A Pseudo Upper Bound for the van der Waerden Function *

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Abstract

For each positive integer n, let the set of all 2-colorings of the interval $[1,n] = \{1,2,\ldots,n\}$ be given the uniform probability distribution, that is, each of the 2^n colorings is assigned probability 2^{-n}. Let f be any function such that $f(k)/\log k \to \infty$ as $k \to \infty$. For convenience we assume that $f(k)2^k$ is always a positive integer. We show that the probability that a random 2-coloring of $[1, f(k)2^k]$ produces a monochromatic k-term arithmetic progression tends to 1 as $k \to \infty$. We call $f(k)2^k$ a *pseudo upper bound* for the van der Waerden function. We also prove the "density version" of this result.

1 Introduction

Let w denote the van der Waerden function. By definition, for each integer $k \geq 1$, $w(k)$ is the smallest positive integer such that every 2-coloring of the interval $[1,w(k)] = \{1,2,\ldots,w(k)\}$ produces a monochromatic k-term arithmetic progression. (Equivalently, for every partition of $[1,w(k)]$ into at most two parts, at least one part contains a k-term arithmetic progression.)

The existence of $w(k)$, $k \geq 1$, was proved by van der Waerden in 1927 [7]. The best known lower bound for $w(k)$ is $w(k) > (2^k/2ek)(1+o(1))$ (see [4]). For p prime, Berlekamp [2] showed that $w(p+1) \geq p2^p$. (For some related lower bounds, see [1, 3, 5].) The best known upper bound (a "wowzer" function, as described in [4]) is due to Shelah [6]. R. L. Graham has offered \$1000 (see [4]) for a proof that $w(k) < 2^{2^{\cdot^{\cdot^{2}}}}$, a tower of height k.

Let f be any function such that $f(k)/\log k \to \infty$ as $k \to \infty$. For convenience we assume that $f(k)2^k$ is always a positive integer.

In this note we show that "almost all" of the 2-colorings of $[1, f(k)2^k]$ produce a monochromatic k-term arithmetic progression.

This means that if S_k is the set of "exceptional" colorings, that is, if $n_k = f(k)2^k$ and S_k is the set of all 2-colorings of $[1,n_k]$ for which there is no monochromatic k-term arithmetic progression, then $|S_k| \cdot 2^{-n_k} \to 0$ as $k \to \infty$.

We then say that $f(k)2^k$ is a *pseudo upper bound* for the van der Waerden function.

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To illustrate the method, consider 2-colorings of the interval $[1,tk]$, where $t = k2^k$, and let T_k denote the set of all those 2-colorings of $[1,tk]$ for which none of the t intervals $[1,k]$, $[k+1,2k]$, …, $[(t-1)k+1,tk]$ is monochromatic. Then

$$|T_k| \cdot 2^{-tk} = (2^k - 2)^t \cdot 2^{-tk} = \left(1 - \frac{1}{2^{k-1}}\right) < e^{-2k} \to 0 \text{ as } k \to \infty.$$

Therefore "almost all" of the 2-colorings of $[1,k^2 2^k]$ produce a monochromatic k-term arithmetic progression; moreover, this monochromatic progression is one the intervals $[1,k]$, $[k+1,2k]$,…, $[(t-1)k+1,tk]$.

The proof of Theorem 1 below consists of a refinement of the above simple argument, in order to reduce $k^2 2^k$ to $f(k)2^k$.

In the argument above, we used certain (pairwise disjoint) arithmetic progressions with common difference 1. In the proofs of Theorems 1 and 2, we use certain progressions with common differences $1, k, k^2, \ldots, k^s$, any two of which have at most one point in common.

Theorem 2 is the "density version" of Theorem 1. (Note that Theorem 1 follows from the case $\varepsilon = 1/2$ of Theorem 2.)

2 Results

Theorem 1. *Let f be any function such that $f(k)/\log k \to \infty$ as $k \to \infty$. For convenience we assume that $f(k)2^k$ is always a positive integer. For each positive integer n, let the set of all 2-colorings of $[1,n] = \{1,2,\ldots,n\}$ be given the uniform probability distribution, that is, each of the 2^n colorings is assigned probability 2^{-n}. Then the probability that a random 2-coloring of $[1, f(k)2^k]$ produces a monochromatic k-term arithmetic progression, tends to 1 as $k \to \infty$.*

Proof. For each positive integer $k \geq 2$, let $n_k = f(k)2^k$. Let S_k denote the set of all those 2-colorings of $[1,n_k]$ which do *not* produce any monochromatic k-term arithmetic progression. We will show that $\lim_{k\to\infty} |S_k| \cdot 2^{-n_k} = 0$.

