On a Certain Kind of Generalized Number-Theoretical Möbius Function

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Abstract

The classical Möbius function appears in many places in number theory and in combinatorial theory. Several different generalizations of this function have been studied. We wish to bring to the attention of a wider audience a particular generalization which has some attractive applications. We give some new examples and applications, and mention some known results.

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This paper is dedicated to the memory of Professor Gian-Carlo Rota

1 Introduction

We define a generalized Möbius function μ_α for each complex number α. (When $\alpha = 1$, μ_1 is the classical Möbius function.) We show that the set of functions μ_α forms an Abelian group with respect to the Dirichlet product, and then give a number of examples and applications, including a generalized Möbius inversion formula and a generalized Euler function. Special cases of the generalized Möbius functions studied here have been used in [6–8]. For other generalizations see [1, 5]. For interesting survey articles, see [2, 3, 11].

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Let us recall that the classical Möbius function $\mu(n)$ is defined for positive integers n in the following way: $\mu(1) = 1$. If n is not square free then $\mu(n) = 0$. If n is square free and r is the number of distinct primes dividing n, then $\mu(n) = (-1)^r$ [9].

For any integer r, a Möbius function of order r may be defined by using binomial coefficients, namely for each positive integer n,

$$\mu_r(n) = \prod_{p|n} \binom{r}{\partial_p(n)} (-1)^{\partial_p(n)}$$

where p runs through all the prime divisors of n, and $\partial_p(n) = \mathrm{ord}_p n$ denotes the highest power k of p such that p^k divides n. Obviously $\mu_1(n) = \mu(n)$. For more details, see [7].

We now define a generalized Möbius function μ_α for each complex number α, by setting

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\partial_p(n)} (-1)^{\partial_p(n)}$$

At the end of the paper, we mention a particularly interesting application of the case where α is real.

2 Group-theoretic properties

Recall that the classical Möbius function is multiplicative; i.e., if m and n are relatively prime, then $\mu(mn) = \mu(m)\mu(n)$. It is easily seen that the definition of μ_α implies that this property extends to the Möbius function of order α, giving us the following lemma.

Lemma 1. *For each complex number α, μ_α is a multiplicative function.*

Next, we recall the definition of the Dirichlet product (or convolution) of two arithmetic functions f and g (cf. [1,4]).

Definition 1. *Given two arithmetic functions f and g, the Dirichlet (convolution) product $f * g$ is again an arithmetic function which is defined by*

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d),$$

where the summations are taken over all positive divisors d of n.

Evidently, the product is commutative: $f * g = g * f$. Using a little algebra one easily shows that the following associative law also holds: $(f * g) * h = f * (g * h)$. That is, for all positive integers n, $((f * g) * h)(n) = (f * (g * h))(n)$. Moreover, the convolution $f * g$ is a multiplicative function whenever f and g are multiplicative functions.

Definition 2. *Let*

$$M = \{\mu_\alpha : \alpha \in \mathbb{C}\}$$

where $\mathbb{C}$ denotes the set of complex numbers. The set M may be called the set of generalized Möbius functions of complex order.

Lemma 2. *For any given numbers α and β in $\mathbb{C}$, we have*

$$\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$$

Proof. It is required to show that for all positive integers n,

$$(\mu_\alpha * \mu_\beta)(n) = \sum_{d|n} \mu_\alpha(d)\mu_\beta\left(\frac{n}{d}\right) = \mu_{\alpha+\beta}(n).$$

Since μ_α and μ_β are multiplicative (by Lemma 1), the Dirichlet product $\mu_\alpha * \mu_\beta$ is also multiplicative. Thus, it suffices to consider the case $n = p^k$, where p is prime and k is a positive integer. We easily find

$$\begin{aligned}(\mu_\alpha * \mu_\beta)(p^k) &= \sum_{d|p^k} \mu_\alpha(d)\mu_\beta\left(\frac{p^k}{d}\right) = \sum_{i=0}^{k} \mu_\alpha(p^i)\mu_\beta(p^{k-i}) \\ &= \sum_{i=0}^{k} \binom{\alpha}{i}(-1)^i \binom{\beta}{k-i}(-1)^{k-i} \\ &= (-1)^k \binom{\alpha+\beta}{k} = \mu_{\alpha+\beta}(p^k),\end{aligned}$$

since the relation $(1+x)^\alpha(1+x)^\beta = (1+x)^{\alpha+\beta}$ implies

$$\binom{\alpha+\beta}{k} = \sum_{i=0}^{k} \binom{\alpha}{i}\binom{\beta}{k-i}.$$

□

Notice that μ_0 is the Möbius function of order zero that gives the values

$$\mu_0(n) = \prod_{p|n} \binom{0}{\partial_p(n)}(-1)^{\partial_p(n)} = \begin{cases} 1 & n = 1, \\ 0 & n > 1. \end{cases}$$

Let us denote μ_0 by δ. Since from Lemma 2 we have $\mu_\alpha * \delta = \delta * \mu_\alpha = \mu_\alpha$ for all α, we call it the identity element with respect to the Dirichlet product operation $*$.

