Monochromatic structures in colorings of the positive integers and the finite subsets of the positive integers

Tom Brown*

Citation data: T.C. Brown, *Monochromatic structures in colorings of the positive integers and the finite subsets of the positive integers*, 15th MCCCC (Las Vegas, NV, 2001). J. Combin. Math. Combin. Comput. **46** (2003), 141–153.

Abstract

We discuss van der Waerden's theorem on arithmetic progressions and an extension using Ramsey's theorem, and the canonical versions. We then turn to a result (Theorem 6 below) similar in character to van der Waerden's theorem, applications of Theorem 6, and possible canonical versions of Theorem 6. We mention several open questions involving arithmetic progressions and other types of progressions.

1 van der Waerden's theorem on arithmetic progressions

One of the great results in combinatorics is the following theorem.

Theorem 1. *(van der Waerden's theorem on arithmetic progressions) If N is finitely colored (= finitely partitioned) then some color class (= cell of the partition) contains arbitrarily large arithmetic progressions $P = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$.*

Van der Waerden's original proof is in [30]. The most famous proof (essentially van der Waerden's own proof) is in [20]. See also [31]. The shortest proof is in [18]. The clearest proof is probably in [22]. A topological proof can be found in [15]. For other proofs, see [1, 6, 12, 13, 23–25, 29, 32].

The "canonical" version of van der Waerden's theorem is the following, due to Erdős and Graham [14].

Theorem 2. *Given $f : N \to \omega = \{0, 1, 2, \ldots\}$, there exist arbitrarily large arithmetic progressions P such that $f|_P$ is either constant or 1-1.*

An equivalent form of van der Waerden's theorem is the following.

Theorem 3. *For all $k \geq 1$ there exists a (smallest) $\omega(k)$ such that every 2-coloring of $[1, \omega(k)]$ produces a monochromatic k-term arithmetic progression.*

*Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada V5A 1S6. tbrown@sfu.ca.

Using the definition of $\omega(k)$ from the preceding theorem, the only known values of $\omega(k)$ are $\omega(1) = 1$, $\omega(2) = 3$, $\omega(3) = 9$, $\omega(4) = 35$, $\omega(5) = 178$. For values involving more than two colors, see [2,4,11,19,26].

Berlekamp [7] showed in 1961 that $k2^k < \omega(k+1)$ if k is prime.

Erdős asked in 1961 whether or not $\lim_{k\to\infty} \frac{\omega(k)}{2^k} = \infty$, and offered US \$25 for an answer. This question is still open, and the prize is still available.

Szabo [28] showed in 1990 that $\frac{2^k}{k^\varepsilon} < \omega(k)$, $k > k(\varepsilon)$.

Gowers showed in 1998 [16,17] that $\omega(k) < 2^{2^{2^{2^{2^{k+9}}}}}$.

Graham asked in 1998 whether $\omega(k) < 2^{k^2}$, and offers US \$1000 for an answer.

Using Ramsey's theorem, the following "extended" van der Waerden's theorem can be proved.

Theorem 4. *(Extended van der Waerden's theorem) If $P_f(\mathbb{N})$ (the collection of all finite subsets of $\mathbb{N}$) is finitely colored, then for every $n \geq 1$ there exist an infinite set $Y \subseteq \mathbb{N}$ (Y depends on n) and an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$ such that the set $[Y]^a \cup [Y]^{a+d} \cup [Y]^{a+2d} \cup \cdots \cup [Y]^{a+(n-1)d}$ is monochromatic. (Here $[Y]^k$ denotes the set of all k-element subsets of Y.)*

Proof. Let $g : P_f(\mathbb{N}) \to [1,r]$ be an r-coloring of $P_f(\mathbb{N})$. Let n be given. By van der Waerden's theorem, choose m large enough that every r-coloring of $[1,m]$ produces a monochromatic n-term arithmetic progression. Using Ramsey's theorem, choose infinite sets $X_1, X_2, \ldots, X_m$ in turn so that $Y = X_m \subseteq X_{m-1} \subseteq \cdots \subseteq X_2 \subseteq X_1 \subseteq \mathbb{N}$ and g is constant on each of $[X_k]^k$, $1 \leq k \leq m$. ($[X_k]^k$ denotes the set of all k-element subsets of X_k.) Let us suppose that (for each k) $g(A) = a_k$ for all A in $[X_k]^k$.

