Approximations of additive squares in infinite words

Tom Brown

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Abstract
We show that every infinite word \( \omega \) on a finite subset of \( \mathbb{Z} \) must contain arbitrarily large factors \( B_1B_2 \) which are “close” to being additive squares. We also show that for all \( k > 1 \), \( \omega \) must contain a factor \( U_1U_2\cdots U_k \) where \( U_1, U_2, \cdots, U_k \) all have the same average.

1 Introduction
If \( S \) is a finite subset of \( \mathbb{Z} \) and \( \omega \in S^\mathbb{N} \), we write \( \omega = x_1x_2x_3\cdots \). For any (finite) factor \( B = x_i x_{i+1} \cdots x_{i+n} \) of \( \omega \), we write \( |B| \) for the length of \( B \) (here \( |B| = n+1 \)), and we write
\[
\sum B = x_i + x_{i+1} + \cdots + x_{i+n}.
\]
If \( B_1B_2 \) is a factor of \( \omega \) with
\[
|B_1| = |B_2| \quad \text{and} \quad \sum B_1 = \sum B_2,
\]
we say that \( B_1B_2 \) is an additive square contained in \( \omega \). For example, if \( \omega = 2135126 \cdots \) (a word on the alphabet \( S = \{1,2,3,4,5,6\} \)), then \( \omega \) contains the additive square \( B_1B_2 \), where \( B_1 = 135, B_2 = 126 \), with \( |B_1| = |B_2| = 3 \) and \( \sum B_1 = \sum B_2 = 9 \).

A celebrated result of Keränen [11] (see also [12]) is that there exist infinite words \( \omega \) on an alphabet of 4 symbols which contain no abelian square, that is, \( \omega \) contains no factor \( B_1B_2 \) where \( B_1, B_2 \) are permutations of one another. (For early background, see [3].)

After Keränen’s result, it was natural to consider the question of whether an infinite word \( \omega \) on 4 (or more) integers must contain an additive square.

Freedman [7] showed that if \( a, b, c, d \in \mathbb{Z} \) (or more generally if \( a, b, c, d \) belong to any field of characteristic 0) and \( a + d = b + c \), then every word of length 61 on \( \{a, b, c, d\} \) contains an additive square.

*Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada. tbrown@sfu.ca
Cassaigne, Currie, Schaeffer, and Shallit [4] showed that there is an infinite word $\omega$ on the alphabet \{0, 1, 3, 4\} which contains no additive cube, that is, $\omega$ contains no factor $B_1B_2B_3$ such that $|B_1| = |B_2| = |B_3|$ and $\sum B_1 = \sum B_2 = \sum B_3$.

A few remarks follow.

For each $k \geq 1$, let $g(1, 2, \cdots, k)$ denote the length of a longest word on \{1, 2, \cdots, k\} which does not contain an additive square. (We allow $g(1, 2, \cdots, k) = \infty$.) Then the following three statements are equivalent:

1. For all $k \geq 1$, $g(1, 2, \cdots, k) < \infty$.
2. For all $k \geq 1$, and all infinite words $\omega$ on \{1, 2, \cdots, k\}, $\omega$ contains arbitrarily large additive squares.
3. Let $x_1 < x_2 < x_3 \cdots$ be any sequence of positive integers such that, for some $M$, $0 < x_{i+1} - x_i < M$ for all $i \geq 1$. Then there exist $i < j < k$ such that both $\{i, j, k\}$ and $\{x_i, x_j, x_k\}$ are arithmetic progressions. (Statement 3 is equivalent to statement 1 via van der Waerden’s theorem on arithmetic progressions [15].)

Finally, let us denote by $g(a, b, c, d)$ the length of a longest word on \{a, b, c, d\} which does not contain an additive square. Then the statement

$$\lim_{n \to \infty} g(1, n, n^2, n^3) = \infty$$

is equivalent (by standard combinatorial arguments) to the result of Keränen stated above.

The question concerning the presence of additive squares seems to have appeared in print for the first time in a paper by Pirillo and Varricchio [13]. Other related material can be found in [1, 2, 4, 5, 7, 9, 10, 14].

In this note we show that for every finite subset $S$ of $\mathbb{Z}$ there is a constant $C$ (which depends only on $S$) such that every infinite word $\omega$ on $S$ contains arbitrarily long factors $UV$ such that

$$|U| = |V| \text{ and } |\sum U - \sum V| \leq C.$$ 

We also show that for every infinite word $\omega$ on a finite subset of $\mathbb{Z}$ there must exist, for every $k > 1$, a factor $B_1B_2\cdots B_k$ of $\omega$ such that $B_1, B_2, \cdots, B_k$ all have the same average. Here, the average of a factor $B$ is $\frac{1}{|B|} \sum B$.

