# Sequences with Translates Containing Many Primes

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**Abstract**

Garrison [3], Forman [2], and Abel and Siebert [1] showed that for all positive integers $k$ and $N$, there exists a positive integer $\lambda$ such that $n^k+\lambda$ is prime for at least $N$ positive integers $n$. In other words, there exists $\lambda$ such that $n^k+\lambda$ represents at least $N$ primes.

We give a quantitative version of this result. We show that there exists $\lambda \le x^k$ such that $n^k+\lambda$, $1 \le n \le x$, represents at least $\left(\frac{1}{k}+o(1)\right)\pi(x)$ primes, as $x \to \infty$. We also give some related results.

# 1 Introduction

In [1], Abel and Siebert make the wonderful observation that if $A=\{a_n\}$ is a sequence of natural numbers and $A(x)=\sum_{a_n\le x} 1$, then

$$\sum_{\lambda\le 2x}\sum_{a_n\le x}\sum_{p=a_n+\lambda} 1 \ge [\pi(2x)-\pi(x)]A(x),$$

where $p$ denotes a prime and $\pi(x)$ denotes the number of primes $p\le x$. They used this inequality, together with Chebyshev's inequalities, to show that if $\limsup_{x\to\infty}\frac{A(x)}{\log x}=\infty$, then for all $N$ there exists $\lambda$ such that $a_n+\lambda$ represents at least $N$ primes. Forman [2] obtained the same result with methods different from those of Abel and Siebert.

Earlier, Sierpenski [5] showed that $n^2+\lambda$ represents arbitrarily many primes. Then Garrison [3] extended this to $n^k+\lambda$. Forman [2] and Abel and Siebert [1] showed that $g(n)+\lambda$ represents arbitrarily many primes, where $g(x)$ is any polynomial with integer coefficients and positive leading coefficient.

In this note we consider sums of the form

$$S(x)=\sum_{\lambda\le 2x}\sum_{a_n\le x}\sum_{p=a_n+\lambda} f(b_m) \text{ and } T(x)=\sum_{\lambda\le x}\sum_{a_n\le x}\sum_{p=a_n+\lambda} f(b_m),$$

where $A=\{a_n\}$ and $B=\{b_m\}$ are given sequences of natural numbers and $f$ is a given nonnegative function defined on the natural numbers. In particular, if $B$ is the sequence of primes and $f\equiv 1$, then $T(x)=(1+o(1))A(x)\pi(x)$. This implies that if $A=\{n^k : n\ge 1\}$, then $T(x)=(1+o(1))x^{\frac{1}{k}}\pi(x)$. It follows that there exists a positive integer $\lambda\le x^k$ such that $n^k+\lambda$, $n\le x$, represents at least $(\frac{1}{k}+o(1))\pi(x)$ primes.

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## 2 Results

**Theorem 1.** *Let $A = \{a_n\}$, $B = \{b_m\}$ be sequences of natural numbers, and let $f$ be a nonnegative function defined on the natural numbers. Let $A(x) = \sum_{a_n \le x} 1$, $B(x) = \sum_{b_m \le x} f(b_m)$.*

*Assume that $B(x) = (1+o(1))x^{\alpha}\varphi(x)$, where $\varphi$ is monotonic and $\lim_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} = 1$. Let $S(x)$ denote the sum*

$$S(x) = \sum_{\lambda \le 2x} \sum_{a_n \le x} \sum_{b_m = a_n + \lambda} f(b_m)$$

*Then*

$$(2^{\alpha} - 1 + o(1))A(x)B(x) \le S(x) \le (3^{\alpha} + o(1))A(x)B(x).$$

*Proof.* For the lower bound, we start with Abel and Seibert's inequality

$$S(x) \ge [B(2x) - B(x)]A(x).$$

Next,

$$\frac{B(2x) - B(x)}{B(x)} = \frac{(1+o(1))(2x)^{\alpha}\varphi(2x)}{(1+o(1))x^{\alpha}\varphi(x)} - 1 \to 2^{\alpha} - 1,$$

hence $B(2x) - B(x) = (2^{\alpha} - 1 + o(1))B(x)$.

