MATH 155 — Quiz #2 Solutions

March 22, 2000

Problem 1 *Binomial and normal distribution.* Assume the probability of passing *Math-155* with a grade of C or higher is 80%.

1. **Write down** the formula for the exact probability, that in a class of 220 students 170 or more students obtain a grade of C or better. Do not evaluate this formula.

2. Approximate the binomial distribution by the normal distribution, and obtain an approximate probability of the event that in a class of 220 students 170 or more students obtain a grade of C or better. (Do not forget the continuity correction!) Give your answer as a formula involving an integral, and evaluate the integral (approximately) using the table below. If the appropriate value is not listed, use the nearest value, or try linear interpolation.

The density function g of a *standard normal random variable* is given by $g(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$. The normal curve area $A(z) = \int_0^z g(s)ds$ is tabulated below:

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.00000	.03983	.07926	.11791	.15542	.19146	.22575	.25804	.28814	.31594
1	.34134	.36433	.38493	.40320	.41924	.43319	.44520	.45543	.46407	.47128
2	.47725	.48214	.48610	.48928	.49180	.49379	.49534	.49653	.49744	.49813
3	.49865	.49903	.49931	.49952	.49966	.49977	.49984	.49989	.49993	.49995

1. Set $p = 0.8$, $q = 1 - p = 0.2$, $N = 220$, and $K = 170$. The probability according to the binomial distribution is given by **[2 points]**

$$P(N_{\text{mark}\geq C}) = \sum_{j=K}^{N} \binom{N}{j} p^j (1-p)^{N-j} = \sum_{j=170}^{220} \binom{220}{j} 0.8^j 0.2^{220-j} = \sum_{i=0}^{50} \binom{220}{i} 0.2^i 0.8^{N-i}.$$

A correct answer written in any form using this formula is acceptable. This probability equals approximately 86.28% (evaluating the sum is not part of the question).

2. The mean μ and variance σ^2 of this binomial distribution $p = 0.8$, $N = 220$, is given by

$$\mu = Np = 176, \qquad \sigma^2 = Np(1-p) = 35.2 \Rightarrow \sigma = \sqrt{35.2} \approx 5.933.$$

(**[1 point]** each for μ and σ.) We are asked for the probablity of having 170 or more "successes", and this can be approximated by the normal distribution **[2 points]**:

$$P_{binomial}(X \geq 170) \approx P_{normal}(Z > z), \qquad Z = \frac{X - \mu}{\sigma}, \quad z = \frac{169.5 - \mu}{\sigma}.$$

Evaluating, we obtain

$$z = -1.0956;$$

In the table, we find a value of $A(1.1) = .36433$. **[1 point.]** This gives a probabilty for our event of $0.5 + 0.36433$, i.e., 86.433% **[1 point.]** (a more accurate evaluation of the integral gives 86.34%). Deduct 1 point if continuity correction is missing.

Problem 2 The region R bounded by $y = x^2$, $x = 0$, $x = 2$, and the x-axis ($y = 0$) is revolved about the x-axis. Find the volume of the solid formed by this revolution. Sketch the solid.

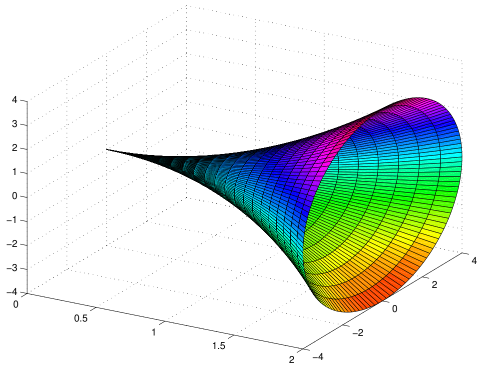

Figure 1: Solid formed by revolution

The volume of a solid formed by revolving the curve $y = f(x)$ about the x-axis is **[1 point]**

$$V = \pi \int_a^b [f(x)]^2 \, dx.$$

Hence, the volume here is simply

$$V = \pi \int_0^2 \left[x^2\right]^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32}{5}\pi.$$

(**[1 point]** each for lower and upper bound of integration, integrand, finding antiderivative and evaluating at the bounds.) Note that an answer like $V = 20.1062$ is only an approximate answer, you may lose a point. Lose 1 point for missing π. For a picture of this solid see Figure 1. **[2 points]** for reasonable sketch.

Problem 3 The lifespan (measured in days) of a certain species of plant in a given environment is a continuous random variable X with density function

$$f(x) = \frac{1}{200} e^{-x/200}.$$

Determine

(a) the probability distribution function $P(X < x)$.

(b) the average (or expected) lifespan of the plants, $E(X)$. You must write down the appropriate formula, not only the final answer!