2 Adjacent equal length blocks with nearly equal sums

Here we exploit the fact that if $U, V$ are words on a 2-element subset of $\mathbb{Z}$, then $UV$ is an additive square ($|U| = |V|$ and $\sum U = \sum V$) if and only if $UV$ is an abelian square ($U$ and $V$ are permutations of one another).

**Theorem 2.1.** For every finite subset $S$ of $\mathbb{Z}$ there exists a constant $C$ (depending only on $S$) such that every infinite word $\omega$ on $S$ contains arbitrarily long factors $UV$ such that

$$|U| = |V| \text{ and } |\sum U - \sum V| \leq C.$$ 

**Proof.** First assume that $S$ is a finite subset of $\mathbb{N}$, and let $\omega = x_1x_2x_3\cdots$ be an infinite word on $S$. Let $1^s$ denote a string of $s$ 1s of length $x_i$ (e.g., $1^4 = 1111$), and let $\omega'$ be the binary word $1^{x_1}01^{x_2}01^{x_3}0\cdots$, which we write for convenience as $x_10x_20x_30\cdots$. By a theorem of Entringer, Jackson, and Schatz [6] the word $\omega'$ contains arbitrarily large abelian squares $U'V'$, and hence arbitrarily large factors $U'V'$ with $|U'| = |V'|$ and $\sum U'' = \sum V''$. Re-numbering the indices for convenience, such a square $U'V'$, since each of $U'$ and $V'$ must contain the same number, say $k$, of 0s, has the form

$$U' = \alpha_30\alpha_30\alpha_30\cdots, V' = \alpha_40\alpha_40\alpha_40\cdots.$$
where $\alpha_1 + \alpha_2 = x_1, \alpha_3 + \alpha_4 = x_{k+1}, \alpha_5 + \alpha_6 = x_{2k+1}$. (All the $\alpha_i$ are non-negative integers.) Since $U'V'$ is an additive square,

$$\alpha_2 + \sum_{i=2}^{k} x_i + \alpha_3 = \alpha_4 + \sum_{i=k+2}^{2k} x_i + \alpha_5,$$

or (using $\alpha_1 + \alpha_2 = x_1$ and $\alpha_3 + \alpha_4 = x_{k+1}$)

$$|\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{2k+1} x_i| = |\alpha_1 - 2\alpha_3 + \alpha_5| \leq 2 \max S.$$

With $U = x_1x_2 \cdots x_k, V = x_{k+1}x_{k+2} \cdots x_{2k}$, we have the factor $UV$ of $\omega$ with

$$|U| = |V| \text{ and } |\sum U - \sum V| \leq 2 \max S.$$

When $S$ is a finite subset of $\mathbb{Z}$ which contains non-positive integers, translate $S$ to the right by $|\min S| + 1$ and apply the above argument, to get arbitrarily large factors $UV$ such that

$$|U| = |V| \text{ and } |\sum U - \sum V| \leq 2(|\min S| + 1 + \max S).$$

\[ \square \]

3 Adjacent factors with equal averages

**Theorem 3.1.** For any finite subset $S$ of $\mathbb{Z}$, any infinite word $\omega$ on $S$, and any $k > 1$, there exists a factor $U_1U_2 \cdots U_k$ with

$$\frac{1}{|U_1|} \sum U_1 = \frac{1}{|U_2|} \sum U_2 = \cdots = \frac{1}{|U_k|} \sum U_k.$$

**Proof.** Let $\omega = x_1x_2x_3 \cdots$ be a given infinite word on the set of integers $S = \{s_1, s_2, \cdots, s_t\}$. Consider the infinite sequence of points in the plane $P_i = (i, x_1 + x_2 + \cdots + x_i), i \geq 1$. Since $P_{i+1} - P_i = (1, x_{i+1}) \in \{(1, s_j) : 1 \leq j \leq t\}$, a theorem of Gerver and Ramsey [8] asserts that the set $\{P_i : i \geq 1\}$ contains, for any given $k > 1$, $k + 1$ collinear points $P_1, P_2, \cdots, P_{k+1}$. For $1 \leq j \leq k$, let $U_j = x_{j+1}x_{j+2} \cdots x_{j+k+1}$. The slope of the line segment joining $P_{ij}$ and $P_{ij+1}$ is $\frac{1}{|U_j|} \sum U_j$. Since this slope is the same for each choice of $j$, we have

$$\frac{1}{|U_1|} \sum U_1 = \frac{1}{|U_2|} \sum U_2 = \cdots = \frac{1}{|U_k|} \sum U_k.$$

\[ \square \]

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**Remark.** The author has learned that Theorem 3.1 was proved independently by Jeffrey Shallit.
References