For the upper bound, we write

$$\begin{aligned} S(x) &= \sum_{a_n \le x} \sum_{a_n + 1 \le b_m \le a_n + 2x} f(b_m) \\ &= \sum_{a_n \le x} [B(a_n + 2x) - B(a_n)] \le \sum_{a_n \le x} B(a_n + 2x). \end{aligned}$$

We now estimate $B(a_n + 2x)$ from above.

Let $a$ be an integer, $1 \le a \le x$. Since $\varphi$ is monotonic, $x \le a + x \le 2x$, and $\frac{\varphi(x)}{\varphi(x)} = 1$, $\frac{\varphi(2x)}{\varphi(x)} \to 1$, it follows that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\frac{\varphi(a+x)}{\varphi(x)} < 1 + \varepsilon, \ \ x > N, \ 1 \le a \le x.$$

From this it follows that $\frac{\varphi(3x)}{\varphi(x)} = \frac{\varphi(3x)}{\varphi(2x)} \cdot \frac{\varphi(2x)}{\varphi(x)} \to 1$.

Now since $2x \le a + 2x \le 3x$, $\varphi$ is monotonic, and $\frac{\varphi(2x)}{\varphi(x)} \to 1$, $\frac{\varphi(3x)}{\varphi(x)} \to 1$, it follows that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\frac{\varphi(a+2x)}{\varphi(x)} < 1 + \varepsilon, \ \ x > N, \ 1 \le a \le x.$$

It now follows that for any $a = a(x)$, $1 \le a \le x$, and any $\varepsilon > 0$,

$$\frac{B(a+2x)}{B(x)} = \frac{(1+o(1))(a+2x)^{\alpha}\varphi(a+2x)}{(1+o(1))x^{\alpha}\varphi(x)} < 3^{\alpha} + \varepsilon$$

for sufficiently large $x$. Hence, independent of the choice of $a$, $1 \le a \le x$,

$$B(a+2x) \le (3^{\alpha} + o(1))B(x),$$

and

$$S(x) \le \sum_{a_n \le x} B(a_n + 2x) \le (3^a + o(1))A(x)B(x).$$

□

Now we let $B$ be the sequence of primes.

**Theorem 2.** *Let $A = \{a_n\}$ be a sequence of natural numbers. Then*

$$S(x) = \sum_{\lambda \le 2x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 \ge (1 + o(1))A(x)\pi(x),$$

*where $p$ denotes a prime. Hence there exists $\lambda$, $1 \le \lambda \le 2x$, such that $a_n + \lambda$, $1 \le a_n \le x$ represents at least $(\frac{1}{2} + o(1))\frac{A(x)}{x}\pi(x)$ primes.*

*Proof.* This proof is a direct application of the method of Abel and Siebert. We have

$$S(x) = \sum_{\lambda \le 2x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 \ge (\pi(2x) - \pi(x))A(x) = (1 + o(1))A(x)\pi(x),$$

or

$$\frac{1}{2x} \sum_{\lambda=1}^{2x} \left( \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 \right) \ge \left( \frac{1}{2} + o(1) \right) \frac{A(x)}{x} \pi(x),$$

so at least one $\lambda$, $1 \le \lambda \le 2x$, has the required property. □

We now improve this result by using part of the method of Theorem 1.

**Theorem 3.** *Let $A = \{a_n\}$ be a sequence of natural numbers. Then*

$$T(x) = \sum_{\lambda \le x} \sum_{a_n \le x} \sum_{p = a_n + \lambda} 1 = (1 + o(1))A(x)\pi(x),$$

*where $p$ denotes a prime. Hence there exists $\lambda$, $1 \le \lambda \le x$, such that $a_n + \lambda$, $1 \le a_n \le x$ represents at least $(1 + o(1))\frac{A(x)}{x}\pi(x)$ primes.*

*Proof.* As in the proof of Theorem 1, we write

$$T(x) = \sum_{a_n \le x} \sum_{a_n + 1 \le p \le a_n + x} 1 = \sum_{a_n \le x} [\pi(a_n + x) - \pi(a_n)].$$

