The Ramsey property for collections of sequences not containing all arithmetic progressions

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Abstract

A family $\mathscr{B}$ of sequences has the Ramsey property if for every positive integer k, there exists a least positive integer $f_{\mathscr{B}}(k)$ such that for every 2-coloring of $\{1,2,\ldots,f_{\mathscr{B}}(k)\}$ there is a monochromatic k-term member of $\mathscr{B}$. For fixed integers $m>1$ and $0 \le q < m$, let $\mathscr{B}_{q(m)}$ be the collection of those increasing sequences of positive integers $\{x_1,\ldots,x_k\}$ such that $x_{i+1}-x_i \equiv q \pmod m$ for $1 \le i \le k-1$. For t a fixed positive integer, denote by $\mathscr{A}_t$ the collection of these arithmetic progressions having constant difference t. Landman and Long showed that for all $m \ge 2$ and $1 \le q < m$, $\mathscr{B}_{q(m)}$ does not have the Ramsey property, while $\mathscr{B}_{q(m)} \cup \mathscr{A}_m$ does. We extend these results to various finite unions of $\mathscr{B}_{q(m)}$'s and $\mathscr{A}$'s. We show that for all $m \ge 2$, $\bigcup q = 1m-1\mathscr{B}_{q(m)}$ does not have the Ramsey property. We give necessary and sufficient conditions for collections of the form $\mathscr{B} \cup (\bigcup_{t\in T} \mathscr{A}_t)$ to have the Ramsey property. We determine when collections of the form $\mathscr{B}_{a(m_1)} \cup \mathscr{B}_{b(m_2)}$ have the Ramsey property. We extend this to the study of arbitrary finite unions of $\mathscr{B}_{q(m)}$'s. In all cases considered for which $\mathscr{B}$ has the Ramsey property, upper bounds are given for $f_{\mathscr{B}}$.

1 Introduction

One of the oldest and most well-known theorems of Ramsey theory is van der Waerden's theorem, which says that for all k and r, there is a least positive integer $w(k,r)$ such that for every partition of $\{1,2,\ldots,w(k,r)\}$ into r classes, at least one of the classes contains an arithmetic progression of length k [13]. The estimation of $w(k,r)$ has been, and remains, one of the more intriguing and difficult Ramsey theory problems. In particular, it is still unknown whether $w(k,2)$ (usually denoted more simply by $w(k)$) is bounded above by a tower of k's having height k, while the best lower bounds for $w(k)$ are of a much smaller order of magnitude (see [3] for a general discussion).

More generally, if $\mathscr{B}$ represents any particular collection of sequences, we say $\mathscr{B}$ has the *Ramsey property* if for every positive integer k, there exists a least positive integer $f_{\mathscr{B}}(k)$ such that for every partition of $\{1,2,\ldots,f_{\mathscr{B}}(k)\}$ into two classes, at least one of the classes will contain a k-term member of $\mathscr{B}$. Clearly, whenever $\mathscr{B} \subseteq \mathscr{C}$ and $\mathscr{B}$ has the Ramsey property, we know that $\mathscr{C}$ has the Ramsey property

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and that $f_{\mathscr{C}}(k) \le f_{\mathscr{B}}(k)$ for all k. In particular, any collection $\mathscr{C}$ of sequences that contains the arithmetic progressions has the Ramsey property; the corresponding Ramsey functions $f_{\mathscr{C}}$ have been studied for a variety of such collections $\mathscr{C}$ [2,4,6–12].

For collections not containing the arithmetic progressions, the behavior of the associated Ramsey functions can be somewhat unpredictable. Of course, if the collection is too small, it will not have the Ramsey property. For example, if d is a fixed positive integer, let $\mathscr{A}_d$ consist only of those arithmetic progressions for which the difference between consecutive terms is d. Then it is a trivial matter to partition the positive integers into two classes so as to avoid even 2-term members of $\mathscr{A}_d$, and hence $\mathscr{A}_d$ does not have the Ramsey property. In fact, it is known that if T is any finite set of positive integers, then the family $\bigcup_{d\in T}\mathscr{A}_d$ does not have the Ramsey property [1]. For fixed integers $m \ge 2$ and $0 \le q < m$ define a $q \pmod m$-*sequence* to be an increasing sequence of positive integers $\{x_1,\dots,x_k\}$ such that $x_{i+1} - x_i \equiv q \pmod m$ for $1 \le i \le k-1$. It was shown in [5] that, for fixed m and q with $q \ne 0$, the family of all $q \pmod m$-sequences, $\mathscr{B}_{q(m)}$, does not have the Ramsey property (the associated Ramsey function $f_{q(m)}(k)$ is undefined for all $k > 2$). This is not surprising, if we think of a $q \pmod m$-sequence as an arithmetic progression, modulo m, with constant difference q. Less expectedly, it turns out [5] that for all m and q, the family $\mathscr{B}_{q(m)} \cup \mathscr{A}_m$ does have the Ramsey property, and its associated Ramsey function, which we denote by $f_{q(m),m}(k)$, satisfies

