On the partition function of a finite set *

T. C. Brown
Department of Mathematics
Simon Fraser University
Burnaby, BC
Canada V5A 1S6
email: tbrown@sfu.ca

Wun-Seng Chou
Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529, ROC
email: macws@ccvax.sinica.edu.tw

and

Peter J-S Shiue†
Department of Mathematical Sicences
University of Nevada at Las Vegas
Las Vegas, Nevada 89154-4020, USA
email: shiue@nevada.edu

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Abstract

Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of k relatively prime positive integers. Let $p_A(n)$ denote the partition function of n with parts in A, that is, p_A is the number of partitions of n with parts belonging to A.

We survey some known results on $p_A(n)$ for general k, and then concentrate on the cases $k = 2$ (where the exact value of $p_A(n)$ is known for all n), and the more interesting case $k = 3$. We also describe an approach using the cycle indicator formula.

Let $A = \{a, b, c\}$, where a, b, c are pairwise relatively prime. It has long been known (Ehrhart, J. Reine Angew. Math. 226 (1967), 1–29) that the problem of finding the value of $p_A(n)$ reduces to the problem of finding the value of $p_A(r)$, where $0 \le r < abc$. Sertöz and Özlük (Istanbul Tek. Üniv. Bül. 39 (1986), 41–51) have handled the case $abc - a - b - c < r < abc$. Our main contribution is a recursive method for computing the value of $p_A(r)$ in the case $r \le abc - a - b - c$.

1 Introduction

Let n be a positive integer. A *partition* of n is a representation of n as a sum of positive integers. The order of the terms of this sum does not matter. The *partition function*, denoted by $p(n)$, counts the number of

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partitions of n. For example, $p(4) = 5$, since 4 has exactly 5 partitions: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4.

Now, let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k relatively prime positive integers. A *partition of n with parts in A* is a representation of n as a sum of not necessarily distinct elements of A. Again, the order of the terms of this sum does not matter. The *partition function* in this situation, denoted by $p_A(n)$, counts the number of partitions of n with parts in A, see Stanley [37]. Obviously, $p_A(n)$ is the number of non-negative integer solutions $(x_1, x_2, \ldots, x_k)$ of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n.$$

as mentioned by Comtet [8]. It is well known that for all sufficient large n the equation has a solution. Trivially, if $A = \{1, 2, \ldots, n\}$, then $p_A(n) = p(n)$ (see [25]).

The famous problem of Frobenius is to find the largest natural number g such that $p_A(g) = 0$, that is, the largest natural number g which cannot be expressed in the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k$, where the x_i are non-negative integers.

The Frobenius problem has a long history (See, for example, [16, 31]). Sylvester [38] completely solved the problem for $k = 2$ in 1882, and Glaisher [13] simplified the proof in 1909: When $A = \{a_1, a_2\}$ and a_1, a_2 are relatively prime, then every $n \geq (a_1 - 1)(a_2 - 1)$ can be expressed in the form $n = a_1x + a_2y$, where x, and y are non-negative integers, and $a_1a_2 - a_1 - a_2$ cannot be so expressed. Thus the number g in this case is $g = a_1a_2 - a_1 - a_2$.

When $k = 3$, no closed-form expression for g is known, except in some special cases, although there do exist explicit algorithms for calculating it. See for example [7, 9, 15, 19, 20, 32, 33].

It seems very difficult to calculate g when $k \geq 4$ (however, see [35]). In the general case, various upper bounds are known (for instance, see [6]), and closed-form expressions are known in a few special cases, for example in the case that $a_1, a_2, \cdots, a_k$ is an arithmetic progression (See [31]). In fact, it has been long conjectured that the Frobenius problem is NP-hard, and this is proved by Ramirez-Alfonsin [29].