It is not hard to show that for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$1 - \varepsilon < \frac{\pi(a+x) - \pi(a)}{\pi(x)} < 1 + \varepsilon, \;\; x > N, \; 1 \le a \le x.$$

(For fixed $\varepsilon$, divide $[1,x]$ into subintervals of length $\varepsilon x$, and use the Prime Number Theorem to estimate $\frac{\pi(a+x)-\pi(a)}{\pi(x)}$ when $a \in [(i-1)\varepsilon x, i\varepsilon x]$.)

Summing this over all $a_k$, $a_k \le x$, gives

$$(1-\varepsilon)A(x)\pi(x) < T(x) < (1+\varepsilon)A(x)\pi(x), \quad x > N,$$

or $T(x) = (1+o(1))A(x)\pi(x)$. The rest follows as in the proof of Theorem 2. □

**Corollary.** *Let $k \ge 1$ be given. Then there exists a positive integer $\lambda \le x^k$ such that $n^k + \lambda$, $n \le x$, represents at least $(\frac{1}{k} + o(1))\pi(x)$ primes.*

*Proof.* Setting $a_n = n^k$ in Theorem 3, and replacing $x$ by $x^k$ in the conclusion of Theorem 3 shows that there exists $\lambda$, $1 \le \lambda \le x^k$, such that $n^k + \lambda$, $1 \le n^k \le x^k$, represents at least

$$(1+o(1))\frac{(x^k)^{\frac{1}{k}}}{x^k}\pi(x^k) = (1+o(1))\frac{x}{x^k}\frac{x^k}{\log x^k} = (1+o(1))\frac{x}{k\log x} = \left(\frac{1}{k} + o(1)\right)\pi(x)$$

primes. □

We now apply our methods to the case when $B$ is the sequence of square-free numbers.

**Theorem 4.** *Let $A = \{a_n\}$ be a given sequence of natural numbers. Let $A(x) = \sum_{a_n \le x} 1$, and let $\alpha$ be any fixed real number with $\frac{1}{2} < \alpha < 1$. Let $\varepsilon > 0$ be given. Then for all sufficiently large $x$, there exists $\lambda$, $1 \le \lambda \le x^\alpha$, such that more than $(\frac{6}{\pi^2} - \varepsilon)A(x)$ of the numbers $a_n + \lambda$, $a_n \le x$, are square-free.*

*Proof.* Let $B = \{b_m\}$ be the sequence of square-free numbers, and let $B(x) = \sum_{b_m \le x} 1$. It is known (see [4]) that

$$B(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Let $\alpha$ be fixed, $1/2 < \alpha < 1$, and let $L$ denote the number $L = [x^\alpha]$.

Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}
\sum_{\lambda=1}^{L} \sum_{a_n \le x} \sum_{b_m = a_n + \lambda} 1 &= \sum_{a_n \le x} \sum_{\lambda \le L} \sum_{b_m = a_n + \lambda} 1 \\
&= \sum_{a_n \le x} \sum_{a_n + 1 \le b_m \le a_n + L} 1 \\
&= \sum_{a_n \le x} (B(a_n + L) - B(a_n)) \\
&= \sum_{a_n \le x} \left(\frac{6L}{\pi^2} + O(\sqrt{x+L})\right) \\
&= \sum_{a_n \le x} \frac{6L}{\pi^2}(1 + o(1)) \\
&> \left(\frac{6}{\pi^2} - \varepsilon\right) L \sum_{a_n \le x} 1 \\
&= \left(\frac{6}{\pi^2} - \varepsilon\right) LA(x)
\end{aligned}$$

holds for sufficiently large $x$. Hence there exists at least one $\lambda$, $1 \le \lambda \le L = [x^\alpha]$, for which

$$\sum_{a_n \le x} \sum_{b_m = a_n + \lambda} 1 > \left( \frac{6}{\pi^2} - \varepsilon \right) A(x),$$

which was to be proved. □

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