$$f_{q(m),m}(k) = mk^2(1+o(1)).$$

In this paper we consider several situations that are more general than those dealt with in [5]. If $S = \{a_i \pmod{m_i}) : 1 \le i \le n\}$ is any set of congruence classes, we consider the family of sequences that are $a_i \pmod{m_i}$-sequences for some i. For example, if

$$S = \{1 \pmod 4, 2 \pmod 3)\},$$

then the sequences $\{1,2,11\}$ and $\{1,9,11\}$ belong to the family, while $\{1,2,4\}$ and $\{1,9,10\}$ do not. We ask: (i) For which choices of S will the family of sequences have the Ramsey property? (ii) For such S, what is the rate of growth of the associated Ramsey function $f_S(k)$?

In particular, we look at the family of arithmetic progressions modulo m, i.e., $\bigcup_{q=1}^{m-1}\mathscr{B}_{q(m)}$. It is clear that for almost all choices of k, m and q, the number of k-term arithmetic progressions modulo m is considerably greater than the number of sequences belonging to $\mathscr{B}_{q(m)} \cup \mathscr{A}_m$. Yet, in Section 2 we show that the arithmetic progressions modulo m do not have the Ramsey property. As a corollary, if T is a set of positive integers, we are able to characterize those families $\mathscr{B}_{q(m)} \cup (\bigcup_{t\in T}\mathscr{A}_t)$ which have the Ramsey property, and we give upper bounds for the associated Ramsey functions $f_{q(m),T}$, which generalize known results about $f_{q(m),m}$ from [5].

In Section 3, we first consider $|S| = 2$, and characterize those S that give rise to the Ramsey property. We also give an upper bound for the associated Ramsey function. We then look at the function f_S for general S. We give sufficient conditions for $f_S(k)$ to be defined for all k, and show that for a large variety of S's, these conditions are also necessary. In Section 4 we consider a special case involving partitions into $r \ge 2$ classes, and obtain an upper bound for the associated Ramsey function, where S consists of r congruences. Section 5 includes some observations, based on computer output, concerning the sharpness of the bounds.

We make use of the following additional notation and terminology.

If $i < j$, the symbol $[i, j]$ denotes the set $\{i, i+1, \ldots, j\}$.

A (k-term) *arithmetic progression* (or simply *a.p.*) is a sequence $\{a+id : i = 0, 1, \ldots, k-1\}$ where $a, d > 0$. For a fixed $t > 0$, an a.p. is called a *t-a.p.* if $d = t$. If $m \geq 2$ and $0 \leq q < m$, we call an increasing sequence $X = \{x_1, \ldots, x_k\}$ a *k-term* $q\,(\mathrm{mod}\ m)$*-sequence* if $x_{i+1} - x_i \equiv q\,(\mathrm{mod}\ m)$ for $1 \leq i \leq k-1$. If we drop the requirement that the sequence be increasing, we will call it a *semi-*$q\,(\mathrm{mod}\ m)$*-sequence*. For example $\{1, 2, 18, 4, 15, 1\}$ is a 6-term semi $1\,(\mathrm{mod}\ 5)$-sequence, but it is not a $1\,(\mathrm{mod}\ 5)$-sequence.

An *r-coloring* of a set A of positive integers is a function $\chi : A \to \{1, 2, \ldots, r\}$ (i.e., it is a partition of A into r classes). A coloring χ is *monochromatic* on a set Y if χ is constant on Y. Since most of this paper deal with 2-colorings, we sometimes use the word *coloring* to indicate a 2-coloring.

