) obtaining at most 230 heads?

(ii) getting at least 180 heads, but not more than 235 heads?

(iii) obtaining no more than 180 heads **and** no more than 215 tails?

The density function g of a *standard normal random variable* is given by $g(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$. The normal curve area $A(z) = \int_0^z g(s)ds$ is tabulated below:

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.00000	.03983	.07926	.11791	.15542	.19146	.22575	.25804	.28814	.31594
1	.34134	.36433	.38493	.40320	.41924	.43319	.44520	.45543	.46407	.47128
2	.47725	.48214	.48610	.48928	.49180	.49379	.49534	.49653	.49744	.49813
3	.49865	.49903	.49931	.49952	.49966	.49977	.49984	.49989	.49993	.49995

Solution of Problem 2.

Solution of Problem 2 – continued.

Problem 3 *Differential equations.*

(a) What is the general solution to the differential equation

$$y'(t) = 1 + y(t)^2 \qquad ?$$

(b) Consider the linear first order differential equation

$$y'(t) + 2ty(t) = 2te^{-t^2}. \tag{1}$$

(i) Find the general solution of the corresponding homogeneous equation

$$y'(t) + 2ty(t) = 0.$$

(ii) Use variation of constants to find the general solution of equation (1).

(iii) Find the particular solution of equation (1) for which $y(0) = 1$.

Solution of Problem 3.

Solution of Problem 3 – continued.

Problem 4 *Integrals, areas.*

(a) What is the area between the curves $y = x^2$ and $y = x^3$ and the two vertical lines $x = 0$ and $x = 2$?

(b) Use integration by parts to evaluate the integral

$$\int xe^{-2x}\,dx.$$

(c) What is

$$\int_0^{\pi^2} \frac{1}{2} \cos\sqrt{x}\,dx \quad ?$$

Solution of Problem 4.

Solution of Problem 4 – continued.

Problem 5 *Maxima and minima of functions of two variables. Critical points.*

(a) A packing crate holding 480 cubic feet is to be constructed from three different materials. The cost of constructing the bottom of the crate is \$10 per square foot, the cost of the sides is \$8 per square foot, and the cost of the top is \$5 per square foot. What are the dimensions of the crate that will minimize the total cost of building it? (Find the critical point of the cost function, and show that you have indeed a minimum. Note: $480 = 15 \times 32 = 15 \times 2^5$; $16 = 2^4$)

(b) Consider the function

$$f(x,y) = x^3 + 3x^2y + \frac{4}{3}y^3 - y.$$

(i) Find all the critical points of this function f (Hint: There are four in total).

(ii) Classify all the critical points of this function. Which are local minima, local maxima, and which are saddle points.

Solution of Problem 5.

Solution of Problem 5 – continued.

Problem 6 *True or false. Give a good reason.* (**No credit** *if answer is not justified!)*

1. The quadratic equation $k^2 + 2k + 1 = 0$ has $k = -1$ as its only solution. Therefore, the general solution to the differential equation $y'' + 2y' + y = 0$ is the function $y(t) = c_1 e^{-t}$.

2. The quadratic equation $k^2 + 2k + 2 = 0$ has no real solution. Therefore, the differential equation $y'' + 2y' + 2y = 0$ has no solution.

3. The function $f(x, y)$ has a local minimum at (x_0, y_0), if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, and $f_{xx}(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$.

4. If A and B are two events, than the probability p of the event $A \cap B$ is $p(A \cap B) = p(A)p(B)$.

5. $\int_0^{2\pi} \cos^2 x \, dx = \pi$.