Monochromatic Homothetic Copies of $\{1,1+s,1+s+t\}$

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Abstract

For positive integers s and t, let $f(s,t)$ denote the smallest positive integer N such that every 2-coloring of $[1,N]=\{1,2,\ldots,N\}$ has a monochromatic homothetic copy of $\{1,1+s,1+s+t\}$.

We show that $f(s,t)=4(s+t)+1$ whenever s/g and t/g are not congruent to 0 (modulo 4), where $g=\gcd(s,t)$. This can be viewed as a generalization of part of van der Waerden's theorem on arithmetic progressions, since the 3-term arithmetic progressions are the homothetic copies of $\{1,1+1,1+1+1\}$. We also show that $f(s,t)=4(s+t)+1$ in many other cases (for example, whenever $s>2t>2$ and t does not divide s), and that $f(s,t)\leq 4(s+t)+1$ for all s,t.

Thus the set of homothetic copies of $\{1,1+s,1+s+t\}$ is a set of triples with a particularly simple Ramsey function (at least for the case of two colours), and one wonders what other "natural" sets of triples, quadruples, etc., have simple (or easily estimated) Ramsey functions.

1 Introduction

Van der Waerden's Theorem on Arithmetic Progressions [5] states that for every positive integer k there exists a smallest positive integer $w(k)$ such that for every 2-coloring of $[1,w(k)]=\{1,2,\ldots,w(k)\}$, there is a monochromatic k-term arithmetic progression. (In other words, if $[1,w(k)]$ is partitioned in any way into two parts A and B, then either A or B must contain a k-term arithmetic progression.) The only known non-trivial values of $w(k)$ are $w(3)=9$, $w(4)=35$, $w(5)=178$. Furthermore the estimation of the function $w(k)$ for large k is one of the most outstanding (and presumably one of the most difficult) problems in Ramsey theory. For a discussion of this, see [2].

The function $w(k)$ is often called the *Ramsey function* for the set of k-term arithmetic progressions. Landman and Greenwell [3,4] considered the Ramsey function $g(n)$ of the set of all n-term sequences that are homothetic copies (see the definition below) of $\{1,2,2+t,2+t+t^2,\ldots,2+t+t^2+\cdots+t^{n-2}\}$ for some positive integer t. They obtained a lower bound for $g(n)$ and an upper bound for $g^{(r)}(3)$, where the (r) indicates that r colours are used. Other "substitutes" for the set of k-term arithmetic progressions were introduced in [1].

In contrast, in this paper we consider the Ramsey function associated with a much smaller set of sequences, namely the set of homothetic copies of $\{1,1+s,1+s+t\}$ for given positive integers s and t.

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A *homothetic copy* of $\{1, 1+s, 1+s+t\}$ is any set of the form $\{x, x+ys, x+ys+yt\}$, where x and y are positive integers. From now on, let us agree to use the term "(s,t)-progression" to refer to a homothetic copy of $\{1, 1+s, 1+s+t\}$.

Instead of considering 3-term arithmetic progressions, as in the case $k=3$ of van der Waerden's theorem, we consider the set of (s,t)-progressions for given positive integers s and t. (Note that the $(1,1)$-progressions are the 3-term arithmetic progressions.)

For positive integers s and t we define $f(s,t)$ to be the smallest positive integer N such that every 2-colouring of $[1,N]$ has a monochromatic (s,t)-progression. Note that $f(s,t) = f(t,s)$. We will use this fact several times.

We show that for all positive integers s and t, if $s/g \not\equiv 0$ and $t/g \not\equiv 0 \pmod 4$, where $g = \gcd(s,t)$, then $f(s,t) = 4(s+t)+1$. A special case of this is $w(3) = f(1,1) = 9$. Thus this result can be viewed as a generalization of the case $k=3$ of van der Waerden's theorem.

We also show that $f(s,t) \leq 4(s+t)+1$ for all s and t, and we show that even if $s/g \equiv 0$ or $t/g \equiv 0$ (mod 4), the equality $f(s,t) = 4(s+t)+1$ still holds, except for a small number of possible exceptions. For example, we are unable to find the exact value of $f(4m,1)$, although we show in Theorem 4 that $4(4m+1) \leq f(4m,1) \leq 4(4m+1)+1$. The remaining cases where $f(s,t)$ is unknown are described in Section 4.

