Recap: SDE

Given functions $f$ and $g$, the stochastic process $X(t)$ is a solution of the SDE

$$dX(t) = f(X(t))dt + g(X(t))dW(t)$$

if $X(t)$ solves the integral equation

$$X(t) - X(0) = \int_0^t f(X(s)) \, ds + \int_0^t g(X(s)) \, dW(s)$$

Discretize the interval $[0, T]$: let $\Delta t = T/N$ and $t_n = n \Delta t$

Compute $X_n \approx X(t_n)$

Initial value $X_0$ is given

Euler–Maruyama

Exact solution:

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(X(s)) \, ds + \int_{t_n}^{t_{n+1}} g(X(s)) \, dW(s)$$

Euler–Maruyama:

$$X_{n+1} = X_n + \Delta t f(X_n) + \Delta W_n g(X_n)$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$

(Left endpoint Riemann sums)

In MATLAB, $\Delta W_n$ becomes `sqrt(Dt)*randn`
\[ f(x) = \mu x \quad \text{and} \quad g(x) = \sigma x, \mu = 2, \sigma = 0.1, X(0) = 1 \]

Solution: \[ X(t) = X(0)e^{(\mu - \frac{1}{2} \sigma^2)t} + \sigma W(t) \]

Disc. Brownian path with \( \delta t = 2^{-8} \), E-M with \( \Delta t = 4\delta t \):

\[ |X_N - X(T)| = 0.69 \]
Reducing to \( \Delta t = 2\delta t \) gives \( |X_N - X(T)| = 0.16 \)
Reducing to \( \Delta t = \delta t \) gives \( |X_N - X(T)| = 0.08 \)

\[ f(x) = \mu x \quad \text{and} \quad g(x) = \sigma x, \mu = 2, \sigma = 0.1, X(0) \]

Solution has \( \mathbb{E}[X(T)] = e^{\mu T} \)

Convergence?

\( X_n \) and \( X(t_n) \) are random variables at each \( t_n \)

In what sense does \( |X_n - X(t_n)| \to 0 \) as \( \Delta t \to 0? \)

There are many, non-equivalent, definitions of convergence for sequences of random variables

The two most common and useful concepts in numerical SDEs are

- **Weak convergence:** error of the mean
- **Strong convergence:** mean of the error

**Weak Convergence**

Weak convergence: capture the average behaviour

Given a function \( \Phi \), the weak error is

\[ e_{\Delta t}^{\text{weak}} := \sup_{0 \leq t_n \leq T} |\mathbb{E}[\Phi(X_n)] - \mathbb{E}[\Phi(X(t_n))]| \]

\( \Phi \) from e.g. set of polynomials of degree at most \( k \)

Converges weakly if \( e_{\Delta t}^{\text{weak}} \to 0 \), as \( \Delta t \to 0 \)

Weak order \( p \) if \( e_{\Delta t}^{\text{weak}} \leq K \Delta t^p \), for all \( 0 < \Delta t \leq \Delta t^* \)

In practice we estimate \( \mathbb{E}[\Phi(X_n)] \) by Monte Carlo simulation over many paths \( \Rightarrow "1/\sqrt{M}" \) sampling error

Least squares fit: power is 1.011 (Confidence intervals smaller than graphics symbols)
Suggests weak order \( p = 1 \)
Weak Euler–Maruyama

\[ X_{n+1} = X_n + \Delta t f(X_n) + \Delta W_n g(X_n) \]

where \( P(\Delta W_n = \sqrt{\Delta t}) = \frac{1}{2} = P(\Delta W_n = -\sqrt{\Delta t}) \)

E.g., use \( \sqrt{D_t} \text{sign}(\text{randn}) \)
or \( \sqrt{D_t} \text{sign}(\text{rand}-0.5) \)

Least squares fit: power is 1.03

Strong Convergence

**Strong convergence**: follow paths accurately

**Strong error** is

\[ e_{\Delta t}^{\text{strong}} := \sup_{0 \leq t_n \leq T} \mathbb{E}[|X_n - X(t_n)|] \]

**Converges strongly** if \( e_{\Delta t}^{\text{strong}} \to 0 \), as \( \Delta t \to 0 \)

**Strong order** \( p \) if \( e_{\Delta t}^{\text{strong}} \leq K \Delta t^p \), for all \( 0 < \Delta t \leq \Delta t^* \)

\[ f(x) = \mu x \text{ and } g(x) = \sigma x, \mu = 2, \sigma = 1, X(0) = X_0 \]

**Solution**: \( X(t) = X(0) e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W(t)} \)

\( M = 5,000 \) disc. Brownian paths over \([0, 1]\) with \( \delta t = 2^{-11} \)

For each path apply EM with \( \Delta t = \delta t, 2\delta t, 4\delta t, 16\delta t, 32\delta t, 64\delta t \)

Record \( \mathbb{E}[|X_N - X(1)|] \) for each \( \delta t \)

Least squares fit: power is 0.51
Strong Convergence

Generally EM has strong order \( p = \frac{1}{2} \) on appropriate SDEs

Can prove using Ito’s Lemma, Ito isometry and Gronwall

Note: strong convergence \( \Rightarrow \) weak convergence, but this doesn’t recover the optimal weak order

Strong Convergence

Euler–Maruyama has

\[
\mathbb{E}[|X_n - X(t_n)|] \leq K \Delta t^{\frac{1}{2}}
\]

Markov inequality says

\[
P(|X| > a) \leq \frac{\mathbb{E}[|X|]}{a}, \quad \text{for any } a > 0
\]

Taking \( a = \Delta t^{\frac{1}{2}} \) gives

\[
P(|X_n - X(t_n)| \geq \Delta t^{\frac{1}{2}}) \leq K \Delta t^{\frac{1}{2}}, \text{ i.e.}
\]

\[
P(|X_n - X(t_n)| < \Delta t^{\frac{1}{2}}) \geq 1 - K \Delta t^{\frac{1}{2}}
\]

Along any path error is small with high prob.