Let $S = \{a_i\,(\mathrm{mod}\ m_i)\colon\ 1 \leq i \leq n\}$ be a set of congruence classes where $1 \leq a_i < m_i$ for all i, and let T be a set of positive integers. The symbol $f_{S,T}(k)$ will denote the least positive integer such that whenever $[1, f_{S,T}(k)]$ is 2-colored, there will be a k-term monochromatic $a_i\,(\mathrm{mod}\ m_i)$-sequence for some $i \in \{1, \ldots, n\}$ or a k-term monochromatic t-a.p., for some $t \in T$. If $f_{S,T}(k)$ does not exist, we write $f_{S,T}(k) = \infty$. If S consists of a single congruence class $q\,(\mathrm{mod}\ m)$, we denote $f_{S,T}$ by $f_{q(m),T}$; if, in addition, $T = \{t\}$, we use $f_{q(m),t}$. If $T = \phi$, we use the symbol f_S rather than $f_{S,T}$; or in case $|S| = 1$ or 2, simply $f_{a(m_1)}$ or $f_{a(m_1),b(m_2)}$, respectively.

As a more general notation, we use $f_{q(m),T}(k_1, k_2)$ to denote the Ramsey function corresponding to k_1-term $q\,(\mathrm{mod}\ m)$-sequences or k_2-term t-a.p.'s for some $t \in T$. If $\vec{k} = (k_1, k_2, \ldots, k_n)$, $f_S(\vec{k})$ represents the least positive integer so that every 2-coloring of $[1, f_S(\vec{k})]$ will contain a monochromatic k_i-term $a_i\,(\mathrm{mod}\ m)$-sequence for some $a_i\,(\mathrm{mod}\ m_i) \in S$.

Finally, when considering a Ramsey function associated with r-colorings we use the symbol $f^{(r)}$ in place of f.

2 Using a Single Modulus

In [5] it was shown that when $q \neq 0$ the Ramsey number $f_{q(m)}(k)$ corresponding to $q\,(\mathrm{mod}\ m)$-sequences is infinite for all $k \geq 3$, while $f_{q(m),m}$, the Ramsey function corresponding to $q\,(\mathrm{mod}\ m)$-sequences or m-a.p.'s, is always finite. More specifically, the following upper bounds for $f_{q(m),m}$ were obtained.

Theorem 1 (Landman and Long)**.** *Let $m \geq 2$, $1 \leq q < m$, and $k, n \geq 2$. Let $e = \gcd(q, m)$. If m/e is even, then*

$$f_{q(m),m}(k, n) \leq m(k-1)(n-1) + (k-2)q + 1 \tag{1}$$

If m/e is odd, then

$$f_{q(m),m}(k, n) \leq m[(k-2)(n-1) + 1] + kq - e + 1 \tag{2}$$

Theorem 1 deals with the case in which for some fixed m and q, $S = \{q\,(\mathrm{mod}\ m)\}$ and $T = \{m\}$. We now consider the case in which $S = \{a\,(\mathrm{mod}\ m) : 1 \leq a \leq m-1\}$ and $T = \phi$, i.e., the case of the a.p.'s modulo m. As the next theorem shows, the size of the collection of sequences does not, in itself, tell us whether the corresponding Ramsey function is defined.

Theorem 2. *Let $m \geq 2$ and $S = \{a\,(\mathrm{mod}\ m) : 1 \leq a \leq m-1\}$. Then $f_S(k) = \infty$ whenever $k > \lceil m/2 \rceil$.*

Proof. Let $m \geq 2$ and consider the following 2-coloring, χ, of the positive integers. For $1 \leq x \leq \lceil m/2 \rceil$ let $\chi(x) = 0$, and for $\lceil m/2 \rceil + 1 \leq x \leq m$ let $\chi(x) = 1$. When $x > m$, let $\chi(x) = \chi(\bar{x})$, where $x \equiv \bar{x} \pmod{m}$ and $1 \leq \bar{x} \leq m$.

We will show that, with respect to χ, the maximum size of a monochromatic $q \pmod{m}$-sequence is bounded above by $\left\lceil \frac{m}{2\gcd(q,m)} \right\rceil \leq \left\lceil \frac{m}{2} \right\rceil$ for each q, $1 \leq q \leq m-1$. From this it follows immediately that $f_S(k) = \infty$ if $k > \left\lceil \frac{m}{2} \right\rceil$.