2 Upper bounds

First we give a simple proof of the weak bound $f(s,t) \leq 9s+8t$, which is subsequently refined (in Theorem 2 below) to give the stronger bound $f(s,t) \leq 4(s+t)+1$. The equality $w(3) = 9$ will be used in our proof of this weak bound, but will not be used again.

We prove $f(s,t) \leq 9s+8t$ by contradiction. Assume that $f(s,t) > 9s+8t$, and let $[1, 9s+8t]$ be 2-coloured, using the colours Red and Blue, in such a way that there is no monochromatic (s,t)-progression. Since $w(3) = 9$, the set $\{s, 2s, 3s, \ldots, 9s\}$ contains a monochromatic (say in the colour Red) 3-term arithmetic progression. Let us suppose, in order to simplify our notation, that this Red progression is $\{s, 5s, 9s\}$. (In all other cases, the argument is essentially the same.)

Consider the (s,t)-progressions $\{s, 5s, 5s+4t\}$, $\{5s, 9s, 9s+4t\}$, $\{s, 9s, 9s+8t\}$. Since by assumption none of these is monochromatic, and s, $5s$, $9s$ are all Red, it follows that $\{5s+4t, 9s+4t, 9s+8t\}$ is a Blue (s,t)-progression, a contradiction, completing the proof.

The following theorem will be useful in obtaining both upper and lower bounds for $f(s,t)$.

Theorem 1. *Let s,t,c be positive integers. Then $f(cs,ct) = c(f(s,t)-1)+1$.*

Proof. Let $M = f(s,t)$. Let B be a 2-colouring of $[1, c(M-1)+1]$. Since every 2-colouring of $[0, M-1]$ contains a monochromatic (s,t)-progression, every 2-colouring of $\{1, c+1, 2c+1, \ldots, (M-1)c+1\}$ contains a monochromatic (cs,ct)-progression. Thus, $f(cs,ct) \leq c(M-1)+1$.

On the other hand, we know there is a 2-colouring, B, of $[1, M-1]$ that contains no monochromatic (s,t)-progressions. Define B' on $[1, c(M-1)]$ by $B'([c(i-1)+1, ci]) = B(i)$, for $i = 1, \ldots, M-1$. We will show that B' avoids monochromatic (cs,ct)-progressions, which will complete the proof.

Assume, by way of contradiction, that x_1, x_2, x_3 is a (cs, ct)-progression, contained in $[1, c(M-1)]$, that is monochromatic under B'. Then there exists $r > 0$ such that $x_3 - x_2 = rct$, $x_2 - x_1 = rcs$. Let $y_j = \lceil x_j/c \rceil$ for $j = 1,2,3$. Then $y_3 - y_2 = \lceil x_3/c \rceil - \lceil x_2/c \rceil = rt$, and similarly $y_2 - y_1 = rs$.

Hence y_1, y_2, y_3 is an (s,t)-progression. Also, $B(y_j) = B(\lceil x_j/c \rceil) = B'(x_j)$, for each j. This contradicts our assumption that there is no monochromatic (s,t)-progression relative to the colouring B. $\square$

Note that this proof easily extends to a proof that if $f(a_1, \dots, a_k) = M$, then $f(ca_1, \dots, ca_k) = c(M-1)+1$, where $f(a_1, \dots, a_k)$ denotes the least positive integer N such that every 2-coloring of $[1,N]$ will contain a monochromatic homothetic copy of $\{1, 1+a_1, 1+a_1+a_2, \dots, 1+a_1+a_2+\cdots+a_k\}$.

Theorem 2. *For all positive integers s and t, $f(s,t) \le 4(s+t)+1$.*

Proof. Let s,t be given. We may assume without loss of generality that $s \le t$. We may also assume that $\gcd(s,t) = 1$, for if we knew the result in this case then, with $g = \gcd(s,t)$, Theorem 1 would give $f(s,t) = g\lceil f(s/g, t/g) - 1 \rceil + 1 \le g[4(s/g + t/g) + 1 - 1] + 1 = 4(s+t)+1$.

