Arithmetic Progressions in Sequences With Bounded Gaps

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Abstract

Let $G(k,r)$ denote the smallest positive integer g such that if $1 = a_1, a_2, \ldots, a_g$ is a strictly increasing sequence of integers with bounded gaps $a_{j+1} - a_j \leq r$, $1 \leq j \leq g-1$, then $\{a_1, a_2, \ldots, a_g\}$ contains a k-term arithmetic progression. It is shown that $G(k,2) > \sqrt{\frac{k-1}{2}}\left(\frac{4}{3}\right)^{\frac{k-1}{2}}$, $G(k,3) > \frac{2^{k-2}}{ek}(1+o(1))$, $G(k,2r-1) > \frac{r^{k-2}}{ek}(1+o(1))$, $r \geq 2$.

For positive integers k, r, the van der Waerden number $W(k,r)$ is the least integer such that if $w \geq W(k,r)$, then any partition of $[1,w]$ into r parts has a part that contains a k-term arithmetic progression. The celebrated theorem of van der Waerden [4] proves the existence of $W(k,r)$. The best known upper bound for $W(k,2)$ is enormous whereas the best known lower bound for $W(k,2)$ (see [1]) is

$$W(k,2) > \frac{2^k}{2ek}(1+o(1)) \tag{1}$$

where e is the base of the natural logarithm.

Let $G(k,r)$ denote the smallest positive integer g such that if $1 = a_1, a_2, \ldots, a_g$ is a strictly increasing sequence of integers with bounded gaps $a_{j+1} - a_j \leq r$, $1 \leq j \leq g-1$, then $\{a_1, a_2, \ldots, a_g\}$ contains a k-term arithmetic progression. In [3], Rabung notes that van der Waerden's theorem implies the existence of $G(k,r)$ for all k, r and conversely.

Nathanson makes the following quantitative connection between $W(k,r)$ and $G(k,r)$ [2, Theorem 4]:

$$G(k,r) \leq W(k,r) \leq G((k-1)r+1, 2r-1). \tag{2}$$

In particular, $W(k,2) \leq G(2k-1,3)$, which suggests that it is no easier to find a reasonable upper bound for $G(k,3)$ than it is for $W(k,2)$.

However, $G(k,2)$ "escapes" Nathanson's inequalities in the sense that an upper bound for $G(k,2)$ does not immediately give an upper bound for $W(k,2)$.

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Setting $r = 2$ and combining (1) and (2) gives

$$G(k,3) > \frac{2^{\frac{k+1}{2}}}{e(k+1)}(1+o(1)),$$

but again $G(k,2)$ "escapes" in that no lower bound for $G(k,2)$ can be deduced from Nathanson's inequalities.

In this note we obtain an exponential lower bound for $G(k,2)$ and improved lower bounds for $G(k,r)$, $r > 2$. The Lovász Local Lemma is used when $r > 2$. However when $r = 2$ this method fails, and elementary counting arguments are used.

Theorem 1. *For all $k \geq 1$,*

$$G(k,2) > \sqrt{\frac{k-1}{2}}\left(\frac{4}{3}\right)^{\frac{k-1}{2}}.$$

Proof. We use the following notation. For each positive integer n, let

$$\Omega_n = \{\alpha = a_1, a_2, \ldots, a_n : a_1 = 1,\ 1 \leq a_{j+1} - a_j \leq 2,\ 1 \leq j \leq n-1\},$$

and let $\mathscr{S}_n$ be the set of all k-term arithmetic progressions contained in $[1, 2n-1]$.

Let $i \in [1, 2n-1]$ and $\alpha \in \Omega_n$. We say that i occurs in $\alpha = a_1, a_2, \ldots, a_n$ if $i \in \{a_1, a_2, \ldots, a_n\}$. Similarly, for any subset I of $[1, 2n-1]$, we say that I occurs in α if $I \subseteq \{a_1, a_2, \ldots, a_n\}$ and will write $I \subseteq \alpha$.

Let $k \geq 3$ be fixed and give Ω_n the uniform probability distribution. The idea of the proof is to show that for any k-term arithmetic progression $S \in \mathscr{S}_n$, $\Pr(S \subseteq \alpha) \leq \left(\frac{3}{4}\right)^{k-1}$.

For each i, $1 \leq i \leq 2n-1$, let $A_i = \{\alpha \in \Omega_n : i \text{ occurs in } \alpha\}$. Then $\Pr(A_1) = 1$, $\Pr(A_2) = 1/2$ and $\Pr(A_3) = \frac{3}{4}$.