Let $h : [1,m] \to [1,r]$ be defined by setting $h(k) = a_k$, $1 \leq k \leq m$. By the choice of m, there exist positive integers a, d such that $h(a) = h(a+d) = h(a+2d) = \cdots = h(a+(n-1)d)$. This means that g is constant on the set $[Y]^a \cup [Y]^{a+d} \cup [Y]^{a+2d} \cup \cdots \cup [Y]^{a+(n-1)d}$, and the proof is complete. □

The following was pointed out by Shi Lingsheng, a student of Hans Juergen Proemel: Suppose X is any finite collection of subsets of $\mathbb{N}$ such that every finite coloring of $\mathbb{N}$ produces arbitrarily large monochromatic elements of X. Then for every finite coloring of $P_f(\mathbb{N})$ and every n, there exists an infinite set Y (Y depends on n) and an element A of X of size n, such that $\bigcup [Y]^x$ is monochromatic, where the union is over all x in A. The proof is exactly as above. Thus for example, using Folkman's theorem, for every finite coloring of $P_f(\mathbb{N})$ and every n, there exists an infinite set Y (Y depends on n) and distinct positive integers $a_1, a_2, \ldots, a_n$ such that $\bigcup [Y]^x$ is monochromatic, where the union is over all x such that x is a sum of distinct $a_1, a_2, \ldots, a_n$.

It would be of interest to find a canonical version of Theorem 4. (One would likely need to use the canonical version of van der Waerden's theorem above, as well as the canonical version of Ramsey's theorem.)

This has not yet been done. However, one can get a canonical theorem (Theorem 5 below) for a structure somewhat smaller (but still very large!) than $[Y]^a \cup [Y]^{a+d} \cup [Y]^{a+2d} \cup \cdots \cup [Y]^{a+(n-1)d}$.

From the set $[Y]^a \cup [Y]^{a+d} \cup [Y]^{a+2d} \cup \cdots \cup [Y]^{a+(n-1)d}$ one can easily construct a forest F with the following properties:

1. Each vertex of F is a finite subset of Y.

2. F has n levels, and each vertex at a level i has $a + id$ elements, $0 \leq i \leq n-1$.

3. Vertex y at level $i+1$ covers vertex x at level i iff $x \subset y$.

4. If y,z cover x then $y \cap z = x$.

5. Each element of Y appears in at most one tree of the forest F.

6. Each vertex not at level $n-1$ has infinitely many immediate successors.

7. F is the union of infinitely many non-empty trees.

Let us call such a structure an "arithmetic ω-forest of height n."

Diana Piguetova, a student of Jarik Nešetřil, has proved the following result.

Theorem 5. *If $g : P_f(\mathbb{N}) \to \omega$ is an arbitrary coloring, then for every $n \geq 1$ there exists an arithmetic ω-forest F of height n on which the coloring g has one of the following patterns:*

1. *$g|_F$ is constant.*

2. *Each tree is monochromatic, and different trees have different colors.*

3. *Each level in the whole forest is monochromatic, all of different colors.*

4. *Each level in each tree is monochromatic, all of different colors.*

5. *$g|_F$ is 1-1.*

2 A theorem involving "almost arithmetic progressions"

Van der Waerden's theorem guarantees large arithmetic progressions, but says nothing about the common difference of these progressions. In fact, Beck [3] showed that there exists a 2-coloring of $\mathbb{N}$ such that for all large d, there do not exist large monochromatic arithmetic progressions with common difference d — in fact if P is a monochromatic arithmetic progression with common difference d, then $|P| < 2\log d$.

In the opposite direction, there exists a 2-coloring of $\mathbb{N}$ for which there does not even exist a monochromatic 4-term arithmetic progression $\{a, a+d, a+2d, a+3d\}$ with $d > a/3$. (The 1/3 here cannot be replaced by 1/32.) See [10].

The next theorem, first proved in [8] (see also [9]), shows that one can control the common difference, but at the expense of not insisting on an arithmetic progression.