Let q be fixed ($1 \leq q \leq m-1$), and let $d = \gcd(q,m)$, $s = m/d$. Now regard $1, 2, \ldots, m$ as the elements of the m-element cyclic group (where $\bar{x} + \bar{y} = \overline{x+y}$), and let H be the s-element cyclic subgroup of $[1,m]$. By elementary group theory, we know that

$$H = \{\overline{q}, \overline{2q}, \ldots, \overline{sq}\} = \{d, 2d, \ldots, sd\}. \tag{3}$$

Now let $x_1 < x_2 < \cdots < x_s$ be any s-term $q \pmod{m}$-sequence. Then $x_{i+1} = x_i + d_i$ for $1 \leq i \leq s-1$, with $d_i \equiv q \pmod{m}$, so that

$$\{\overline{x_1}, \ldots, \overline{x_s}\} = \{\overline{x_1}, \overline{x_1 + q}, \ldots, \overline{x_1 + (s-1)q}\} = \overline{x_1} + H.$$

From (3) we see that the subset $\overline{x_1} + H$ of $[1,m]$ forms an arithmetic progression with $s = m/d$ terms and common difference d. Exactly $\left\lfloor \frac{s}{2} \right\rfloor$ of these terms will be less than or equal to $m/2$, and hence exactly $\left\lceil \frac{s}{2} \right\rceil$ terms will form a monochromatic set with respect to χ. Since $\chi(x_i) = \chi(\overline{x_i})$, this shows that if $x_1 < \cdots < x_t$ is a monochromatic $q \pmod{m}$-sequence, then $t \leq \left\lceil \frac{s}{2} \right\rceil$, as required. [Note: one can also show that if $1 \leq q \leq m/2$, the maximum size of a monochromatic $q \pmod{m}$-sequence, with respect to this same coloring, is exactly $\left\lceil \frac{m}{2q} \right\rceil$. We omit the details.] □

As a consequence of Theorems 1 and 2, we are able to characterize, for given q and m, those sets T such that $f_{q(m),T}(k)$ exists for all k. We also generalize the bounds of Theorem 1, which deals with the case of $T = \{m\}$, to any set T containing some multiple of m.

Corollary 1. *Let $m \geq 2$, $1 \leq q \leq m-1$, and let T be a set of positive integers.*

(a) $f_{q(m),T}(k) < \infty$ for all k if and only if $cm \in T$ for some positive integer c.

(b) Let $d = \gcd(q,m)$, k, $n \geq 2$, and $c \geq 1$. Assume that $cm \in T$.

If m/d is even, then

$$f_{q(m),T}(k,n) \leq cm(k-1)(n-1) + (k-2)q + 1. \tag{4}$$

If m/d is odd, then

$$f_{q(m),T}(k,n) \leq cm(k-2)(n-1) + m + kq - d + 1. \tag{5}$$

Proof. Assume no multiple of m belongs to T. Then for each $t \in T$, every t-a.p. is an $a \pmod{m}$-sequence for some $a \in \{1, \ldots, m-1\}$. It then follows from Theorem 2 that $f_{q(m),T}(k) = \infty$ for all $k > m/2$.

Now assume $cm \in T$. Note that if $\{x_1, \ldots, x_{(n-1)c+1}\}$ is an m-a.p., then $\{x_{ic+1} : i = 0, \ldots, n-1\}$ is a cm-a.p. Hence,

$$f_{q(m),cm}(k,n) \leq f_{q(m),m}(k,(n-1)c+1).$$

Inequalities (4) and (5) follow from (1) and (2), respectively. □

3 Using an Arbitrary Set of Congruence Classes

We now direct our attention to the function f_S for general S. The case in which $|S| = 1$ was considered in [5]. The authors showed that if S consists of the single congruence class $a \pmod{m}$, then as long as $a \neq 0$ and $k \geq 3$, $f_S(k) = \infty$. They also showed that the only case in which $|S| = 1$ and $f_S(3) < \infty$ is if $q = 0$, and that in fact, for all $k \geq 2$

$$f_{0(m)}(k) = 2m(k-1)+1. \tag{6}$$

We also see by Theorem 2, that if $S = \{a_1 \pmod{m}, \ldots, a_r \pmod{m} : a_i \neq 0 \text{ for } 1 \leq i \leq r\}$, then $f_S(k) = \infty$ whenever $k > m/2$.

Before dealing with the most general set S, we first consider the case in which $|S| = 2$. We now characterize those sets $S = \{a \pmod{m_1}, b \pmod{m_2} : a, b \neq 0\}$ for which the corresponding collection of sequences has the Ramsey property (if $a = 0$ or $b = 0$ we see by (6) that $f_S(k)$ does exist for all k), and provide an upper bound for the corresponding Ramsey function.