To show that $\Pr(A_i) \leq 3/4$ for $i > 3$, partition A_i so that it is the disjoint union $A_i = A_i^0 \cup A_i^1 \cup A_i^2$ where $A_i^0 = \{\alpha \in A_i : a_n = i\}$ and for $m = 1, 2$, $A_i^m = \{\alpha \in A_i : a_j = i \Longrightarrow a_{j+1} = i + m\}$. Now $|A_i^1| = |A_i^2|$ and so $\Pr(A_i^m) \leq \frac{1}{2}\Pr(A_i)$, $1 \leq m \leq 2$. Moreover, A_{i+2} is the disjoint union of A_{i+1}^1 and A_i^2, and thus

$$\begin{aligned} \Pr(A_{i+2}) &= \Pr(A_{i+1}^1) + \Pr(A_i^2) \\ &\leq \frac{1}{2}\left(\Pr(A_{i+1}) + \Pr(A_i)\right). \end{aligned}$$

It follows by induction that for $i = 2, 3, \ldots, 2n-1$,

$$\Pr(A_i) \leq \frac{3}{4}. \tag{3}$$

Note that inequality (3) is independent of n. That is, for every $n \geq 1$ and every $i = 2, 3, \ldots, 2n-1$,

$$|\{\alpha \in \Omega_n : i \text{ occurs in } \alpha\}| \leq \frac{3}{4}|\Omega_n| = \frac{3}{4} \cdot 2^{n-1}. \tag{4}$$

Let n be fixed and let I be a non-empty subset of $\{2, 3, \ldots, 2n-1\}$. Let m be the largest element of I and define $A_I = \bigcap_{i \in I} A_i$. We proceed to show that $\Pr(A_I) \leq \left(\frac{3}{4}\right)^{|I|}$.

Define $\tilde{A}_I = \{\tilde{\alpha} = a_1, a_2, \ldots, a_s : a_1 = 1, a_s = m, s \le n, I \text{ occurs in } \alpha\}$ and for each $\tilde{\alpha} \in \tilde{A}_I$, $\tilde{\alpha} = a_1, a_2, \ldots, a_s$, define $B_{\tilde{\alpha}}$ to be the set of all 2^{n-s} "continuations" of $a_1, a_2, \ldots, a_s$. That is, let $B_{\tilde{\alpha}} = \{\alpha \in \Omega_n : \alpha = a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n\}$. Then A_I is the disjoint union

$$A_I = \bigcup_{\tilde{\alpha} \in \tilde{A}_I} B_{\tilde{\alpha}} \tag{5}$$

Let j be such that $m < j \le 2n-1$. We now want to estimate the number of sequences in $B_{\tilde{\alpha}}$ in which j occurs. For each $\tilde{\alpha} \in \tilde{A}_I$, $\tilde{\alpha} = a_1, a_2, \ldots, a_s$, we can map $B_{\tilde{\alpha}}$ onto Ω_{n-s+1} by dropping $a_1, a_2, \ldots, a_{s-1}$ and then shifting $m-1$ units to the left. That is, we map $\alpha \in B_{\tilde{\alpha}}$, $\alpha = a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n$, into $\beta = 1, b_{k+1} - (m-1), \ldots, b_n - (m-1)$. Clearly j occurs in α if and only if $j - (m-1)$ occurs in β.

Using (4) we therefore have

$$\begin{aligned} |\{\alpha \in B_{\tilde{\alpha}} : j \text{ occurs in } \alpha\}| &= |\{\beta \in \Omega_{n-s+1} : j-(m-1) \text{ occurs in } \beta\}| \\ &\le \frac{3}{4}|\Omega_{n-s+1}| \\ &= \frac{3}{4}\cdot 2^{n-s} \\ &= \frac{3}{4}|B_{\tilde{\alpha}}|. \end{aligned} \tag{6}$$

Combining (5) and (6) we obtain

$$\begin{aligned} |A_I \cap A_j| &= \sum_{\tilde{\alpha} \in \tilde{A}_I} |B_{\tilde{\alpha}} \cap A_j| \\ &\le \sum_{\tilde{\alpha} \in \tilde{A}_I} \frac{3}{4}|B_{\tilde{\alpha}}| \\ &= \frac{3}{4}|A_I| \end{aligned} \tag{7}$$

Hence by induction (using (3) and (7)), $\Pr(A_I \cap A_j) \le \frac{3}{4}\Pr(A_I) \le \left(\frac{3}{4}\right)^{|I|+1}$. Note also that $\Pr(A_{I\cup\{1\}}) \le \left(\frac{3}{4}\right)^{|I|}$. In particular, for all $S \in \mathscr{S}_n$, $\Pr(S \subseteq \alpha) \le \left(\frac{3}{4}\right)^{k-1}$.