Theorem 6. *for every finite coloring of N, there exist a fixed d and arbitrarily large monochromatic sets $A = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ with $\max\{a_{j+1} - a_j | j = 1, 2, \ldots, n-1\} = d$.*

Note that when A is very large compared to d, then the elements of A are approximately equally spaced, so we might call A an "almost arithmetic progression."

Theorem 6 differs from van der Waerden's theorem in a number of ways:

1. It does not directly imply, and is not directly implied by, van der Waerden's theorem.

2. It does not have a "density version." (See [5] for example.)

3. It has an extremely simple proof, by induction on the number of colors.

4. The d in the conclusion is fixed.

5. No canonical version is known.

An application of Theorem 6 is the following result, proved in [8, 9].

Theorem 7. *Let S,T be semigroups and let $\varphi : S \to T$ be a homomorphism. Assume that T is locally finite, and that for every idempotent e in T, $\varphi^{-1}(e)$ is locally finite. Then S is locally finite.*

(For groups, this is an old theorem of O. Schmidt: A locally finite extension of a locally finite group is locally finite.)

Sketch of proof. Some simple considerations reduce the proof to the following case. Let $\varphi : S \to G$, where G is a finite group, and assume that $\varphi^{-1}(e)$ is locally finite. Assume that S is generated by $W = \{w_1, w_2, \ldots, w_t\}$. It is necessary to show that S is finite. It suffices for this (by a simple compactness argument) to show that every *sequence* $s = x_1x_2x_3\ldots$ of elements of W contains a "contractible" factor $x_{j+1}x_{j+2}\cdots x_{j+k}$, that is, a factor $x_{j+1}x_{j+2}\cdots x_{j+k}$ which equals the product of fewer than k elements of W.

Define the finite coloring f of $\mathbb{N}$ by $f(m) = \varphi(x_1x_2\cdots x_m)$ for all $m \in \mathbb{N}$. Then, by Theorem 6, we have a *fixed* d and, for every n, a monochromatic set $A = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ with $\max\{a_{j+1} - a_j | j = 1, 2, \ldots, n-1\} = d$.

Define $g_1 = x_1\cdots a_1$, $g_2 = x_{a_1+1}\cdots x_{a_2}$, $g_3 = x_{a_2+1}\cdots x_{a_3}, \ldots, x_{a_{n-1}}\cdots x_{a_n}$.

Then $f(a_1) = f(a_2) = \cdots = f(a_n)$ means $\varphi(g_1) = \varphi(g_1g_2) = \varphi(g_1g_2g_3) = \cdots = \varphi(g_1g_2g_3\cdots g_n)$, so $e = \varphi(g_2) = \varphi(g_3) = \varphi(g_4) = \cdots = \varphi(g_n)$, or $g_2, g_3, g_4, \ldots, g_n \in \varphi^{-1}(e)$, and $|g_i| \leq d$. Since $\varphi^{-1}(e)$ is locally finite, and there are only finitely many possibilities for $g_2, g_3, \ldots, g_n$ (independent of n), when n is large enough the factor $g_2g_3\cdots g_n$ of s will be contractible. Hence S is finite. □

Theorem 6, together with some algebra, also implies [27] that every torsion semigroup of matrices over an arbitrary field F is locally finite [21].

The historical results here are:

- 1911 Schur – Every torsion group of matrices over $\mathbb{C}$ is locally finite.

- 1965 Kaplansky – Every torsion group of matrices over an arbitrary field F is locally finite.

- 1971 Brzowski, Culik II & Gabrielian – There is an infinite semigroup S on two generators satisfying the identity $x^2 = x^3$ for all $x \in S$.

- 1975 McNaughton & Zalcstin – Every torsion semigroup of matrices over F is locally finite.

The questions on groups of matrices were inspired by "Burnside's Problem:"

- 1902 Burnside: Is every torsion group G locally finite? Yes, in the case $x^2 = 1$, and in the case $x^3 = 1$.

- 1940 Sanov – Yes, if $x^4 = 1$.

