On the canonical version of a theorem in Ramsey Theory

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Abstract

We show that the constant colorings and the one-to-one colorings are insufficient for a canonical version of a certain theorem in Ramsey theory.

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1 Results

Definition 1. *If* $A = \{a_1 < a_2 < \ldots < a_n\} \subset \omega = \{0,1,2,\ldots\}$, $n \geq 2$, *the* gap size *of* A *is* $gs(A) = \max\{a_{j+1} - a_j : 1 \leq j \leq n-1\}$. *If* $|A| = 1$, $gs(A) = 1$.

Theorem 2. *If* ω *is finitely coloured, there exist a fixed* $d \geq 1$ *(*d *depends only on the colouring) and arbitrarily large monochromatic sets* A *with* $gs(A) = d$.

This fact first appeared in [1]. Various applications appear in [2–6].

Definition 3. *The colouring* $f : \omega \to \omega$ *is defined by writing each number* n *in binary and then projecting onto the odd powers of two. Explicitly, if* $n = \sum_{i\ odd} \varepsilon_i 2^i + \sum_{j\ even} \varepsilon_j 2^j$ *, then* $f(n) = \sum_{i\ odd} \varepsilon_i 2^i$. *(By default,* $f(0) = 0$.*)*

Theorem 4. *For every* $A \subset \omega$ *(with* f *as above),*

$$\sqrt{|A|/8gs(A)} < |f(A)| < \sqrt{8|A|gs(A)}.$$

According to Theorem 4, there clearly do not exist a fixed d and arbitrarily large sets A with $gs(A) = d$ such that either $|f(A)| = 1$ or $|f(A)| = |A|$.

Definition 5. *For* $k \geq 0$, *an* aligned block *of size* 2^k *(respectively* 4^k*) is a set of* 2^k *(respectively* 4^k*) consecutive non-negative integers whose smallest element is* $m2^k$ *(respectively* $m4^k$*), for some* $m \geq 0$.

Proof of Theorem 4 The following facts follow directly from the definition of the colouring f.

Each aligned block of size 4^k has exactly 2^k colours, each appearing 2^k times.

Consecutive aligned blocks of size 4^k are either identical or have no colour in common.

Every set of 4^s consecutive non-negative integers has at least 2^s colours. Hence every set of 4^s consecutive aligned blocks of size 2^k has at least 2^s aligned blocks of size 2^k, no two of which have a common colour.

Now let $A \subset \omega$.

Choose s minimal so that A is contained in the union of two aligned blocks of size 4^s. (Two blocks are needed in case A contains $4^k - 1$ and 4^k.) Then $4^{s-1} < |A|gs(A)$, hence $|f(A)| \leq 2 \cdot 2^s < 2\sqrt{2|A|gs(A)}$.

Next, given $A \subset \omega$, choose k so that $2^{k-1} < gs(A) \leq 2^k$, and choose t minimal so that A is contained in the union of t consecutive aligned blocks of size 2^k. By the choice of k, A intersects each of these t blocks. Let $4^s \leq t < 4^{s+1}$. Since 2^s of these blocks have no colours in common, $2^s \leq |f(A)|$. Hence $|A| \leq t2^k < 4 \cdot 4^s \cdot 2 \cdot 2^{k-1} < 8|f(A)|^2 gs(A)$, or $\sqrt{|A|/8gs(A)} < |f(A)|$.

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