For each S in $\mathscr{S}_n$, let E_S denote the event "$S \subseteq \alpha$". The probability that some S in $\mathscr{S}_n$ occurs in α satisfies

$$\begin{aligned} \Pr\Big(\bigcup_{S \in \mathscr{S}_n} E_S\Big) &\le \sum_{S \in \mathscr{S}_n} \Pr(E_S) \\ &\le |\mathscr{S}_n| \left(\frac{3}{4}\right)^{k-1} \\ &\le \frac{(2n-1)^2}{2(k-1)} \left(\frac{3}{4}\right)^{k-1}. \end{aligned}$$

If $n < \frac{1}{2} + \sqrt{\frac{k-1}{2}} \left(\frac{4}{3}\right)^{\frac{k-1}{2}}$, then $\frac{(2n-1)^2}{2(k-1)} \left(\frac{3}{4}\right)^{k-1} < 1$ and hence $\Pr(\bigcap_{S \in \mathscr{S}_n} \overline{E_S}) > 0$. That is, there exists $\alpha \in \Omega_n$ that does not contain a k-term arithmetic progression. Therefore $G(k,2) > \sqrt{\frac{k-1}{2}} \left(\frac{4}{3}\right)^{\frac{k-1}{2}}$. □

The proof of Theorem 1 can easily be modified to show that $G(k,r) > \sqrt{\frac{k-1}{2}}\left(\frac{1}{p}\right)^{\frac{k-1}{2}}$ where $p = p(r) = \frac{1}{r}\left(1+\frac{1}{r}\right)^{r-1}$, for all $k \geq 3$, $r \geq 2$. But this is much weaker than the following result.

Theorem 2. *For all $k \geq 1$, $r \geq 2$,*

$$G(k, 2r-1) > \frac{r^{k-2}}{ek}(1+o(1)).$$

Before proving Theorem 2, we state the form of the Lovász Local Lemma we use([1]).

Lovász Local Lemma. Let $A_1, \ldots, A_m$ be events with $\Pr(A_i) \leq p$ for all i. Suppose that each A_i is mutually independent of all but at most d of the other A_j's. If $ep(d+1) < 1$, then $\Pr(\cap \overline{A}_i) > 0$.

Proof of Theorem 2. To simplify the notation, we carry out the proof only in the case $r = 2$. The proof for the general case is essentially the same.

Fix $k \geq 1$ and fix n. Let $\mathscr{M}$ be the set of all sequences $\alpha = a_1, a_2, \ldots, a_n$ such that $a_i \in \{2i-1, 2i\}$, $1 \leq i \leq n$. Thus α contains exactly one of the two elements in each of the *blocks* $[1,2]$, $[3,4]$, $\ldots$, $[2n-1, 2n]$.

Let the symbols S, T denote k-term arithmetic progressions contained in $[1, 2n]$ with common differences at least two. Give $\mathscr{M}$ the uniform probability distribution and again let E_S denote the event "$S \subseteq \alpha$". Then $|\mathscr{M}| = 2^n$ and $|\{\alpha \in \mathscr{M} : S \subseteq \alpha\}| = 2^{n-k}$, so $\Pr(E_S) = 2^{-k}$.

The event E_S is mutually independent of all the other events E_T for all T that have no blocks in common with S (that is, for no i, $1 \leq i \leq n$, is it true that $[2i-1, 2i] \cap S \neq \emptyset$ and $[2i-1, 2i] \cap T \neq \emptyset$). To see this, note that a random $\alpha \in \mathscr{M}$ can be constructed by randomly and independently choosing each element a_i from $[2i-1, 2i]$ with uniform probability. Thus even if we know the chosen element of α for each block besides those of S, the probability of E_S remains unchanged, and any assumption on the events E_T for T that have no blocks in common with S is determined by these chosen elements.

For each S, the number of T such that S and T do have a block in common is bounded above by $4nk$. (To see this note that the number of k-term arithmetic progressions in $[1, 2n]$ which contain any given element of $[1, 2n]$ is bounded above by $2n$ (in fact, by about $(\log 2)(2n)$). Since S meets k blocks, T will have a block in common with S only if T contains one of the $2k$ elements of these k blocks.)

Now we can apply the Lovász Local Lemma with $p = 2^{-k}$, $d = 4nk$. If $n < \frac{2^{k-2}}{ek}(1-\varepsilon)$, then $ep(d+1) < 1$, so $\Pr(\cap \overline{E_S}) > 0$. Therefore if $n < \frac{2^{k-2}}{ek}(1-\varepsilon)$, there is $\alpha \in \mathscr{M}$, $\alpha = a_1, a_2, \ldots, a_n$, which contains no k-term arithmetic progression. Since $a_{j+1} - a_j \leq 3$ for all j, this shows that $G(k,3) > \frac{2^{k-2}}{ek}(1+o(1))$. $\square$

References

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