This paper is devoted to th study of $p_A(n)$ when $k = 2$ and 3. Our main contribution is a recursive method for computing the value $p_A(n)$ when $n \leq a_1a_2a_3 - a_1 - a_2 - a_3$ where a_1, a_2, a_3 ar pairwise relatively prime integers. We also provide a short proof of a known result when $k = 2$ (see Theorem 4.1). Our proof yields a complete explicit formula for $p_A(n)$ in the case $k = 2$ (see Corollary 4.3).

In Sections 2 and 4, we survey some known results on $p_A(n)$ for general k. In Section **??**, we forcus our attention on the cases $k = 2$ and $k = 3$ (see [10, 11] for some results concerning the case $k = 4$). Section 5 describes an approach using the cycle indicator formula.

2 Asymptotic formula for $p_A(n)$ and $p(n)$

If $A = \{a_1, a_2, \ldots, a_k\}$ is a set of k relatively prime positive integers, it is known that

$$p_A(n) \sim \frac{n^{k-1}}{a_1a_2 \cdots a_k(k-1)!}$$

(see [40, pp. 134, Problem 15C]). A proof of this result appears in [26], Problem 27. The proof there is based on the generating function of $p_A(n)$. Elementary proofs are given in [24, 36, 41]. For the case

$A = \{1, 2, \cdots, k\}$, an elementary proof of this formula was given by Erdős [12].

For the unrestricted partition function $p(n)$, Rademacher [28] (see also [2]) gives an asymptotic formula as

$$p(n) \sim \frac{\exp(\pi(2/3)^{1/2} n^{1/2})}{4\sqrt{3}n},$$

a result which was proved earlier by Hardy and Ramanujan [17]. Erdős [12] gave an elementary proof of the relation

$$p(n) \sim \frac{a \cdot \exp(\pi(2/3)^{1/2} n^{1/2})}{n},$$

but was unable to show that $a = \frac{1}{4\sqrt{3}}$. Krätzel [21] proved the bound $p(n) \leq 5^{n/4}$, with equality only when $n = 4$.

3 Recurrence relation for $p_A(n)$ and $p(n)$

Apostol [2] (see also [1]) shows by analytical methods that

$$np_A(n) = \sum_{k=1}^{n} \sigma_A(k) p_A(n-k),$$

where $\sigma_A(n)$ denotes the sum of those divisors of n which belong to A.

This result generalizes a result of Euler, who proves this identity for the case $A = \{1, 2, \ldots, k\}$. This result holds for an arbitrary set A of positive integers, not necessarily finite. Hence when A is the set of all positive integers, this becomes

$$np(n) = \sum_{k=1}^{n} p(n-k)\sigma(k).$$

Bell [4] shows that if $A = \{a_1, a_2, \ldots, a_k\}$ and a is the least common multiple of $\{a_1, a_2, \ldots, a_k\}$, then

$$p_A(an+b) = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1},$$

where $c_0, c_1, c_2, \ldots, c_k$ are constants independent of n and b, $0 \leq b < a$. (See also [27, 41].)

The constants are fully determined if $p_A(an+b)$ is known for k different values of n. This can be done using Lagrange's interpolation formula. For example, if $A = \{a_1, a_2, a_3\}$, then

$$\begin{aligned} 2p_A(an+b) &= (n-2)(n-3)p_A(a+b) - 2(n-1)(n-3)p_A(2a+b) \\ &\quad + (n-1)(n-2)p_A(3a+b). \end{aligned}$$

This formula does not however provide an effective way to calculate $p_A(n)$. Later, Kuriki [22] proves a somewhat different recursion formula for $p_A(n)$.

Although there are a number of algorithms for finding the largest number not representable in the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k$ (see for example [14, 23, 35]), it would be of interest to find a fast algorithm for calculating $p_A(n)$.