- 1952 Green & Rees – [Every group with $x^n = 1$ is locally finite] $\Leftrightarrow$ [Every semigroup with $x^{n+1} = x$ is locally finite.]
- 1957 M. Hall Jr. – Yes, if $x^6 = 1$.
- 1964 Golod & Shafarevich – No, if $x^{n(x)} = 1$. (6 pages. See the book *Noncommutative Rings*, by I. N. Herstein, Mathematical Association of America, Washington, DC, 1994.)
- 1965 Novikov & Adian – No, if $x^n = 1$, for odd $n \geq 4381$. (300+ pages.)
- 1975 Adian – No, if $x^n = 1$, for odd $n \geq 665$.
- 1992 Lysionok – No, if $x^n = 1$, for all $n \geq 213$.

(See the book *Around Burnside*, by A. I. Kostrikin, Springer-Verlag, Berlin, 1990.)

3 On the canonical version of Theorem 6

In this section we describe a 2-coloring f of ω which shows that the constant colorings and the 1-1 colorings are not sufficient for a canonical version of Theorem 6. That is, there does not exist a fixed d and arbitrarily large sets $A = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ with $\max\{a_{j+1} - a_j | j = 1, 2, \ldots, n-1\} = d$, such that $f|_A$ is either constant or 1-1. In fact, even the "almost constant" colorings (c colors are allowed, where c is a constant) and the "almost 1-1" colorings (at most c-to-1, where c is a constant) are not enough. (See Theorem 8 below.) We omit the proofs.

Let S denote the set of all sums of distinct even powers of 2, including 0 as the empty sum. Thus $S = \{0, 1, 4, 5, 16, 17, 20, 21, 64, \ldots\}$.

Let T denote the set of all sums of distinct odd powers of 2, including 0 as the empty sum. Thus $T = \{0, 2, 8, 10, 32, 34, 40, 42, \ldots\}$.

Order the elements of S and T so that $S = \{s_0 < s_1 < \cdots\}$ and $T = \{t_0 < t_1 < \cdots\}$.

Then, for each j, $f^{-1}(j)$ is defined by $f^{-1}(j) = S + t_j = \{s + t_j | s \in S\}$. Then

$$\begin{array}{ccccccccccc} 0 & = & f(0) & = & f(1) & = & f(4) & = & f(5) & = & \cdots \\ 1 & = & f(2) & = & f(3) & = & f(6) & = & f(7) & = & \cdots \\ 2 & = & f(8) & = & f(9) & = & f(12) & = & f(13) & = & \cdots \\ 3 & = & f(10) & = & f(11) & = & f(14) & = & f(15) & = & \cdots \\ & \cdots & & & & & & & & & \end{array}$$

(Note that in [13], 4 colors are used, 4 times each. In general, in $[0, 4^k - 1]$, 2^k colors are used, 2^k times each.)

Since for all $n \geq 0$, $f^{-1}(n)$ is a translate of $f^{-1}(0) = S$, we have a partition of ω into infinitely many translates of the infinite set S. Other such partitions of ω could be obtained by partitioning the powers of 2 differently. For example, we could let S be the set of all sums of distinct prime powers of 2 including 2^0 and 2^1, and let T be the set of all sums of distinct composite powers of 2.

The set S above (S = set of sums of distinct even powers of 2) is the "Moser-de Bruijn sequence." (Neil Sloane's sequence #A000695. See http://www.oeis.org.) Setting $f = f(0)f(1)f(2)f(3)\cdots$, we have $f = [(0011001122332233)^2(4455445566776677)^2]^2\cdots$.

Definition 1. *For $A = \{a_1 < a_2 < a_3 < \cdots < a_n\}$ with $\max\{a_{j+1} - a_j | j = 1,2,\ldots,n-1\} = d$, we say that the gap size of A is d, and write $gs(A) = d$. If $|A| = 1$, we set $gs(A) = 1$.*

Theorem 8. *With f defined as above, and any $A \subset \omega$, $\sqrt{|A|/8gs(A)} < |f(a)| < \sqrt{8|A|gs(a)}$.*

4 Some open questions

Conjecture 1. *if g is an arbitrary coloring of ω, then there exists a fixed d and arbitrarily large sets A with $gs(A) = d$, such that either*

(i) at most $\sqrt{|A|}$ colors appear in A; or

(ii) each color appears at most $\sqrt{|A|}$ times in A.

