MATH 155 — Lecture #5

January 17, 2000

Partial Fractions.

Find

$$\int \frac{x^5 - 2x^4 + 2x^2 - x + 2}{x^3 - x} dx.$$

Step 1: Find “entire” and “fractional” part of the integrand through long division:

$$\begin{array}{cccccccccccccccccc}
(& x^5 & - & 2x^4 & & & + & 2x^2 & - & x & + & 2) & / & (x^3 - x) & = & x^2 - 2x + 1 \\
 & x^5 & & & - & x^3 & & & & & & & & & & \\
\hline
 & & - & 2x^4 & + & x^3 & + & 2x^2 & - & x & + & 2 & & & & \\
 & & - & 2x^4 & & & + & 2x^2 & & & & & & & & \\
\hline
 & & & & + & x^3 & & & - & x & + & 2 & & & & \\
 & & & & & x^3 & & & - & x & + & 2 & & & & \\
\hline
 & & & & & & & & & & & 2 & & & &
\end{array}$$

Therefore,

$$\frac{x^5 - 2x^4 + 2x^2 - x + 2}{x^3 - x} = x^2 - 2x + 1 + \frac{2}{x^3 - x}.$$

To integrate the fractional part we need to find the partial fractions of

$$\begin{array}{rcl}
\dfrac{2}{x^3 - x} = \dfrac{2}{x(x+1)(x-1)} & = & \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1} \\
2 & = & A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \\
\text{Set } x = 0: \qquad 2 & = & -A \\
\text{Set } x = 1: \qquad 2 & = & 2B \\
\text{Set } x = -1: \qquad 2 & = & 2C
\end{array}$$

Therefore,

$$A = -2, \qquad B = C = 1.$$

Solving this system gives $A = 2$, $B = -1$.

$$\int \frac{2}{x^3 - x} dx = \int \left(-\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x-1} \right) dx = -2 \ln|x| + \ln|x+1| + \ln|x-1| + C = \ln \frac{|x^2 - 1|}{x^2} + C,$$

and

$$\begin{aligned}
\int \frac{x^5 - 2x^4 + 2x^2 - x + 2}{x^3 - x} dx &= \int (x^2 - 2x + 1) dx + \int \frac{2}{x^3 - x} dx \\
&= \frac{1}{3}x^3 - x^2 + x + \ln \frac{|x^2 - 1|}{x^2} + C.
\end{aligned}$$