4 Cases $|A| = 2$ and $|A| = 3$

In the first part of this section, we consider the case $|A| = 2$. It is quite well known that $p_A(n) = \left[\frac{n}{ab}\right]$ or $\left[\frac{n}{ab}\right] + 1$ (see [25]). However, one unified formulae has been obtained as stated in the following theorem. This theorem is proved independently by Sertöz in 1998 [34], Tripathi in 2000 [39] and Beck and Robins [3]. Their proofs involve generating functions. There is also a simple direct proof, which we give below. We then give a simple algorithm for calculating $p_A(n)$ based on the proof of this theorem.

Theorem 4.1. *Let $A = \{a,b\}$ with $(a,b) = 1$. Define $a'(n)$ and $b'(n)$ by $a'(n)a \equiv -n \mod b$, with $1 \le a'(n) \le b$ and $b'(n)b \equiv -n \mod a$ with $1 \le b'(n) \le a$, respectively. Then for all $n \ge 1$,*

$$p_A(n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1.$$

Proof. It is well known (see for example Brown and Shiue [5]) that for all $n \ge 0$, if $n = qab + r$ with $0 \le r < ab$ then $p_A(n) = q + p_A(r)$, that for all $0 < n < ab$, $p_A(n) = 0$ or 1, that $p_A(n) = 1$ for $ab - a - b < n < ab$, and that $p_A(n) = 0$ if $n = ab - a - b$. Therefore to prove the theorem we may assume that $0 < n < ab - a - b$.

Note that ab divides $aa'(n) + bb'(n) + n$, since each of a and b divides $aa'(n) + bb'(n) + n$. Also, $0 < aa'(n) + bb'(n) + n < 3ab$, so that either $aa'(n) + bb'(n) + n = ab$ or $aa'(n) + bb'(n) + n = 2ab$. Now we only need to show that

(i) $aa'(n) + bb'(n) + n = ab$ implies $p_A(n) = 0$;

(ii) $aa'(n) + bb'(n) + n = 2ab$ implies $p_A(n) = 1$.

If $aa'(n) + bb'(n) + n = ab$ and $as + bt = n$ for some $s,t \ge 0$, then $aa'(n) + bb'(n) + as + bt = ab$, or $a(a'(n) + s) + b(b'(n) + t) = ab$, so $a|(b'(n) + t)$ and $b|(a'(n) + s)$. Since $0 < b'(n) + t \le a$ and $0 < a'(n) + s \le b$, this gives $a = b'(n) + t$ and $b = a'(n) + s$, hence $2ab = ab$, a contradiction. This proves (i). To prove (ii), simply note that if $aa'(n) + bb'(n) + n = 2ab$, then $n = a(b - a'(n)) + b(a - b'(n))$. □

This theorem is easy to generalize to the case $(a,b) = d$ in the following corollary. We omit its trivial proof.

Corollary 4.2. *Let $A = \{a,b\}$ with $(a,b) = d$. If d divides n, define $a'(n)$ and $b'(n)$ by $a'(n)\frac{a}{d} \equiv -\frac{n}{d} \mod \frac{b}{d}$ and $b'(n)\frac{b}{d} \equiv -\frac{n}{d} \mod \frac{a}{d}$, respectively, as those in Theorem 4.1. Then for all $n \ge 1$,*

$$p_A(n) = \begin{cases} 0 & \text{if } d \text{ does not divide } n \\ \frac{n + aa'(n) + bb'}{\operatorname{lcm}\{a,b\}} - 1 & \text{if } d \text{ divides } n. \end{cases}$$

From the statement and the proof of Theorem 4.1, if $(a,b) = 1$, we can compute $p_A(n)$ in the following

Algorithm 4.3. *Let $A = \{a,b\}$ with $(a,b) = 1$. Let $n = qab + r$ with $0 \le r < ab$. If $ab - a - b < r < ab$, then $p_A(n) = q + 1$. If $r = ab - a - b$, then $p_A(n) = q$. For the remaining value of r, we have $p_A(n) = q$ if $aa'(r) + bb'(r) + r = ab$ and $p_A(n) = q + 1$ if $aa'(r) + bb'(r) + r = 2ab$. (Here $a'(r)$ and $b'(r)$ are defined as in the statement of the theorem.)*

We now give examples using this corollary. We do not write down all partitions and only compute the number $p_A(n)$ instead.