Consider the following set of 20 triples contained in $[1, 4(s+t)+1]$, which are all (s,t)-progressions:

$$\{1, s+1, s+t+1\}, \{s+1, 2s+1, 2s+t+1\}$$

$$\{2s+1, 3s+1, 3s+t+1\}, \{3s+1, 4s+1, 4s+t+1\}$$

$$\{1, 2s+1, 2s+2t+1\}, \{s+1, 3s+1, 3s+2t+1\}$$

$$\{2s+1, 4s+1, 4s+2t+1\}, \{1, 3s+1, 3s+3t+1\}$$

$$\{s+1, 4s+1, 4s+3t+1\}, \{1, 4s+1, 4s+4t+1\}$$

$$\{s+t+1, 2s+t+1, 2s+2t+1\}, \{2s+t+1, 3s+t+1, 3s+2t+1\}$$

$$\{3s+t+1, 4s+t+1, 4s+2t+1\}, \{s+t+1, 3s+t+1, 3s+3t+1\}$$

$$\{2s+t+1, 4s+t+1, 4s+2t+1\}, \{s+t+1, 4s+t+1, 4s+4t+1\}$$

$$\{2s+2t+1, 3s+2t+1, 3s+3t+1\}, \{3s+2t+1, 4s+2t+1, 4s+3t+1\}$$

$$\{2s+2t+1, 4s+2t+1, 4s+4t+1\}, \{3s+3t+1, 4s+3t+1, 4s+4t+1\}$$

It is straightforward to check (under the assumptions that $s \le t$ and $\gcd(s,t) = 1$) that except in the cases $s = 1, 1 \le t \le 3$, the 15 integers which appear in these 20 triples are distinct. It is then a simple matter to check all 2-colourings of these 15 integers and verify that each 2-colouring has a monochromatic triple from the above list of 20 triples. (If one identifies these 15 integers with the numbers 1,2,...,15 via the correspondence

$$1 \leftrightarrow 1, s+1 \leftrightarrow 2, 2s+1 \leftrightarrow 3, 3s+1 \leftrightarrow 4, 4s+1 \leftrightarrow 5,$$

$$s+t+1 \leftrightarrow 6, 2s+t+1 \leftrightarrow 7, 3s+t+1 \leftrightarrow 8, 4s+t+1 \leftrightarrow 9,$$

$$2s+2t+1 \leftrightarrow 10, 3s+2t+1 \leftrightarrow 11, 4s+2t+1 \leftrightarrow 12,$$

$$3s+3t+1 \leftrightarrow 13, 4s+3t+1 \leftrightarrow 14, 4s+4t+1 \leftrightarrow 15,$$

the resulting set of 20 triples contained in $[1,15]$ has a particularly pleasing form.) The cases $s=1, 1 \le t \le 3$ can be checked separately. In all cases we obtain $f(s,t) \le 4(s+t)+1$. □

3 Lower bounds and exact values for $f(s,t)$

Theorem 3. *Let s,t be positive integers, and let $g = \gcd(s,t)$. If $s/g \not\equiv 0$ and $t/g \not\equiv 0 \pmod 4$ then $f(s,t) = 4(s+t)+1$.*

Proof. The proof splits naturally into two cases.

Case 1. Assume that s/g and t/g are both odd. In view of Theorem 2, we only need to show that $f(s,t) \ge 4(s+t)+1$.

First, assume $g=1$. Now colour $[1,4(s+t)]$ as

$$101010\cdots10\ 010101\cdots01,$$

where each of the two long blocks has length $2(s+t)$. Assume x,y,z is a monochromatic (s,t)-progression. Then $y = x+ds$ and $z = y+dt$, for some positive integer d. Let B_1 and B_2 represent $[1,2(s+t)]$ and $[2(s+t)+1,4(s+t)]$, respectively.

In case d is odd, then x and y have opposite parity, and y and z have opposite parity. Since x and y have the same colour and opposite parity, x is in B_1, while y is in B_2. Hence z is in B_2, so that y and z cannot have the same colour, a contradiction.

If d is even, then x, y and z all have the same parity, so they all must be in the same B_i. But then $d(s+t) = z-x \le 2(s+t)$, and hence $d=1$, a contradiction.

If g is unequal to 1, then by Theorem 1 and the case in which $g=1$, $f(s,t) = g[f(s/g,t/g)-1]+1 \ge g[4(s/g+t/g)+1-1]+1 = 4(s+t)+1$. This finishes the proof of Case 1.

Case 2. Assume without loss of generality that $s/2 \equiv 2 \pmod 4$. First we assume that $g=1$. Then $s \equiv 2 \pmod 4$ and t is odd.