(c) the probability that a given plant will live longer than 100 days. Write down the appropriate formula as well as the final answer.

x	e^{-x}
0.0	1.00000
-0.1	0.90484
-0.2	0.81873
-0.3	0.74082
-0.4	0.67032
-0.5	0.60653
-0.6	0.54881
-0.7	0.49659
-0.8	0.44933
-0.9	0.40657
-1.0	0.36788

(a) For $x < 0$ we have $P(X < x) = 0$ **[1 point]**. For $x \geq 0$, we obtain **[2 points]**

$$P(X < x) = F(x) = \int_0^x f(t)dt = \int_0^x \frac{1}{200}e^{-t/200}dt = \left[-e^{-t/200}\right]_0^x = 1 - e^{-x/200}.$$

Note, that for $x = 0$, the probability $F(0) = P(X < 0) = 0$; the function $F(x)$ is monotonically increasing, and approaches the value 1 as $x \to \infty$.

(b) **[3 points]**

$$\begin{aligned}
\mu = E(X) &= \int_0^\infty x f(x)dx \\
&= \int_0^\infty \underbrace{x}_{\downarrow} \underbrace{\frac{1}{200}e^{-x/200}}_{\uparrow} dx \\
&= \underbrace{\left[x(-e^{-x/200})\right]_0^\infty}_{=0} - \int_0^\infty (-e^{-x/200})dx \\
&= \left[-200e^{-x/200}\right]_0^\infty = 0 + 200 = 200.
\end{aligned}$$

(c) **[2 points]**

$$P(X > 100) = \int_{100}^\infty f(x)dx = 1 - F(100) = 1 - (1 - e^{-100/200}) = e^{-0.5} =\approx 0.60653,$$

i.e., this probability is roughly equal to 60.65%.

Problem 4

(a) Find the general solution to the **first order linear** differential equation

$$2\frac{dy}{dt} + y = 3.$$

(b) What is the equilibrium solution ($y' = 0$) of this equation?

(c) Find the solution of

$$2\frac{dy}{dt} + y = 3,$$

for which $y(0) = 1$.

(a) **[2 points]** to obtain correct answer, **[2 points]** for getting there, or saying why answer is not correct. There are two obvious ways of solving this differential equation. First, use separation of variables:

$$\begin{aligned}
2\frac{dy}{dt} + y &= 3 \\
\frac{dy}{3-y} &= \frac{1}{2}dt \\
\int \frac{dy}{3-y} &= \frac{1}{2}\int dt \\
-\ln(3-y) &= \frac{t}{2} + C \\
3 - y &= Ke^{-t/2} \\
y &= 3 - Ke^{-t/2}.
\end{aligned}$$

Alternatively, one can treat this equation as a general linear first order equation. The homogeneous equation

$$\frac{dy}{dt} + \frac{1}{2}y = 0$$

has the solution $y(t) = Ce^{-t/2}$. Variation of constants gives

$$\begin{aligned}
2C'(t)e^{-t/2} - C(t)e^{-t/2} + C(t)e^{-t/2} &= 3 \\
C'(t) &= \frac{3}{2}e^{t/2} \\
C(t) &= 3e^{t/2} + K
\end{aligned}$$

Thus, we obtain

$$y(t) = (3e^{t/2} + K)e^{-t/2} = 3 + Ke^{-t/2}$$

as the solution.

(b) **[2 points]** Setting $y' = 0$ leaves simply the equation $y = 3$, which is the sought after equilibrium solution. Alternatively, one can see from (a), that $y = 3$ is the only constant solution of the differential equation.

(c) **[2 points]** Since $y(t) = 3 + Ke^{-t/2}$ is the general solution, the condition $y(0) = 1$ implies $K = -2$, therefore, we get

$$y(t) = 3 - 2e^{-t/2}.$$

Problem 5 Find the general solution of

(a) The general solution of

$$y''(t) + 2y'(t) - 3y(t) = 0$$

is

$$y(t) = c_1 e^{at} + c_2 e^{bt}.$$

What values must be assigned to a and b ?

(b) The second order differential equation

$$y''(t) + 4y'(t) + 13y(t) = 0$$

has

$$y(t) = e^{at}\left(\cos(\omega t) + 2\sin(\omega t)\right)$$

as one of its solutions. What are the values of a and ω?

Solution: For each of (a) and (b), **[2 points]** for correct approach, **[1 point]** each for the correct value of each constant.

(a) Trying to find solutions of the form e^{kt} we obtain as a condition on k that

$$k^2 + 2k - 3 = 0 \qquad \Rightarrow \qquad (k+3)(k-1) = 0 \qquad \Rightarrow \qquad k = -3 \text{ or } k = +1.$$

Therefore, $a = -3$, $b = 1$ or vice versa ($a = 1$, $b = -3$).

(b) Trying to find solutions of the form e^{kt} we obtain as a condition on k that

$$k^2 + 4k + 13 = 0 \qquad \Rightarrow \qquad (k+2)^2 + 9 = 0 \qquad \Rightarrow \qquad (k+2)^2 = -9.$$

Hence, $k = -2 \pm 3i$, so $a = -2$, and $\omega = 3$. Note, that this problem can also be solved simply by plugging

$$y(t) = e^{at}\left(\cos(\omega t) + 2\sin(\omega t)\right)$$

into the differential equation $y''(t) + 4y'(t) + 13y(t) = 0$.