Example 4.4. *[34] Let $n = 123456789012345$ and $A = \{a,b\}$, where $a = 1234567$, $b = 12345678$. Write $q = 8$ and $r = 1524255800937$. Then we have $n = q \cdot ab + r$. Moreover, $a'(r) = 462963$ and $b'(r) = 1064806$. Hence, $aa'(r) + bb'(r) + r = 15241566651426 = ab$. By Corollary 4.3, we have $p_A(n) = 8$.*

We now consider the case $|A| = 3$ in the remaining part of this section. The case is a little bit more complicated. First of all, we need the following lemma. In this lemma and afterwards, $u'_v(t)$ will denote the number $1 \le u'_v(t) \le v$ satisfying $uu'_v(t) \equiv -t \mod v$, whenever $u, v \ge 1$ and t are integers satisfying $(u,v) = 1$.

Lemma 4.5. *Let $A = \{a,b,c\}$, where a,b, and c are relatively prime positive integers. Write $d_3 = (a,b)$, $d_1 = (b,c)$, and $d_2 = (c,a)$. Then for any integer $n > 0$, the number $n' = n - (d_1 - a'_{d_1}(n))a - (d_2 - b'_{d_2}(n))b - (d_3 - c'_{d_3}(n))c$ is divisible by $d_1d_2d_3$. Moreover, $p_A(n) = p_{A'}(\frac{n'}{d_1d_2d_3})$, where $A' = \{\frac{a}{d_2d_3}, \frac{b}{d_3d_1}, \frac{c}{d_1d_2}\}$ and where we use the convention that $p_{A'}(0) = 1$ and $p_{A'}(\frac{n'}{d_1d_2d_3}) = 0$ if $n' < 0$.*

Proof. If $ax + by + cz = n$ with $x,y,z \ge 0$, then d_3 divides $n - cz = ax + by$. Since $d_3 - c'_{d_3}(n)$ is the smallest nonnegative integer u such that d_3 divides $n - uc$, $z = d_3z' + (d_3 - c'_{d_3}(n))$ for some nonnegative integer z'. Similarly, $x = d_1x' + (d_1 - a'_{d_1}(n))$ and $y = d_2y' + (d_2 - b'_{d_2}(n))$ for some nonnegative integers x' and y', respectively. So, $ax + by + cz = n$ with $x,y,z \ge 0$ if and only if $a(x - (d_1 - a'_{d_1}(n))) + b(y - (d_2 - b'_{d_2}(n))) + c(z - (d_3 - c'_{d_3}(n))) = n'$ with $x - (d_1 - a'_{d_1}(n)), y - (d_2 - b'_{d_2}(n)), z - (d_3 - c'_{d_3}) \ge 0$. This implies that $d_1d_2d_3$ divides n'. Moreover,

$$\frac{a(x - (d_1 - a'_{d_1}(n)))}{d_1d_2d_3} + \frac{b(y - (d_2 - b'_{d_2}(n)))}{d_1d_2d_3} + \frac{c(z - (d_3 - c'_{d_3}(n)))}{d_1d_2d_3} = \frac{n'}{d_1d_2d_3}.$$

This implies $p_A(n) = p_{A'}(\frac{n'}{d_1d_2d_3})$. □

From this lemma, it is enough to consider, afterwards, the set $A = \{a,b,c\}$ such that positive integers a,b, and c are relatively prime in pairs, i.e., $(a,b) = (b,c) = (c,a) = 1$. The following two theorems are quite well-known.