By Theorem 2, we only need to provide a 2-colouring of $[1,4(s+t)]$ that contains no monochromatic (s,t)-progression. Let C be the colouring $11001100\cdots1100$ (i.e., $s+t$ consecutive blocks each having the form 1100).

We proceed by contradiction. Assume that x,y,z is a monochromatic (s,t)-progression. So there exists a $d>0$ such that $y-x = ds$ and $z-y = dt$. By the way C is defined, if $C(i) = C(j)$ and $j-i$ is even, then 4 divides $j-i$. Now since $z-x = d(s+t) \le 4(s+t)-1$, we must have that $d<4$. The case $d=2$ is impossible, for if $d=2$, then $C(z) = C(x)$, $z-x = d(s+t)$ is even, but 4 does not divide $z-x$, a contradiction. Hence d is odd. But then, since $s \equiv 2 \pmod 4$, $y-x$ is even yet 4 doesn't divide $y-x$, again a contradiction.

This shows that $f(s,t) \ge 4(s+t)+1$ in the case $g=1$.

If g is unequal to 1, we proceed just as at the end of Case 1. □

Suppose that $s/g \equiv 0 \pmod 4$, where $g = \gcd(s,t)$. Then t/g is odd, and in the case $t/g = 1$, that is, t divides s, we have the following result.

Theorem 4. *Let m,t be positive integers. Then either*

$$f(4mt,t) = 4(4mt+t)-t+1 \text{ or } f(4mt,t) = 4(4mt+t)+1.$$

Proof. By Theorem 1, it is sufficient to show that $4(4m+1) \leq f(4m,1) \leq 4(4m+1)+1$. By Theorem 2, we only need to show that $4(4m+1) \leq f(4m,1)$. Thus it suffices to find a 2-colouring of $[1,16m+3]$ that avoids monochromatic $(4m,1)$-progressions. Let χ be the colouring $1A0B0C1D0$, where

$$A = 00110011\cdots0011 \text{ has length } 4m$$
$$B = 11001100\cdots11 \text{ has length } 4m-2$$
$$C = 11001100\cdots1100 \text{ has length } 4m$$
$$D = 00110011\cdots0011 \text{ has length } 4m.$$

Assume x,y,z is a monochromatic $(4m,1)$-progression. We shall reach a contradiction. We know there exists a positive integer d such that $y-x=4md$ and $z-y=d$. Hence, $d(4m+1) \leq 16m+2$, so that $d \leq 3$. Let

$$S_1 = [2,4m+1] \text{ (corresponds to } A \text{ above)}$$
$$S_2 = [4m+3,8m] \text{ (corresponds to } B \text{ above)}$$
$$S_3 = [8m+2,12m+1] \text{ (corresponds to } C \text{ above)}$$
$$S_4 = [12m+3,16m+2] \text{ (corresponds to } D \text{ above).}$$

Case 1. $d=1$. Then y,z belong to the same S_i, for some $1 \leq i \leq 4$. Denote by $S(i,j)$ the jth element of S_i. We see that $y = S(i,j)$ for some odd j. Note that for each even p, if $i=2$ or 4, then $\chi(S(i-1),p)$ is unequal to $\chi(S(i,p-1))$. Now if $i=2$ or $i=4$, then $x = y-4m = S(i-1,j+1)$, so that (by the preceding remark), $\chi(x)$ is different from $\chi(y)$, a contradiction. Now if $i=3$ and $j>1$, then $y-4m = S(2,j-1)$, and $\chi(x) = \chi(y-4m)$ is unequal to $\chi(y)$, a contradiction. If $i=3$ and $j=1$, then $x=4m+2$ and $y=8m+2$, and these again have different colours.

Case 2. $d=2$. Then $y-x=8m$ and $z-y=2$. If $\chi(y)=\chi(z)$ then y must be one of the following: $4m+1, 8m, 12m+1$; and since $y-x=8m$, this reduces the possibilities for y to only $12m+1$. However we see that $\chi(4m+1)$ is unequal to $\chi(12m+1)$, a contradiction.