We are now ready to show that M is an Abelian group.

Theorem 1. *$(M, *)$ is an Abelian group with identity element $\delta = \mu_0$.*

Proof. By Lemma 2 we see that M is closed with respect to the operation $*$. Moreover, we also have

$$\mu_\alpha * \mu_\beta = \mu_\beta * \mu_\alpha \qquad (\alpha, \beta \in \mathbb{C}),$$

$$(\mu_\alpha * \mu_\beta) * \mu_\gamma = \mu_\alpha * (\mu_\beta * \mu_\gamma) \qquad (\alpha, \beta, \gamma \in \mathbb{C}),$$

$$\mu_\alpha * \delta = \delta * \mu_\alpha = \mu_\alpha \qquad \mu_\alpha * \mu_{-\alpha} = \mu_{-\alpha} * \mu_\alpha = \delta \qquad (\alpha \in \mathbb{C}),$$

Thus, the theorem is proved. □

Of course, if G is any additive subgroup of $\mathbb{C}$, then $M_G = \{\mu_\alpha : \alpha \in G\}$ is a subgroup of M.

3 Corollaries, examples and applications

Corollary 1. *(Generalized Möbius inversion formulae).) For all $\alpha \in \mathbb{C}$ and arithmetic functions f, g,*

$$\left[\forall n \in \mathbb{N} \;\; f(n) = \sum_{d|n} \mu_\alpha\left(\frac{n}{d}\right) g(d)\right] \Leftrightarrow \left[\forall n \in \mathbb{N} \;\; g(n) = \sum_{d|n} \mu_{-\alpha}\left(\frac{n}{d}\right) f(d)\right].$$

Proof. In fact, this is equivalent to the statement

$$f = \mu_\alpha * g \Leftrightarrow g = \mu_{-\alpha} * f,$$

which follows from

$$f = \mu_\alpha * g \Leftrightarrow \mu_{-\alpha} * f = \mu_{-\alpha} * \mu_\alpha * g = \delta * g = g.$$

□

Evidently, Corollary 1 with $\alpha = 1$ implies the classical Möbius inversion formulae ($f = \mu * g \Leftrightarrow g = \mu_{-1} * f$), since $\mu_1 = \mu$ and $\mu_{-1} \equiv 1$:

$$\mu_{-1}(n) = \prod_{p|n} \binom{-1}{\partial_p(n)} (-1)^{\partial_p(n)} = \prod_{p|n} \frac{(\partial_p(n))!}{(\partial_p(n))!} = 1.$$

Note here that the Möbius μ-function and $\mu_{-1} \equiv 1$ are inverses of each other under convolution.

Corollary 2. *For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$,*

$$\sum_{d|n} \mu_\alpha(d) = \mu_{\alpha-1}(n).$$

This is equivalent to the statement $(\mu_{-1} * \mu_\alpha)(n) = \mu_{\alpha-1}(n)$. Note that the case $\alpha = 1$ gives the classical identity of Gauss

$$\sum_{d|n} \mu(d) = \mu_0(n) = \delta(n).$$

Corollary 3. *Let f be a completely multiplicative function such that $f(mn) = f(n)f(m)$ for all positive integers m and n, and let r be a positive integer. Then the r-times convolution of $\mu_r f$ with f satisfies*

$$(\mu_r f) * f * f * \cdots * f = \mu_0 f,$$

where $(\mu_\alpha f)(n) = \mu_\alpha(n) f(n)$.