Higher Strong Order

If \( g(x) \) is constant, then EM has strong order \( p = 1 \)

More generally, strong order \( p = 1 \) is achieved by the Milstein method

\[
X_{n+1} = X_n + \Delta tf(X_n) + \Delta W_n g(X_n) + \frac{1}{2}g(X_n)g'(X_n)(\Delta W_n^2 - \Delta t)
\]

(More complicated for SDE systems.)

Even Higher Strong Order: Warning!

Numerical methods for stochastic differential equations
diff_equations
Joshua Wilkie
Physical Review E, 2004

Claims to derive arbitrarily high (strong?) order methods, with a Runge–Kutta approach.

But using only Brownian increments, \( \Delta W_n \), rather than more general integrals like

\[
\int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} dW_1(s) dW_2(t)
\]

there is an order barrier of \( p = 1 \) (Rümelin, 1982).
Beyond Convergence . . .

Numerical methods approximate the continuous by the discrete:
\[ X_n \approx X(t_n), \text{ with } t_{n+1} - t_n =: \Delta t \]

Convergence:
How small is \( X_n - X(t_n) \) at some finite \( t_n \)?

Stability (Dynamics):
Does \( \lim_{n \to \infty} X_n \) look like \( \lim_{t \to \infty} X(t) \)?

Study stability by applying the method to a class of test problems, where information about \( X(t) \) is known.
Hope to show good behavior either for all \( \Delta t > 0 \), or at least for sufficiently small \( \Delta t \).

Stochastic Theta Method

\[ X_{n+1} = X_n + (1 - \theta) \Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + g(X_n) \Delta W_n, \]

where we recall that \( \Delta W_n = W(t_{n+1}) - W(t_n) \),
so \( \Delta W_n = \sqrt{\Delta t} V_n \), with \( V_n \sim \text{Normal}(0,1) \) i.i.d.

\[ X_n \approx X(t_n) \] in the SDE (Itô)

\[ dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0 \]

Stochastic Test Equation

\[ dX(t) = \mu X(t) dt + \sigma X(t) dW(t) \]

(Asset model in math-finance)

Mean-square stability

\[ \lim_{t \to \infty} E(X(t)^2) = 0 \iff 2\mu + \sigma^2 < 0 \]

STM gives \( X_{n+1} = (a + bV_n)X_n \), with
\[ a := \frac{1 + (1 - \theta) \mu \Delta t}{1 - \theta \mu \Delta t}, \quad b := \frac{\sigma \sqrt{\Delta t}}{1 - \theta \mu \Delta t} \]

Saito & Mitsui, SIAM J Num Anal 1996

\[ 0 \leq \theta < \frac{1}{2}: \text{SDE stable } \Rightarrow \text{method stable} \text{ iff } \Delta t < \frac{|2\mu + \sigma^2|}{\mu^2(1 - 2\theta)} \]

\[ \theta = \frac{1}{2}: \text{SDE stable } \Leftrightarrow \text{method stable} \quad \forall \Delta t > 0 \]
\[ \frac{1}{2} < \theta \leq 1: \text{SDE stable } \Rightarrow \text{method stable} \quad \forall \Delta t > 0 \]
Stability Regions

Let $x := \Delta t \mu$ and $y := \Delta t \sigma^2$

SDE stable $\Leftrightarrow y < -2x$
Method stable $\Leftrightarrow y < (2\theta - 1)x^2 - 2x$

Stochastic Test Equation

\[ dX(t) = \mu X(t)dt + \sigma X(t)dW(t) \]

Asymptotic stability

\[ \lim_{t \to \infty} |X(t)| = 0, \text{ with prob. 1 } \Leftrightarrow 2\mu - \sigma^2 < 0 \]

Recall that STM gives $X_{n+1} = (a + bV_n)X_n$, with

\[ a := \frac{1 + (1 - \theta)\mu \Delta t}{1 - \theta \mu \Delta t}, \quad b := \frac{\sigma \sqrt{\Delta t}}{1 - \theta \mu \Delta t} \]

Asymptotic Stability: $\lim_{n \to \infty} |X_n| = 0$, w.p. 1

\[ |X_n| = \left( \prod_{i=0}^{n-1} |a + bV_i| \right) |X_0| \]

SLLN: $\lim_{n \to \infty} |X_n| = 0 \Leftrightarrow \mathbb{E}(\log |a + bV_i|) < 0$

Can be expressed in terms of **Meijer’s G-function**

Difficult to deal with analytically

No simple expression for stability region boundary
Many open questions regarding asymptotic stability

E.g. is there an A-stable method?

Generalizations to nonlinear SDEs are also possible