Case 3. $d=3$. Then $y-x=12m$ and $z-y=3$. Clearly x belongs to $[1,4m]$, so that y belongs to $[12m+1,16m]$. Now $[1,4m]$ has colouring $1\ 0011\cdots001100\ 1$ while $[12m+1,16m]$ has colouring $0100110011\cdots001100$. Hence, since $\chi(x)=\chi(y)$, y belongs to the set $\{12m+3, 12m+5, 12m+7, \ldots, 16m-1\}$. Now z belongs to $[12m+4,16m+3]$, so let's compare the colouring of $[12m+1,16m]$ to that of $[12m+4,16m+3]$: $[12m+1,16m]$ has colouring as noted above, while $[12m+4,16m+3]$ has colouring $0\ 11001100\cdots11\ 0$. Hence, in order for $\chi(y)=\chi(z)$, y must belong to the set $\{12m+1, 12m+2, 12m+4, 12m+6, \ldots, 16m\}$, a contradiction. □

Theorem 5. *Let s,t be positive integers such that $s > t > 1$ and t does not divide s. If $\lfloor s/t \rfloor$ is even or $\lfloor 2s/t \rfloor$ is even, where $\lfloor \cdot \rfloor$ is the floor function, then $f(s,t) = 4(s+t)+1$. If $\lfloor s/t \rfloor$ and $\lfloor 2s/t \rfloor$ are both odd, then $f(s,t) = 4(s+t)+1$ provided s,t satisfy the additional condition $s/t \notin (1.5,2)$.*

Proof. Let s,t satisfy the hypotheses of the theorem. By Theorems 1 and 2, it suffices to show that $f(s,t) \geq 4(s+t)+1$ under the additional assumption that $\gcd(s,t) = 1$, hence throughout the proof we assume $\gcd(s,t) = 1$.

Let $a = \lfloor s/t \rfloor$ and $b = \lfloor 2s/t \rfloor$. Then $s = at + r$, where $0 < r < t$. Also, $2s = 2at + 2r$, so if $2r = t$ we would have $t = 2$. However, since $\gcd(s,t) = 1$, the case $t = 2$ is already covered by Theorem 3. Therefore we assume throughout that proof that $2r \neq t$.

Case 1. We assume that a is even and b is odd. Then $b = 2a+1$, $2r > t$, and $2(s+t) = 2(at+r)+2t = (b-1)t+2r+2t = (b+2)t+(2r-t)$.

Hence we can colour $[1,4(s+t)]$ as follows. Let

$$C = QRQR\cdots QRQJ\ RQRQ\cdots RQRJ',$$

where $Q = 11\cdots1$ and $R = 00\cdots0$ each have length t, $J = 00\cdots0$ and $J' = 11\cdots1$ each have length $2r-t$, and where each of Q and R appears $b+2$ times.

Suppose x,y,z is any (s,t)-progression in $[1,4(s+t)]$ with $y-x = ds$, $z-y = dt$. We will show that $\{x,y,z\}$ is not monochromatic. Clearly $d \leq 3$, since $d(s+t)z - x \leq 4(s+t)-1$.

If $d = 2$, then $z - x = 2(s+t)$, so $C(z) \neq C(x)$. (This is because the colouring on the second half of $[1,4(s+t)]$ is the reversal of the colouring on the first half.)

If $d = 3$, then, since $z = y+3t$ and $C(i) \neq C(i+t)$ for all $i > 2(s+t)$, if $C(y) = C(z)$ we must have $y \leq 2(s+t)$; but then $x = y-3s \leq 2t-s$. However, the conditions $s > t$, $s = at+r$, $0 < r < t$, a even, imply that $s > 2t$, hence $x < 0$, a contradiction.

Now assume that $d = 1$ and $C(y) = C(z)$. Since $z = y+t$, y must occur in the block J, so $C(y) = 0$. Since J has length $2r-t < r$, we see that $y-r$ must occur in the block Q just to the left of block J, so that $y - at - r = x$ also occurs in a block Q, and $C(x) = 1$.

Hence there is no monochromatic (s,t)-progression with respect to the colouring C, therefore $f(s,t) \geq 4(s+t)+1$. This finishes Case 1.

Case 2. We assume that a is odd and b is even. Again we have $s = at+r$, $0 < r < t$, but now $b = 2a$, $2r < t$, and $2(s+t) = (b+2)t+2r$.

Now colour $[1,4(s+t)]$ with the colouring

$$D = QRQR\cdots QRK\ RQRQ\cdots RQK',$$

where Q,R are defined as in Case 1, and $K = 11\cdots1, K' = 00\cdots0$ each have length $2r$.

Assume x,y,z is an (s,t)-progression contained in $[1,4(s+t)]$, with $y-x = ds$, $z-y = dt$; then $d \leq 3$.

