Monochromatic Arithmetic Forests

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Definition 1. *If* $A = \{a_1 < a_2 < \cdots < a_n\} \subset \mathbb{N}$*, where* $\mathbb{N}$ *is the set of positive integers, we say that the* gap size *of* A *is* $\text{gs}(A) = \max\{a_{j+1} - a_j : 1 \le j \le n-1\}$. *(If* $|A| = 1$*, set* $\text{gs}(A) = 1$*.)*

Definition 2. *A subset* X *of* $\mathbb{N}$ *is* piecewise syndetic *if for some* fixed $d \ge 1$ *there are arbitrarily large (finite) sets* $A \subset X$ *such that* $\text{gs}(A) \le d$.

Definition 3. *A subset* X *of* $\mathbb{N}$ *has* property AP *if there are arbitrarily large (finite) sets* $A \subset X$ *such that* A *is an arithmetic progression.*

Fact 1. *If* $\mathbb{N} = X_1 \cup X_2 \cup \cdots \cup X_n$ *then some* X_i *is piecewise syndetic (and hence also has property AP). (The first proofs of Fact 1 appear in [2–4].) However, this result neither implies, nor is implied by, van der Waerden's theorem on arithmetic progressions.*

Fact 2. *If* $X \subset \mathbb{N}$ *and* X *has positive upper density, then* X *has property AP (by Szemerédi's theorem) but* X *need not be piecewise syndetic. (For an example, see [1].)*

The finite version of Fact 1 is:

Theorem A. For all $r \ge 1$ and $f \in \mathbb{N}^{\mathbb{N}}$, there exists $n = n(f,r)$ such that whenever $[1,n]$ is r-colored, there is a monochromatic set A such that $|A| \ge f(\text{gs}(A))$. (Furthermore, $n(f,1) = f(1)+1$ and $n(f,r+1) \le (r+1)f(n(f,r))+1$.)

There are applications of this result to the theory of locally finite semigroups, and in particular to Burnside's problem for semigroups of matrices (see [6]).

The main result of this note is the following generalization of Theorem A:

Theorem A*. For all $r \ge 1$ and $f \in \mathbb{N}^{\mathbb{N}}$, there exists $n^* = n^*(f,r)$ such that whenever $P([1,n^*])$ (the set of all subsets of $[1,n^*]$) is r-colored, there exist $d \ge 1$ and monochromatic copies (all in the same color) of all rooted forests having $f(d)$ vertices. These monochromatic copies have the property that for all pairs of vertices x,y, if vertex y covers vertex x, then $|y| - |x| \le d$. Also, if x,y belong to the same tree then $x \wedge y = x \cap y$; if x,y belong to different trees then $x \cap y = \varnothing$. (Furthermore, $n^*(f,1) = f(1)$ and $n^*(f,r+1) \le n^*(f,r) \cdot f(n^*(f,r))$.)

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(The same remarks concerning $x \cap y$ apply to all of the following results.)

One can prove a similar variation of van der Waerden's theorem:

Theorem W*. For all $r \geq 1$ and $k \geq 1$, there exists $w^* = w^*(k,r)$ such that whenever $P([1,w^*])$ is r-colored, there exist $a \geq 1$ and $d \geq 1$ and monochromatic copies (all in the same color) of all rooted forests having k vertices. These monochromatic copies have the property that every vertex at level $i = 0, 1, \ldots$ has size $a + id$.

The proof of Theorem W* uses van der Waerden's theorem and the finite Ramsey's theorem, in essentially the same way these were combined in the proof of Theorem 2 in [5]. The results that follow use the infinite Ramsey's theorem together with respectively Fact 1, van der Waerden's theorem, and the Erdős-Graham canonical van der Waerden's theorem.

Definition 4. *An* ω-tree of height n *is a rooted tree in which every maximal chain has* $n+1$ *vertices, and every non-maximal vertex has* ω *immediate succesors. An* ω-forest of height n *is a union of* ω *pairwise disjoint* ω*-trees of height* n*.*

Definition 5. *An* arithmetic forest in $[\mathbb{N}]^{<\omega}$ *(the set of all finite subsets of* $\mathbb{N}$*) is a forest for which there exist* $a \geq 1$*,* $d \geq 1$ *such that all vertices at level* i *have size* $a + id$*,* $i \geq 0$*.*

Theorem A.** Let $[\mathbb{N}]^{<\omega}$ be finitely colored. Thene there exists a *fixed* $d \geq 1$ such that for every $n \geq 1$, there is a monochromatic ω-forest $F(n)$ of height n such that for all pairs of vertices x, y in $F(n)$, if y covers x then $|y| - |x| \leq d$.

Theorem W.** Let $[\mathbb{N}]^{<\omega}$ be finitely colored. Then for every $n \geq 1$ there exists a monochromatic arithmetic ω-forest $F(n)$ of height n.

Theorem C.** Let $[\mathbb{N}]^{<\omega}$ be ω-colored. Then for every $n \geq 1$ there exists an arithmetic ω-forest $F(n)$ of height n such that either $F(n)$ is monochromatic or there are distinct colors $c_0, \ldots, c_n$ such that all vertices at level i have color c_i, $0 \leq i \leq n$.

Question. It was observed by J. Walker in [9] that if $k \geq 1$ and $\varepsilon > 0$ are given, then for sufficiently large n, if $S \subseteq P([1,n])$ and $|S| > \varepsilon |P([1,n])|$, S must contain an arithmetic chain of length k. Is it true that for sufficiently large n, if $S \subseteq P([1,n])$ and $|S| > \varepsilon |P([1,n])|$, S must contain arithmetic copies of all k-vertex rooted forests?

Remark. Some related results, and additional references, can be found in [8] and [7].

References

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