Theorem 3. *Let $m_1, m_2 > 1$, let $1 \leq a < m_1$ and $1 \leq b < m_2$, and assume $k > 3$. Let $d = \gcd(m_1, m_2)$ and $m = \operatorname{lcm}\{m_1, m_2\}$. Then $f_{a(m_1),b(m_2)}(k) < \infty$ if and only if $d|a$ or $d|b$. Furthermore, for all $k \geq 3$, if $d|a$ or $d|b$, then $f_{a(m_1),b(m_2)}(k) < m(k-1)^2$.*

Proof. First assume d divides neither a nor b. To prove $f_{a(m_1),b(m_2)}(k)$ does not exist, it suffices to prove $f_{a(d),b(d)}(k)$ does not exist because any coloring of the positive integers that avoids monochromatic $q \pmod{d}$-sequences also avoids monochromatic $q \pmod{m_1}$- and $q \pmod{m_2}$-sequences. Note also that if χ is a 2-coloring of $[1, d]$ that avoids monochromatic semi $q \pmod{d}$-sequences, then the coloring χ' of the set of positive integers defined by

$$\chi'(j) = \chi(i) \text{ if } j \equiv i \pmod{d}$$

will avoid k-term monochromatic $q \pmod{d}$-sequences. Thus, to prove $f_{a(m_1),b(m_2)}(k)$ does not exist it suffices to find a 2-coloring of $[1, d]$ that avoids k-term monochromatic semi $a \pmod{d}$-sequences and k-term monochromatic semi $b \pmod{d}$-sequences. Finally, we can assume $a \leq d/2$ and $b \leq d/2$, because if $q > d/2$, then $S = \{x_1, \ldots, x_k\}$ is a semi $q \pmod{d}$-sequence in $[1, d]$ if and only if $S = \{x_k, \ldots, x_1\}$ is a semi $(d-q) \pmod{d}$-sequence.

We now give a coloring of $[1, d]$ that avoids both 4-term monochromatic semi $a \pmod{d}$-sequences and 4-term monochromatic semi $b \pmod{d}$-sequences. Assuming $a < b$ (we may assume $a \neq b$, since, as stated in the introduction, there is a 2-coloring of the positive integers that avoids monochromatic $a \pmod{d}$-sequences), define the coloring χ recursively:

(i) $\chi(x) = 1$ if $1 \leq x \leq a$

(ii) $\chi(x) \neq \chi(x-a)$ if $a < x \leq b$

(iii) $\chi(x) \neq \chi(x-b)$ if $b < x \leq d$.

It is apparent that there is no monochromatic 3-term semi $b\,(\mathrm{mod}\; d)$-sequence in $[1,d]$. To show there is no monochromatic 4-term semi $a\,(\mathrm{mod}\; d)$-sequence, we consider two cases.

Case I. $a \leq b/2$.
Let $B_i = [ib+1,(i+1)b]$ for $i = 0,\ldots,j-1$ where $j = \lfloor d/b \rfloor$, and let $B_j = [jb+1,d]$ (B_j is empty if b divides d). Assume $x_1,\ldots,x_4$ is a semi $a\,(\mathrm{mod}\; d)$-sequence. Then $x_3 - x_1 \equiv e\,(\mathrm{mod}\; d)$ and $x_4 - x_2 \equiv f\,(\mathrm{mod}\; d)$ where $0 \leq e,\ f \leq b$. Since $a \leq b/2$, some pair x_r, x_{r+1} must be in the same block B_i (for otherwise $x_4 - x_2 > b$ or $x_3 - x_1 > b$). Thus, by definition of χ, $\chi(x_r) \neq \chi(x_{r+1})$.

Case II. $a > b/2$.
Let $A_0 = [1,a]$, let $A_i = [(i-1)b+a+1, ib+a]$ for $i = 1,\ldots,h$ where $h = \lfloor (d-a)/b \rfloor$, and let $A_{h+1} = [hb+1,d]$ (A_{h+1} could be empty). Then each A_i is monochromatic and the A_i's alternate in color. Assume $\{x,y,z\}$ is a monochromatic semi $a\,(\mathrm{mod}\; d)$-sequence. If x and y are in different A_i's, then y must be in A_0; but then $\chi(z) \neq \chi(y)$. Therefore, we assume x and y are in the same A_{i_1}. So $z \notin A_{i_1}$. Thus, $z \in A_0$, and hence $\chi(z+a) \neq \chi(z)$. Thus there is no 4-term semi $a\,(\mathrm{mod}\; d)$-sequence that is monochromatic.