Let $n = n_k = f(k)2^k$. Define the integer $s = s_k$ by $k^{2s} \leq n < k^{2s+2}$. Let $n = k^s q + r$, $0 \leq r < k^s \leq \sqrt{n}$.

Let us assume without loss of generality that $f(k) < k^2$. (The previous discussion has already handled the case $f(k) \geq k^2$.)

Some easy calculations show that, as $k \to \infty$,

$$\frac{s}{k}\frac{k^s}{2^k} \to 0 \text{ and } \frac{s}{k}\frac{k^s q}{2^k} \to \infty.$$

These facts are used later in the proof.

The interval $[1,n]$ consists of q consecutive intervals, $B_1, B_2, \ldots, B_q$, each of length k^s, followed by a single interval of length r, $r < k^s$.

We wish to examine 2-colorings of the interval B_1. Let us identify B_1 (by shifting it one unit to the left) with the interval $[0, k^s - 1]$, and further identify the interval $[0, k^s - 1]$ with the set of s-tuples

$$C = \{x_0 x_1 \cdots x_{s-1} : 0 \leq x_i \leq k-1\},$$

under the correspondence $x_0x_1\cdots x_{s-1} \leftrightarrow \sum_{i=1}^{s-1} x_i k^i$. That is, we identify each integer in $[0, k^s - 1]$ with the s-tuple of the digits in its k-ary expansion.

Under this identification, B_1 may be visualized as the s-dimensional *cube* C, k units on a side. For our purposes, we say that a *line* in the cube C is a set of the form

$$\{x_0 \cdots x_{j-1} y x_{j+1} \cdots x_{s-1} : 0 \le y \le k-1\},$$

where the x_i's are fixed. If the jth coordinate is the "moving" coordinate, then the k points in this line correspond to k integers in B_1 which form an arithmetic progression with common difference k^j.

There are sk^{s-1} lines in the cube C. For each line u in C, let A_i denote the set of 2-colorings of C for which the line u is monochromatic. Then $|A_u| = 2 \cdot 2^{k^s - k}$. Given any two distinct lines u and v, u and v are either disjoint or meet in 1 point. In either case, $|A_u \cap A_v| = 4 \cdot 2^{k^s - 2k}$. Therefore

$$\begin{aligned}
\left|\bigcup_u A_u\right| &\ge \sum_u |A_u| - \sum_{u,v} |A_u \cap A_v| \\
&= sk^{s-1} \cdot 2 \cdot 2^{k^s - k} - \binom{sk^{s-1}}{2} \cdot 4 \cdot 2^{k^s - 2k} \\
&> 2^{k^s} \frac{s}{k} \frac{k^s}{2^k}
\end{aligned}$$

for sufficiently large k, since $(s/k)(k^s/2^k) \to 0$.

Let z denote the number of colorings of the cube C for which none of the sk^{s-1} lines in C are monochromatic. Then (for sufficiently large k)

$$z = 2^{k^s} - \left|\bigcup_u A_u\right| < 2^{k^s}\left(1 - \frac{s}{k}\frac{k^s}{2^k}\right).$$

The set S_k defined at the beginning of the proof has $|S_k| \le z^q 2^r$, so that

$$|S_k| \cdot 2^{-n} < \left(1 - \frac{s}{k}\frac{k^s}{2^k}\right)^q < e^{-(s/k)(k^s q/2^k)}.$$

Since $(s/k)(k^s q/2^k) \to \infty$ as $k \to \infty$, the proof is complete. □

Theorem 2. *Let ε be fixed, $0 < \varepsilon < 1$. Let f be any function such that $f(k)/\log k \to \infty$ as $k \to \infty$. Let $n = f(k)\varepsilon^{-k}$ (we assume that this is always an integer), and let $\mathbf{A}$ be a "random εn-element subset of $[1,n]$," which means that each element of $[1,n]$ belongs to $\mathbf{A}$ with probability ε. Then the probability that $\mathbf{A}$ contains a k-term arithmetic progression, tends to 1 as $k \to \infty$.*