If $d = 2$, then as in Case 1, $D(x) \neq D(z)$.

If $d = 3$, and $D(y) = D(z)$, then as in Case 1, $y \leq 2(s+t)$. In fact, since K and R have opposite colours, $y \leq 2(s+t)-2r$. On the other hand, $y \geq 1+3s \geq 2s+t+r+1$, so y is an element of the last occurrence of R in $[1,2(s+t)]$, hence $D(y)=0$. Then $x=y-3s\leq 2(s+t)-2r-3s<t$, so $D(x)=1$ and $D(x)\neq D(y)$.

Now assume $d=1$ and $D(y)=D(z)$. Then y belongs to the last occurrence of R in $[1,2(s+t)]$, and $y\equiv i \pmod t$, where $2r<i\leq t$. Hence, since a is odd, $x=y-(at+r)$ lies in one of the Q's, and $D(x)=1$, $D(y)=0$.

Thus, no monochromatic (s,t)-progression exists in $[1,4(s+t)]$, hence $f(s,t)\geq 4(s+t)+1$.

Case 3. We assume that both a and b are even. Then $s=at+r$, $b=2a$, $0<2r<t$, and $2(s+t)=(b+2)t+2r$. Note that $a\geq 2$, since $s>t$.

We define the colouring E on $[1,4(s+t)]$ as follows. Let us use the notation $\sim 0=1$ and $\sim 1=0$. Then we define, in turn,

(1) $E(i)=1$, $1\leq i\leq r$,

(2) $E(i)=\sim E(i-r)1$, $r<i\leq t$,

(3) $E(i)=\sim E(i-t)$, $t<i\leq 2(s+t)$,

(4) $E(i)=\sim E(i-2(s+t))$, $2(s+t)<i\leq 4(s+t)$.

That is,

$$E=XYXY\cdots XYL\,YXYX\cdots YXL',$$

where X has length t and consists of $\lfloor t/r\rfloor$ blocks, each block of length r, followed by a single block of length $t-\lfloor t/r\rfloor r$, the blocks alternating in colour; Y is the same as X, except the colours are reversed; L is X restricted to $[1,2r]$; and L' is the same as L, except the colors are reversed.

Let x,y,z be an (s,t)-progression contained $[1,4(s+t)]$, with $y-x=ds$, $z-y=dt$.

If $d=2$, then by (4), $E(x)=\sim E(z)$.

If $d=3$ and $E(y)=E(z)$, then $y\leq 2(s+t)$, hence $x=y-3s\leq 2t-s=2t-(at+r)\leq -r<0$, a contradiction.

If $d=1$ and $E(y)=E(z)$, then $y\leq 2(s+t)$. We consider two subcases.

The first subcase is $y\equiv i \pmod t$, $r+1\leq i\leq t$. Then y and $y-r$ are in the same block (X,Y or L) hence by (2) $E(y)=\sim E(y-r)$. By (3), and the fact that a is even, $E(y)=\sim E(y-r)=\sim E(y-at-r)=\sim E(x)$.

The second subcase is $y\equiv i \pmod t$, $1\leq i\leq r$. Since $E(y)=E(z)=E(y+t)$, y must belong to the block L, that is, $y=(b+2)t+i=(2a+2)t+i$, $1\leq i\leq r$. Since $x=y-s=(2a+2)t+i-at-r=(a+1)t+(i+t-r)$, and $1\leq i+t-r\leq t$, by (3) $E(x)=\sim E(i+t-r)$. Also, since $y=2(s+t)-2r+i$, we have $z=y+t=2(s+t)+(i+t-2r)$, so by (4), $E(z)=\sim E(i+t-2r)$. Since $1\leq i+t-2r\leq t$, (2) gives $E(z)=E(i+t-r)=\sim E(x)$.

Thus, under the colouring E, there is no monochromatic (s,t)-progression in $[1,4(s+t)]$, hence $f(s,t)\geq 4(s+t)+1$.

Case 4. Assume that both a and b are odd, and $s/t\notin(1.5,2)$. It follows that $s=at+r$, $0<r<t$, $b=2a+1$, $t<2r$, and $2(s+t)=(b+2)t+(2r-t)$. Also, $a\geq 3$, as a consequence of the assumption $s/t\notin(1.5,2)$.