(For the coloring f above, with $d = 1$ and $A = [0, 4^k - 1]$, exactly $\sqrt{|A|}$ colors appear in A, and each color appears exactly $\sqrt{|A|}$ times.)

Question 1. The total number of 3-term arithmetic progressions in $[1,n]$ is $(1+o(1))(n^2/4)$. For each $n \geq 1$, let $d(n)$ denote the largest possible size of a collection of 3-term arithmetic progressions in $[1,n]$ such that each two of the intersect in at most one point. There should be a constant d such that $d(n) = (d+o(1))(n^2/4)$. Find the value of d. (Hayri Ardal, a student at Boğaziçi University in Istanbul, has shown that d, if it exists, satisfies $.47 < d < 43/84$.)

Question 2. Color the interval $[1,3n]$ with 3 colors, each color appearing exactly n times. Is it true that there must exist a 3-term arithmetic progression in $[1,3n]$ which has received all three colors? (This problem is due to Radoş Radoičić, a student at MIT. He has shown that if ω is 3-colored, and each color class has density at least 1/6, then there must exist a 3-term arithmetic progression in ω which has received all three colors.)

Question 3. Let $g(K)$ denote the smallest positive integer n such that if the interval $[1,n]$ is partitioned into two parts, then at least one of these parts must contain a k-term increasing set each of whose consecutive differences belongs to the set $\{d, 2d\}$, for some positive integer d.

Let $h(k)$ denote the smallest positive integer n such that if A is any n-element set of positive integers, each of whose consecutive differences belongs to the set $\{1,2\}$, then A must contain a k-term arithmetic progression.

Can one find reasonable upper bounds on $g(k)$ and $h(k)$? If so, this might give a reasonable upper bound on the van der Waerden function $w(k)$, since $w(k) \leq g(h(k))$.

Question 4. Is it not hard to show that if $\{1,2,\ldots,n^2\}$ is 2-colored, then there exists a monochromatic set $\{a_1,a_2,\ldots,a_n\}$ such that the set of consecutive differences $\{a_{j+1}-a_j|1\le j\le n-1\}$ has at most $\sqrt{n}$ elements? Can one find a "small" $g(n)$ such that if $\{1,2,\ldots,g(n)\}$ is 2-colored, then there is a monochromatic set $\{a_1,a_2,\ldots,a_n\}$ such that the set of consecutive differences $\{a_{j+1}-a_j|1\le j\le n-1\}$ has at most $\log n$ elements? For each k can one find (for arbitrarily large n) $a_1<a_2<\cdots<a_n$ with $a_{j+1}-a_j\le\log n, 1\le j\le n-1$, such that $\{a_1,a_2,\ldots,a_n\}$ contains no k-term arithmetic progression?

References

[1] Peter G. Anderson, *A generalization of Baudet's conjecture (van der waerden's theorem)*, Amer. Math. Monthly **83** (1976), 359–361.

[2] J.A. Bate and D.B. Skillicorn, *Sequences, arithmetic-progression free sets and van der waerden numbers*, Proceedings of the Tenth Manitoba Conference on Numerical Mathematics and Computing, Vol. I (Winnipeg, Man., 1980), 1980, Congr. Numer. 30, pp. 203–211.

[3] J. Beck, *A remark concerning arithmetic progressions*, J. Combin. Theory Ser. A **29** (1980), 376–377.

[4] Michael D. Beeler and Patrick E. O'Niel, *Some new van der Waerden numbers*, Discrete Math. **28** (1971), 135–146.

[5] V. Bergelson, N. Hindman, and R. McCutcheon, *Notions of size and combinatorial properties of quotient sets in semigroups*, Topology Proc. **23** (1998), 23–60.

[6] Vitaly Bergelson, Hillel Furstenberg, Neil Hindman, and Yitzhak Katznelson, *An algebraic proof of van der Waerden's theorem*, Enseign. Math. (2) **35** (1989), 209–215.

[7] E.R. Berlekamp, *A construction for partitions which avoid long arithmetic progressions*, Canad. Math. Bull. **11** (1968), 409–414.