Proof. The numbers s and r are defined as in the proof of Theorem 1 (with n now defined by $n = f(k)\varepsilon^{-k}$), and again we write $n = k^s q + r$, $0 \le r < k^s \le \sqrt{n}$. Next, as $k \to \infty$,

$$\frac{s}{k} k^s \varepsilon^k \to 0 \text{ and } \frac{s}{k} k^s \varepsilon^k q \to \infty.$$

(To see this it is convenient to show first that for any $\eta > 0$, the inequalities

$$\left(\frac{1}{2}\log(1/\varepsilon) - \eta\right)\frac{1}{\log k} < \frac{s}{k} < \left(\frac{1}{2}\log(1/\varepsilon) + \eta\right)\frac{1}{\log k}$$

hold for all sufficiently large k. For the right-hand inequality, one again needs to assume that $f(k) < k^2$, and handle the case $f(k) > k^2$ by a separate argument, as in the discussion in the Introduction.)

The cube C is defined as before. Let $\mathbf{B}$ denote a random $\varepsilon|C|$-element subset of C, where each element of C belongs to $\mathbf{B}$ with probability ε. Let $p_u = \Pr[u \subseteq \mathbf{B}] = \varepsilon^k$, where u is any one of the sk^{s-1} lines in C, and let $p_{uv} = \Pr[u \subseteq \mathbf{B} \text{ and } v \subseteq \mathbf{B}]$, where u and v are distinct lines in C. Then $\Pr[u \subseteq \mathbf{B} \text{ for some } u] \geq \sum_u p_u - \sum_{u,v} p_{uv}$.

Through each of the k^s points of C there are s lines, and hence of the $\binom{sk^s-1}{2}$ pairs of lines $\{u,v\}$, exactly $k^s\binom{s}{2}$ pairs meet, and the other pairs are disjoint. Then

$$\begin{aligned}\sum_u p_u - \sum_{u,v} p_{u,v} &= \frac{s}{k}k^s\varepsilon^k - k^s\binom{s}{2}\varepsilon^{2k-1} - \left[\binom{sk^{s-1}}{2} - k^s\binom{s}{2}\right]\varepsilon^{2k} \\ &= \frac{s}{k}k^s\varepsilon^k - k^s\binom{s}{2}\varepsilon^{2k-1} - \frac{1}{2}\frac{s}{k}k^s\varepsilon^k\left(\frac{s}{k}k^s\varepsilon^k - \varepsilon^k\right) + k^s\binom{s}{2}\varepsilon^k.\end{aligned}$$

The remaining inequalities hold for sufficiently large k.

Since $(s/k)k^s\varepsilon^k - \varepsilon^k < 1/2$, we get

$$\sum_u p_u - \sum_{u,v} p_{u,v} > \frac{3}{4}\frac{s}{k}k^s\varepsilon^k - k^s\binom{s}{2}\varepsilon^{2k-1}(1-\varepsilon).$$

Since $\frac{1}{4}(s/k)k^s\varepsilon^k - k^s\binom{s}{2}\varepsilon^{2k-1}(1-\varepsilon) > 0$, we get

$$\sum_u p_u - \sum_{u,v} p_{u,v} > \frac{1}{2}\frac{s}{k}k^s\varepsilon^k.$$

Finally, with $n = k^s q + r$, if each element of $[1,n]$ belongs to $\mathbf{A}$ with probability ε, then

$$\Pr[\text{no } u \subseteq \mathbf{A}] < \left(1 - \frac{1}{2}\frac{s}{k}k^s\varepsilon^k\right)^q < e^{-(1/2)(s/k)k^s\varepsilon^k q} \to 0.$$

□

3 Remarks

Note that in the proofs, the only k-term progressions considered are (some of) those whose common differences have the form k^j, where $0 \leq j \leq s < 1 + (k\log 2)/(2\log k)$.

It would be desirable to get rid of the factor $f(k)$, if possible. To accomplish this, evidently one needs to use a larger collection of k-term progressions. (Using all of the $(s+1)^k - s^k$ combinatorial lines in the cube C, instead of just the sk^{s-1} lines with one moving coordinate, does not lead us to an improvement in $f(k)$.)

Perhaps, by using a sufficiently large set of progressions, one could show that $(1+\alpha)^k$ is a pseudo upper bound for the van der Waerden function, for every $\alpha > 0$.

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