Theorem 4.6 (Ehrhart [10])**.** *Let $A = \{a,b,c\}$, where positive integers a,b, and c are relatively prime in pairs. Let $n = q \cdot abc + r$ with $0 \le r < abc$. Then*

$$p_A(n) = p_A(r) + \frac{q(n + r + a + b + c)}{2}.$$

In particular,

$$p_A(abc) = \frac{abc + a + b + c}{2} + 1.$$

Theorem 4.7 (Sertöz and Özlük [36])**.** *Let $A = \{a,b,c\}$ where a,b, and c are relatively prime in pairs. Let $n = q \cdot abc + r$ with $0 \le r < abc$. Then, for $1 \le x \le a + b + c - 1$,*

$$p_A(abc - x) = \frac{abc + a + b + c}{2} - x.$$

In particular,

$$p_A(abc-a-b-c+1) = \frac{abc-a-b-c}{2}+1.$$

It seems that it is not easy to find a "simple" closed form for computing $p_A(n)$ whenever $n \le abc-a-b-c$. Here, we are going to give a method to compute such $p_A(n)$. For this purpose, we need the following

Proposition 4.8. *Let $A = \{a,b,c\}$ where positive integers a,b,c are pairwise relatively prime and let n be a non-negative integer. Then*

$$p_A(n) = \begin{cases} p_A(n-a-b-c)+q_A(n) & \text{if } n \ge a+b+c \\ q_A(n) & \text{if } 1 \le n < a+b+c \end{cases}$$

where $q_A(n) = p_{A\setminus\{a\}}(n)+p_{A\setminus\{b\}}(n)+p_{A\setminus\{c\}}(n)-\varepsilon_a(n)-\varepsilon_b(n)-\varepsilon_c(n)$ with

$$\varepsilon_d(m) = \begin{cases} 1 & \text{if } d|m \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $E_{\{a,b,c\}}(n) = \{(x,y,z)|x,y,z \ge 0$ are integers, and $xa+yb+zc = n\}$. Let $(x_1,y_1,z_1) \in E_{\{a,b,c\}}(n)$. If $0 < n < a+b+c$ then $x_1y_1z_1 = 0$. Thus, $p_A(n-a-b-c) = |E_{\{a,b,c\}}(n) \setminus \{E_{\{a,b,0\}}(n) \cup E_{\{a,0,c\}}(n) \cup E_{\{0,b,c\}}(n)\}|$ and the result follows by the inclusion-exclusion formula. ☐

In the following corollary the values $p_A(abc-a-b-c)$ and $p_A(abc-a-b-c-1)$ are obtained as particular cases of Proposition 4.8.

Corollary 4.9. *Let $A = \{a,b,c\}$ where a, b and c are positive pairwise relatively prime integers. Then*

$$p_A(abc-a-b-c) = \frac{abc-a-b-c}{2}+1.$$

and

$$p_A(abc-a-b-c-1) = \frac{abc-a-b-c}{2}-1.$$

Proof. From Proposition 4.8, we have $p_A(abc-a-b-c) = p_A(abc)-p_{A\setminus\{a\}}(abc)-p_{A\setminus\{b\}}(abc)-p_{A\setminus\{c\}}(abc)+\varepsilon_a(abc)+\varepsilon_b(abc)+\varepsilon_c(abc)$. By Theorem 4.6, we have that $p_A(abc) = \frac{abc+a+b+c}{2}+1$ and, by Corollary 4.3, we obtain that $p_{A\setminus\{a\}}(abc) = a+1$, $p_{A\setminus\{b\}}(abc) = b+1$, and $p_{A\setminus\{c\}}(abc) = c+1$. Since $\varepsilon_a(abc) = \varepsilon_b(abc) = \varepsilon_c(abc) = 1$ then $p_A(abc-a-b-c) = \frac{abc-a-b-c}{2}+1$.

