Cyclic Reduction Applied to Three Dimensional Convection Diffusion Equations

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Consider the following three dimensional convection diffusion equation

$$-\Delta u + (\sigma, \tau, \mu)^T \ \nabla u = p \ ,$$

on a domain $\Omega \in I\!\!R^3$, subject to Dirichlet type boundary conditions. Standard finite difference discretization of the PDE - for example seven point approximation to the 3-D Laplacian, and upwind or centered difference approximations to the first order derivatives - leads to a large sparse linear system, which may or may not be symmetric, depending on the presence of convection terms: if the vector (σ, τ, μ) is not equal to zero, the matrix of the system is nonsymmetric. It is not necessarily symmetrizable by real similarity transformations, and it is diagonally dominant only in certain circumstances. In this talk we discuss several aspects of solving the system of equations using *cyclic reduction*. In essence, the process includes ordering the unknowns in red/black fashion, and eliminating the unknowns that correspond to one color, then reordering the unknowns that are left and solving the resulting *reduced system*. Note that this system is smaller than the original one, as it involves only half of the original unknowns. The process of reduction is computationally inexpensive because the original system of equations has *property A*.

Elman & Golub conducted an extensive investigation for analogous two dimensional elliptic problems, and showed that performing one step of cyclic reduction, accompanied with reordering of the unknowns, results in a system that is symmetrizable for a large range of the coefficients of the underlying PDE and is faster to converge in comparison to the original system. Full details on the analysis of the two dimensional system can be found in the papers listed in the references. It turns out that the advantages of cyclic reduction carry over to the three dimensional non-self-adjoint case. In the talk we present convergence analysis for block stationary methods, applied to a reduced system based on certain orderings of the grid. Bounds for the iteration matrices are presented, and we show that asymptotically, for a mesh size tending to zero, the spectral radius of the matrix is smaller than that of the iteration matrix associated with the original system of equations. We also show that the reduced system is symmetrizable for large convection terms, whereas the original system is not. The performance of the scheme is better for the reduced system by a significant margin.

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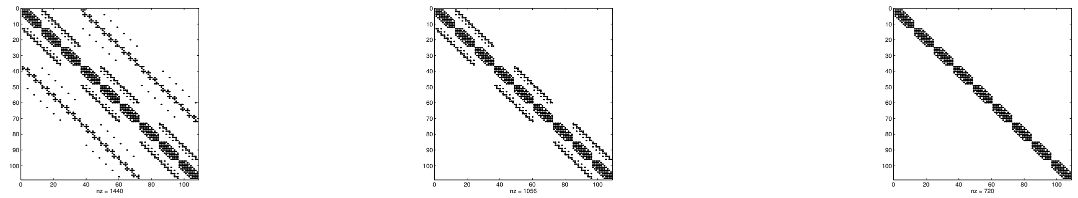
Moreover, there are cases in which the reduced system converges while the original system doesn't.

We focus on an ordering that is unique to three-dimensional problems and is based on gathering unknowns from two horizontal lines and two adjacent planes simultaneously. We call it *two plane ordering*. The sparsity pattern of the reduced matrix resulting from two plane ordering is depicted in Figure 1.1(a). As is evident from the figure, the matrix has an interesting block structure.



(**a**) reduced matrix (**b**) the 2D preconditioner (**c**) the 1D preconditioner

Figure 1.1: Sparsity structure of the reduced matrix and the two suggested preconditioners

Two splittings are compared to each other. Roughly speaking, one of them corresponds to one-dimensional preconditioning [Figure 1.1(c)], and the other corresponds to two-dimensional preconditioning [Figure 1.1(b)]. The latter corresponds to splitting the matrix into $n^2 \times n^2$ blocks and causes it to be block tridiagonal with respect to this splitting, thus having block property A, and so - Young's classical SOR analysis can be applied to it. The system in this case can be solved using inner-outer iterations. However, even though the 2D preconditioner's spectral radius is smaller, the 1D preconditioned system is generally more computationally efficient. Indeed, this splitting leads to an iteration matrix with a very interesting spectral behaviour: numerical evidence indicates that the SOR matrix has a "near-property-A". This phenomenon can be observed by comparing the eigenvalues of the block Jacobi iteration matrix and the eigenvalues of the Gauss Seidel or SOR iteration matrices, and by other means.

Application of cyclic reduction to the variable coefficient case is straightforward, even though the process is more difficult, compared to the constant coefficient case, as the computational molecule depends not only on the PDE's coefficients but also on the coordinates of the associated grid point. However, the sparsity structure of the matrix is identical to the one in the constant coefficient case, and if the operator is separable, the reduced matrix can be symmetrized by a real diagonal similarity transformation. The superiority of the reduced system over the original system is illustrated in numerical examples.

References

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