Now assume d divides (say) a. We first consider the case in which $\frac{m_2}{\gcd(m_2,b)}$ is even. Now $a \neq 0$ implies $m_1 \neq d$, so that there exists a positive integer $c < m_1/d$ such that

$$cm_2 \equiv a\,(\mathrm{mod}\; m_1) \tag{7}$$

By (4), every 2-coloring of $[1, cm_2(k-1)^2 + b(k-2)+1]$ contains either a monochromatic k-term a.p. with difference cm_2 or a monochromatic k-term $b\,(\mathrm{mod}\; m_2)$-sequence. Since

$$\begin{aligned} cm_2(k-1)^2 + b(k-2) + 1 &\leq \left(\frac{m_1}{d} - 1\right) m_2(k-1)^2 + (m_2-1)(k-2) + 1 \\ &< m(k-1)^2, \end{aligned}$$

the result follows from (7). The case in which $\frac{m_2}{\gcd(m_2,b)}$ is odd is easily done in the same way, using (5) instead of (4). □

Remark. Since the coloring χ of $[1,d]$ in the proof of Theorem 3 avoided both 4-term semi $a\,(\mathrm{mod}\; d)$-sequences and 3-term semi $b\,(\mathrm{mod}\; d)$-sequences, we have actually shown the slightly stronger result that if $a \equiv a'\,(\mathrm{mod}\; d)$ and $b \equiv b'\,(\mathrm{mod}\; d)$ with $1 \leq a' \leq b' \leq d-1$, then $f_{a(m_1),b(m_2)}(4,3) = \infty$.

We now direct our attention to the more general case in which $|S| = n$. The following sufficient condition for f_S to exist, and the corresponding upper bound, are immediate from Theorem 3.

Theorem 4. *Let $S = \{a_i\,(\mathrm{mod}\; m_i) : 1 \leq i \leq n\}$ with $1 \leq a_i < m_i$ for each i. For $1 \leq i < j \leq n$, let $d_{ij} = \gcd(m_i,m_j)$ and $m_{ij} = \mathrm{lcm}\{m_i,m_j\}$. If for some pair $i < j$, d_{ij} divides a_i or a_j, then*

$$f_S(k) < m_{ij}(k-1)^2 \text{ for all } k \geq 3 \tag{8}$$

We now wish to consider the converse of Theorem 4. In other words, if S and d_{ij} are defined as in Theorem 4, and if f_S is finite for all k, does it follow that for some $i \neq j$, d_{ij} divides a_i or a_j?

Before proceeding, we give some results which simplify the determination of whether, given S, the corresponding collection of sequences has the Ramsey property.

Lemma 1. *Let $S = \{a_i \pmod{m_i} : 1 \le i \le n\}$. Let $m = \mathrm{lcm}\{m_1, \dots, m_n\}$ and $\vec{k} = (k_1, \dots, k_n)$. Then $f_S(\vec{k}) = \infty$ if and only if there is an m-periodic coloring of the positive integers that avoids monochromatic k_i-term $a_i \pmod{m_i}$-sequences for all i.*

Proof. Clearly, if $f_S(\vec{k}) < \infty$, no m-periodic coloring can avoid monochromatic k_i-term $a_i \pmod{m_i}$-sequences for all i.

Now assume $f_S(\vec{k}) = \infty$. Let C be a coloring of the positive integers that avoids monochromatic k_i-term $a_i \pmod{m_i}$-sequences for all i. Denote C by a sequence of 0's and 1's. For each positive integer j, let C_j represent that portion of the sequence C that is restricted to the interval $[(j-1)m+1, jm]$. There are 2^m possible colorings for each C_j. So at least one of these 2^m colorings must occur infinitely often in $C = C_1C_2\dots$. Say that $B = C_{j_1} = C_{j_2} = \cdots$, where $j_1 < j_2 < \cdots$. Now color the positive integers with the coloring $BBB\dots$, and denote the coloring by D.