Now again, from Proposition 4.8, we have $p_A(abc-a-b-c-1) = p_A(abc-1)-p_{A\setminus\{a\}}(abc-1)-p_{A\setminus\{b\}}(abc-1)-p_{A\setminus\{c\}}(abc-1)+\varepsilon_a(abc-1)+\varepsilon_b(abc-1)+\varepsilon_c(abc-1)$. By Theorem 4.7, we have that $p_A(abc-1)\frac{abc+a+b+c}{2}-1$ and, by Corollary 4.3, we obtain that $p_{A\setminus\{a\}}(abc-1) = p_{A\setminus\{a\}}((a-1)bc+(bc-1)) = a$ (similarly, $p_{A\setminus\{b\}}(abc-1) = b$ and $p_{A\setminus\{c\}}(abc-1) = c$). Since $\varepsilon_a(abc-1) = \varepsilon_b(abc-1) = \varepsilon_c(abc-1) = 0$ then $p_A(abc-a-b-c-1) = \frac{abc-a-b-c}{2}-1$. ☐

Using Proposition 4.8, we will give a method to compute $p_A(n)$ for $n \le abc-a-b-c$ in the following theorem. For this purpose, we need the notation that for positive integers u and v with $(u,v) = 1$, write $v'_u(n)$ instead of $v'(n)$ as in Theorem 4.1.

Theorem 4.10. *Let $A = \{a,b,c\}$ where positive integers a, b and c are pairwise relatively prime. Let n be a positive integer and let t be the largest integer such that $n-t(a+b+c) \geq 0$. Then,*

$$\begin{aligned} p_A(n) = \frac{2n(t+1)s_3 - t(t+1)s_3^2}{2abc} &+ \frac{1}{a}\sum_{i=0}^{t}(b'_a(n-is_3)+c'_a(n-is_3)) \\ &+ \frac{1}{b}\sum_{i=0}^{t}(c'_b(n-is_3)+a'_b(n-is_3)) + \frac{1}{c}\sum_{i=0}^{t}(a'_c(n-is_3)+b'_c(n-is_3)) \\ &- 3(t+1) - \sum_{i=0}^{t}(\varepsilon_a(n-is_3)+\varepsilon_b(n-is_3)+\varepsilon_c(n-is_3)) \end{aligned}$$

where $s_3 = a+b+c$ with $\varepsilon_d(m)$ defined as in Proposition 4.8.

Proof. By applying recursively Proposition 4.8, we have that

$$p_A(n) = \sum_{i=0}^{t-1} q_A(n-is_3) + p_A(n-ts_3) = \sum_{i=0}^{t} q_A(n-is_3)$$

where $q_A(m)$ is defined as in Proposition 4.8. Hence,

$$\begin{aligned} \sum_{i=0}^{t} q_A(n-is_3) = &\sum_{i=0}^{t}(p_{A\setminus\{a\}}(n-is_3)+p_{A\setminus\{b\}}(n-is_3)+p_{A\setminus\{c\}}(n-is_3)) \\ &- \sum_{i=0}^{t}(\varepsilon_a(n-is_3)+\varepsilon_b(n-is_3)+\varepsilon_c(n-is_3)). \end{aligned}$$

The result follows by using Theorem 4.1. □

We give the following example as an illustration of the theorem.

Example 4.11. *Consider $A = \{5,7,11\}$ and $n = 41$. Write $a = 5$, $b = 7$ and $c = 11$ for convenience. Then, $s_3 = a+b+c = 23$. Since $41 = 1\times 23+18$, $t = 1$. It is easy to see that the first term in the theorem equals*