Let $p=t-r$. Then $p<t/2$. Define the colouring F by setting, in turn,

(5) $F(i) = 1$, $1 \le i \le p$,

(6) $F(i) = \sim F(i-p)$, $p < i \le p$,

(7) $F(i) = \sim F(i-t)$, $t < i \le 2(s+t)$,

(8) $F(i) = \sim F(i-2(s+t))$, $2(s+t) < i \le 4(s+t)$,

That is,

$$F = ABAB \cdots ABAM\ BABA \cdots BABM',$$

where A and B are the same as blocks X and Y in Case 3, except that p replaces r; M is B restricted to $[1, 2r-t]$; and M' is the same as M with the colours interchanged.

Let x, y, z be an (s,t)-progression contained in $[1, 4(s+t)]$, with $y - x = ds$, $z - y = dt$.

If $d = 2$, then by (8), $E(x) = \sim E(z)$.

If $d = 3$ and $E(y) = E(z)$, then $y \le 2(s+t)$, hence (since $a \ge 3$) $x = y - 3s \le 2t - s = 2t - (at + r) < 0$, a contradiction.

If $d = 1$ and $E(y) = E(z)$, then $y \le 2(s+t)$, and we again consider two subcases.

The first subcase is $y = ut + i$, $1 \le i \le r$. Then $1 \le i < i + p = i + t - r \le t$, so by (6), $F(i+p) = \sim F(i)$. Using (7) and the oddness of a, we get $F(x) = F(y - at - r) = F(ut - (a+1)t + i + t - r) = F(ut + i + t - r) = F(ut + i + p) = \sim F(ut + i) = \sim F(y)$.

The second subcase is $y = ut + i$, $r + 1 \le i \le t$. Since $F(y) = F(y+t)$ and M has fewer than i elements, y must belong to the last occurrence of the block A in $[1, 2(s+t)]$. Since $2(s+t) = (b+2)t + (2t - r)$, this means that $y = (b+1)t + i$, hence by (7), $F(y) = F(i)$. Since $x = y - at - r = (b+1)t + i - at - r$, we have $F(x) = \sim F(i - r) = F(i + t - r) = F(i + p) = \sim F(i) = \sim F(y)$.

Thus, under the colouring F, there is no monochromatic (s,t)-progression in $[1, 4(s+t)]$, hence $f(s,t) \ge 4(s+t) + 1$. □

4 Remarks

By Theorems 1 and 3, we would know the value of $f(s,t)$ for all s,t provided we knew the value of $f(4m, t)$ when t is odd, and $\gcd(m,t) = 1$. (Here we are using $f(s,t) = f(t,s)$.) Theorem 4 shows $4(4m+1) \le f(4m, 1) \le 4(4m+1) + 1$. Theorem 5 takes care of many of the cases where $t > 1$. For example, Theorem 5 shows that $f(4m, 3) = 4(4m+3) + 1$ whenever 3 does not divide m. By examining the cases not covered by Theorem 5, one sees that these are exactly the cases $f(t+r, t)$ where $0 < r < t < 2r$, and 4 divides t or 4 divides $t + r$.

The computations $f(4,1) = 20$, $f(8,1) = 36$, $f(12,1) = 52$ suggest that perhaps $f(4m, 1) = 4(4m+1)$ for all $m \ge 1$.

For positive integers $r, a_1, \ldots, a_n$, let $f^{(r)}(a_1, \ldots, a_n)$ denote the smallest positive integer N such that every r-colouring of $[1, N]$ ha a monochromatic homothetic copy of $\{1, 1 + a_1, \ldots, 1 + a_1 + \cdots + a_n\}$. Of course $f^{(r)}(a_1, \ldots, a_n)$ always exists (by a statement of van der Warden's theorem which involves any number of colours), but perhaps one can say something about the rate of growth of $f^{(r)}(a_1, \ldots, a_n)$ as a function of $a_1 + \cdots + a_n$. The computations $f^{(2)}(1,1,1) = 35$, $f^{(2)}(1,1,2) = 38$, $f^{(2)}(1,1,3) = 44$,

$f^{(2)}(1,1,4) = 56$, $f^{(2)}(1,1,5) = 59$ suggest that $f^{(2)}(1,1,n)$ does not grow linearly with n. Perhaps $f^{(2)}(1,1,n) \sim c2^n$.

We have no idea of the growth rate of $f^{(3)}(s,t)$ as a function of $s+t$.

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