To complete the proof we will show that the m-periodic coloring D avoids k_i-term monochromatic $a_i \pmod{m_i}$-sequences for all i. Assume by way of contradiction that under D the sequence $x_1, \dots, x_{k_i}$ is a monochromatic $a_i \pmod{m_i}$-sequence. For each s, $1 \le s \le k_i$, let y_s be an element of C_{j_s} such that $y_s \equiv x_s \pmod{m}$. Then the C-color of y_s is the same as the D-color of x_s for each s. Thus, under C, $y_1, \dots, y_{k_i}$ is a k_i-term monochromatic $a_i \pmod{m_i}$-sequence, a contradiction. □

Lemma 2. *Let $S = \{a_i \pmod{m_1} : 1 \le i < m\}$. Let $m = \mathrm{lcm}\{m_1, \dots, m_n\}$ and $\vec{k} = (k_1, \dots, k_n)$. Then $f_S(\vec{k}) < \infty$ if and only if every 2-coloring of $[1,m]$ contains, for some i, a monochromatic k_i-term semi $a_i \pmod{m_i}$-sequence.*

Proof. Assume that for every 2-coloring of $[1,m]$, there is a monochromatic k_i-term semi $a_i \pmod{m_i}$-sequence for some i. Let χ be an m-periodic coloring of the positive integers. Then, under χ, there is, for some i, a monochromatic semi $a_i \pmod{m_i}$-sequence $x_1, \dots, x_{k_i}$ that is contained in $[1,m]$. For $1 \le j \le k_i$, let $y_j \equiv x_j \pmod{m}$ with $y_j \in [(j-1)m+1, jm]$. Then, since χ is m-periodic, $y_1, \dots, y_{k_i}$ is a (increasing) monochromatic $a_i \pmod{m_i}$-sequence that is contained in $[1, mk_i]$. Since χ is an arbitrary m-periodic coloring, by Lemma 1 we know that $f_S(k) < \infty$.

Now assume there is a 2-coloring of $[1,m]$ that avoids monochromatic k_i-term semi $a_i \pmod{m_i}$-sequences. Extend this coloring to an m-periodic 2-coloring of the positive integers. Relative to this extended coloring, there is no monochromatic k_i-term $a_i \pmod{m_i}$-sequence. Hence $f_S(\vec{k}) = \infty$. □

The following result tells us that to determine whether the collection of sequences corresponding to a given S has the Ramsey property, it suffices to check whether the associated Ramsey function is defined at one particular vector $\vec{k}^*$.

Proposition 1. *Let $S = \{a_i \pmod{m_i} : 1 \le i \le n\}$ and $\vec{k}^* = (k_1^*, \dots, k_n^*)$ where $k_i^* = \frac{m_i}{\gcd(a_i, m_i)}$ for $1 \le i \le n$. Then $f_S(k) < \infty$ for all k if and only if $f_S(\vec{k}^*) < \infty$.*

Proof. One direction is trivial. For the other direction, we proceed by contradiction. Suppose $f_S(k) = \infty$ but that $f_S(\vec{k}^*) < \infty$. Let $m = \mathrm{lcm}\{m_1, \dots, m_n\}$. By Lemma 1, let C be an m-periodic coloring of the positive integers which avoids k-term monochromatic $a_i \pmod{m_i}$-sequences for all i.

By the proof of Lemma 2, under the coloring C there exists, for some i, a monochromatic k_i^*-term $a_i \,(\mathrm{mod}\, m_i)$-sequence contained in $[1, mk_i^*]$. Denote this sequence by $x_1, \ldots, x_{k_i^*}$. Note that for each positive integer r,

$$rmk_i^* + x_i - [(r-1)mk_i^* + x_{k_i^*}] \equiv -a_i(k_i^* - 1) \,(\mathrm{mod}\, m)\,.$$

Therefore,

$$rmk_i^* + x_1 - [(r-1)mk_i^* + x_{k_i^*}] \equiv a_i \,(\mathrm{mod}\, m_i)\, e$$

Since C has period m, the sequence

$$x_1, x_2, \ldots, x_{k_i^*}, mk_i^* + x_1, mk_i^* x_2, \ldots, mk_i^* + x_{k_i^*}, 2mk_i^* + x_1,$$
$$2mk_i^* + x_2, \ldots, 2mk_i^* + x_{k_i^*}, \ldots$$

is an infinite monochromatic $a_i \,(\mathrm{mod}\, m_i)$-sequence, contradicting the assumption that under C there is no monochromatic k-term $a_i \,(\mathrm{mod}\, m_i)$-sequence. □

Theorem 3 tells us that, for the case in which $|S| = 2$, the converse of Theorem 4 holds. That is, if d divides neither a_1 nor a_2 then $f_S(k) = \infty$ for k large enough. This converse of Theorem 4 does not, however, hold in the general case of $|S| = n$. In Proposition 2 we exhibit a special class of sets S for which, for all i and j, $\gcd(m_i, m_j)$ divides neither a_i nor a_j, yet $f_S(k)$ is always finite.