This follows easily form Corollary 2 and induction on r. Indeed we have

$$((\mu_r f) * f)(n) = \sum_{d|n} \mu_r(d) f(d) f\left(\frac{n}{d}\right) = f(n) \sum_{d|n} \mu_r(d) = (\mu_{r-1} f)(n).$$

Moreover, it may be of interest to note that

$$\mu_{-2}(n) = (\mu_{-1} * \mu_{-1})(n) = \sum_{d|n} 1 = \tau(n),$$

where $\tau(n)$ denotes the number of positive divisors of n. Thus, $\tau = \mu_{-2}$. Consequently, from $\mu_{-2} * \mu_1 = \mu_{-1}$ and $\mu_{-2} * \mu_2 = \mu_0$, we may obtain the identities

$$\sum_{d|n} \tau(d)\mu\left(\frac{n}{d}\right) = 1 \text{ and } \sum_{d|n} \tau(d)\mu_2\left(\frac{n}{d}\right) = \delta(n).$$

Example 1. *Let $\sigma_r(n)$ denote the sum of the rth powers of the divisors of n. The well-known identity*

$$n^r = \sum \mu(d)\sigma_r\left(\frac{n}{d}\right)$$

can be proved very simply in the following way. Let $i_r(n) = n^r$. Then, since $\mu_{-1} \equiv 1$, we have $i_r * \mu_{-1} = \sigma_r$, and hence

$$(\mu * \sigma_r)(n) = (\mu * i_r * \mu_{-1})(n) = (i_r * \mu * \mu_{-1})(n) = (i_r * \delta)(n) = i_r(n) = n^r.$$

Example 2. *Euler's φ-function may be written as $\varphi = i_1 * \mu_1$. Moreover, using $\tau = \mu_{-2}$ we can easily prove the identity*

$$\sigma(n) = \sum_{d|n} \varphi(d)\tau\left(\frac{n}{d}\right).$$

In fact, these statements follow easily from the relations

$$\varphi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = (i_1 * \mu_1)(n)$$

and $\varphi * \tau = (i_1 * \mu_1) * \mu_{-2} = i_1 * \mu_{-1} = \sigma$ (see Example 1).

Example 3. *Fix a positive integer $r \geq 1$, and define $\varphi_r = i_1 * \mu_r$. Then, if n is 'r-powerful', that is, $\partial_p(n) \geq r$ for every prime divisor p of n, we have*

$$\varphi_r(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)^r.$$

This may be verified as follows:

$$\varphi_r(n) = \sum_{d|n} d\mu_r\left(\frac{n}{d}\right) = n\sum_{d|n}\frac{\mu_r(d)}{d} = n\prod_{p|n}\sum_{j=0}^{r}\binom{r}{j}\left(-\frac{1}{p}\right)^j = n\prod_{p|n}\left(1 - \frac{1}{p}\right)^r.$$

Note that, if $r = 1$, then $\varphi_1 = \varphi$ is the classical Euler function. Thus, φ_r may be called the generalized Euler function of order r. This function has a similar meaning to that of φ, in that ϕ_r counts the number of integers a, $1 \leq a \leq n$, such that a is 'rth-degree prime to n'. (This means that for each prime divisor

p of n, there are $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ and $a \equiv a_0 + a_1 p + \cdots + a_{r-1} p^{r-1} \pmod{p^r}$.) Some related details may be found in [8].

Example 4. *The number of ordered factorizations of n into exactly k factors (see also [4, 13]) is*

$$\mu_{-k}(n) = \prod_{p|n} \binom{\partial_p(n)+k-1}{k-1}.$$

Example 5. *For any given integer $k \geq 1$ one may find a function α_k such that*

$$\varphi(n) = \sum_{d|n} \mu_k\left(\frac{n}{d}\right) \alpha_k(d).$$

(For $k = 1$, of course $\alpha_1 = i_1$.) Indeed, using the generalized Möbius inversion formula and Lemma 2 we obtain $\alpha_k = \mu_{-k} * i_1 * \mu_1 = i_1 * \mu_{1-k}$. A more explicit expression for $\alpha_k(n)$ may also be obtained (see [12, 13]).

Example 6. *Here we would like to mention a remarkable application of μ_α. Rearick [10] has defined the real power f^α of an arithmetic function f with $f(1) > 0$, using his exponential and logarithmic operators* Exp *and* Log, *as $f^\alpha = \mathrm{Exp}(\alpha \mathrm{Log} f)$. Here $(\mathrm{Log} f)(1) = \log f(1)$ and*

$$(\mathrm{Log} f)(n) = (\log n)^{-1} \sum_{d|n} f(d) f^{(-1)}\left(\frac{n}{d}\right) \log d \qquad \text{for } n > 1,$$

where $f^{(-1)}$ is the Dirichlet inverse of f, and $\mathrm{Exp} = (\mathrm{Log})^{-1}$. Recently Haukkanen [6] proved the following result: if f is a completely multiplicative function and α is a real number, then $f^\alpha = \mu_{-\alpha} f$. This result shows that the representation problem for f^α can be nicely solved by using μ_α.

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