$$\frac{2n(t+1)s_3 - t(t+1)s_3^2}{2abc} = \frac{1357}{385}.$$

For positive integers u and v with $(a,b) = 1$, let u_v^{-1} be the multiplicative inverse of u modulo v. It easy to see that $a_b^{-1} = 3$, $a_c^{-1} = 9$, $b_a^{-1} = 3$, $b_c^{-1} = 8$, $c_a^{-1} = 1$, and $c_b^{-1} = 2$. Write $k = 18$. Then, $a'_b(k+is_3) \equiv -a_b^{-1}k - i(1+a_b^{-1}c) \equiv 2+i \pmod{7}$ for $i = 0,1$. Also, $a'_c(k+is_3) \equiv 3+2i \pmod{11}$, $b'_a(k+is_3) \equiv 1+i \pmod{5}$, $b'_c(k+is_3) \equiv 10+3i \pmod{11}$, $c'_a(k+is_3) \equiv 2+2i \pmod{5}$, and $c'_b(k+is_3) \equiv 6+3i \pmod{7}$ for $i = 0,1$. So, $\frac{1}{a}\sum_{i=0^1}(b'_a(k+is_3)+c'_a(k+is_3)) = \frac{9}{5}$, $\frac{1}{b}\sum_{i=0^1}(a'_b(k+is_3)+c'_b(k+is_3)) = \frac{13}{7}$, $\frac{1}{c}\sum_{i=0^1}(a'_c(k+is_3)+b'_c(k+is_3)) = \frac{20}{11}$. Moreover, neither 18 nor 41 is divided by any one of 5, 7 and 11. Hence, $\varepsilon_a(k+is_3) = \varepsilon_b(k+is_3) = \varepsilon_c(k+is_3) = 0$ for $i = 0,1$. Combining all results above together, we have

$$p_A(A)(41) = \frac{1357}{385} + \frac{9}{5} + \frac{13}{7} + \frac{20}{11} - 3(1+1) - 0 = 3.$$

Indeed, there are exactly 3 partitions of 41 with parts in A, namely

$$\begin{aligned}41 &= 5+5+5+5+7+7+7\\ &= 5+5+5+5+5+5+11\\ &= 5+7+7+11+11.\end{aligned}$$

5 The cycle indicator formula

The cycle indicator C_n of the symmetric permutation group of n letters is an effective tool in enumerative combinatorics, which may be written in the form (cf. [30])

$$C_n(t_1,t_2,\dots,t_n) = \sum \frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{t_1}{1}\right)^{k_1}\left(\frac{t_2}{2}\right)^{k_2}\cdots\left(\frac{t_n}{n}\right)^{k_n},$$

where $t_1,t_2,\dots,t_n$ are real numbers and the summation is over all non-negative integer solutions $k_1,k_2,\dots,k_n$ of the equation $k_1+2k_2+\cdots+nk_n = n$.

Let $\sigma(n) = \sum_{d|n} d$. Then Hsu and Shiue [18] obtain

$$p(n) = \frac{1}{n!}C_n(\sigma(1),\sigma(2),\dots,\sigma(n)),$$

where $p(n)$ is the unrestricted partition function from Section 1 above. From this, they obtain by purely combinatorial methods the previously mentioned recurrence relation

$$np(n) = \sum_{k=1}^{n} \sigma(k)p(n-k).$$

The cycle indicator equality above can be generalized in the following way. Let A be any given set of positive integers. (A can be finite or infinite.) Define $p_A(0) = 1$ and $\sigma_A(n) = \sum_{d|n,d\in A} d$. Then Hsu and Shiue [18] obtain

$$p_A(n) = \frac{1}{n!}C_n(\sigma_A(1),\sigma_A(2),\dots,\sigma_A(n)),$$

and consequently they deduce, again by purely combinatorial methods,

$$np_A(n) = \sum_{k=1}^{n} \sigma_A(k)p_A(n-k).$$

As a particular instance, let us take $H = \{2^0,2^1,2^2,\dots\}$, so that $b(n) = p_H(n)$ is the number of *binary partitions* of n. Let $\beta(n) = \sum_{2^i|n} 2^i$. Then the above equations become $b(n) = \frac{1}{n!}C_n(\beta(1),\beta(2),\dots,\beta(n))$ and $nb(n) = \sum_{k=1}^{n} \beta(k)b(n-k)$.

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