[8] T.C. Brown, *On locally finite semigroups (in russian)*, Ukraine Math. J. **20** (1968), 732–738.

[9] ———, *An interesting combinatorial method in the theory of locally finite semigroups*, Pacific J. Math. **36** (1971), 285–289.

[10] T.C. Brown and Bruce M. Landman, *Monochromatic arithmetic progressions with large differences*, Bull. Austral. Math. Soc. **60** (1999), no. 1, 21–35.

[11] Vašek Chvátal, *Some unknown van der waerden numbers*, Combinatorial Structures and Their Applications (New York) (R. et al Guy, ed.), Gordon and Breach, 1970, Proc. Calgary Internat. Conf., Calgary, Alta., 1969, pp. 31–33.

[12] Kevin Compton, *A van der waerden variant*, Electron. J. Combin. **6** (1999), no. 1, 10, Research Paper 22 (electronic).

[13] Walter Deuber, *On van der Waerden's theorem on arithmetic progressions*, J. Combin. Theory Ser. A **32** (1982), 115–118.

[14] P. Erdős and R.L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[15] H. Furstenberg and Y. Katznelson, *Topological dynamics and combinatorial number theory*, J. Analyse Math. **34** (1978), 275–291.

[16] W.T. Gowers, *A new proof of szemerédi's theorem for arithmetic progressions of length four*, Geom. Funct. Anal. **8** (1998), 529–551.

[17] ———, *A new proof of szemerédi's theorem*, Geom. Funct. Anal. **11** (2001), 465–588.

[18] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, 2nd ed., Wiley-Interscience Series in Discrete Mathematics and Optimization. A Wiley-Interscience Publication., John Wiley & Sons, Inc., New York, 1990.

[19] Y.R. Huang and J. S. Yang, *New upper bounds for van der waerden numbers $w(3,n)$*, Chinese Annals of Math. Series A **21** (2000), 631–634.

[20] A.Y. Khinchin, *Three pearls of number theory*, Dover Publications, Inc., Mineola, NY, 1998, Translated fro the Russian by F. Bagemihl, H. Komm, and W. Seidel. Reprint of the 1952 translation.

[21] Robert McNaughton and Yechezkel Zalcstein, *The burnside problem for semigroups*, J. Algebra **34** (1975), 292–299.

[22] George Mills, *A quintessential proof of van der Waerden's theorem on arithmetic progressions*, Discrete Math. **47** (1983), 117–120.

[23] Hans-Jürgen Prömel and Vojtek Rödl, *An elementary proof of the canonizing version of Gallai-Witt's theorem*, J. Combin. Theory Ser. A **42** (1986), 114–149.

[24] Hans-Jürgen Prömel and B.L. Rothschild, *A canonical restricted version of van der Waerden's theorem*, Combinatorica **7** (1987), no. 1, 115–119.

[25] Saharon Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc. **1** (1988), 635–636.

[26] R.S. Stevens and R. Shantaram, *Computer-generater van der Waerden numbers*, J. Combin. Theory Ser. A **33** (1982), 30–35.

[27] H. Straubing, *The burnside problem for semigroups of matrices*, Combinatorics on Words, Progress and Perspectives, Academic Press, 1982, pp. 279–295.

[28] Zoltán Szabó, *An application of lovász' local lemma — a new lower bound for the van der Waerden number*, Random Structures Algorithms **1** (1990), 343–360.

[29] Alan D. Taylor, *A note on van der Waerden's theorem*, J. Combin. Theory Ser. A **33** (1982), 215–219.

[30] B.L. van der Waerden, *Beweis einer baudetschen vermutung*, Nieuw Arch. Wisk. **15** (1927), 212–216.

[31] ______, *How the proof of baudet's conjecture was found*, Studies in Pure Mathematics (Presented to Richard Rado), Studies in Pure Mathematics, Academic Press, London, 1971, pp. 251–260.

[32] Mark Walters, *Combinatorial proofs of the polynomial van der waerden theorem and the polynomial hales-jewitt theorem*, J. London Math. Soc. **61** (2000), no. 2, 1–12.