Proposition 2. *Let $S = \{a_i \,(\mathrm{mod}\, m_i) : i = 1,2,3\}$ where for all $i \neq j$, $\gcd(m_i, m_j)$ divides neither a_i nor a_j. Assume $a_1 = m_1/2$, and that a_1, a_2 are odd, and a_3 is even. Assume $m_2 = 2b$ such that b is odd and $\gcd(a_1, b) = 1$. Assume m_3 is a divisor of $\frac{1}{2}m_1 m_2 = \mathrm{lcm}\{m_1, m_2\}$. Then $f_S(\vec{k}) < \infty$ for all $\vec{k}$.*

Proof. Let $k_1^* = 2$, $k_2^* = \frac{m_2}{\gcd(a_2, m_2)}$, and $k_3^* = \frac{m_3}{\gcd(a_3, m_3)}$. Let $m = \frac{1}{2}m_1 m_2$. By Proposition 1 and Lemma 2, it suffices to show that every 2-coloring of $[1, m]$ contains, for some i, a k_i^*-term monochromatic semi $a_i \,(\mathrm{mod}\, m_i)$-sequence. Suppose that χ is a 2-coloring of $[1, m]$ with no 2-term monochromatic semi $\frac{m_1}{2} \,(\mathrm{mod}\, m_1)$-sequence and not k_2^*-term monochromatic semi $a_2 \,(\mathrm{mod}\, m_2)$-sequence.

Let $S_0 = \{m_1, 2m_1, \ldots, (m/m_1)m_1\}$ and, for $1 \leq j \leq m_1 - 1$, let $S_j = j + S_0 \,(\mathrm{mod}\, m)$. Since there are no 2-term monochromatic semi $\frac{m_1}{2} \,(\mathrm{mod}\, m_1)$-sequences, no member of S_j could have the same color as a member of $S_{j+m_1/2}$, for all j. So each S_j is monochromatic.

Our strategy is to show that one color class consists of even elements of $[1, m]$, while the other color class consists of the odd elements of $[1, m]$. Since a_3 is even, it will then follow that $m_3, m_3 + a_3, \ldots, m_3 + (k_3^* - 1)a_3$ is a monochromatic k_3^*-term semi $a_3 \,(\mathrm{mod}\, m_3)$-sequence. In view of Proposition 1, this will complete the proof.

Let $T_0 = \{m_2, 2m_2, \ldots, (m/m_2)m_2\}$ and, for $1 \leq j \leq m_2 - 1$, let $T_j = j + T_0 \,(\mathrm{mod}\, m)$. Let $0 \leq j \leq m_1 - 2$, and assume $x, y \in S_j \cup S_{j+1}$, with $x \neq y$. In case $x, y \in S_j$ or $x, y \in S_{j+1}$, then $x \equiv y \,(\mathrm{mod}\, m_1)$. Then $x \not\equiv y \,(\mathrm{mod}\, m_2)$ for otherwise $x \equiv y \,(\mathrm{mod}\, m)$. If $x \in S_j$ and $y \in S_{j+1}$ then x and y have opposite parity (since m_1 is even), so that $x \not\equiv y \,(\mathrm{mod}\, m_2)$ (since m_2 is even). Hence, in all cases, no two elements of $S_j \cup S_{j+1}$ are congruent $(\mathrm{mod}\, m_2)$. It then follows, since $S_j \cup S_{j+1}$ has $2(m/m_1) = m_2$ elements, and since all elements of any T_h are congruent $(\mathrm{mod}\, m_2)$, that

$$(S_j \cup S_{j+1}) \cap T_h \neq \varnothing \tag{9}$$

for all $0 \le j \le m_1 - 2$ and all $0 \le h \le m_2 - 1$. It follows from (9) that, for each $1 \le j \le m_1 - 2$, $S_j \cup S_{j+1}$ contains a semi $a_2 \,(\mathrm{mod}\; m_2)$-sequence with $m_2/\gcd(a_2, m_2)$ terms. Since no such sequence can be monochromatic, and since each S_j is monochromatic, S_j and S_{j+1} have opposite colors. Thus, since m_1 is even, one color class consists of the even elements of $[1,m]$ and the other class consists of the odd elements of $[1,m]$, as claimed. □

Specific examples of sets S which satisfy the hypotheses of Proposition 2 include the following: $S = \{a\,(\mathrm{mod}\; 2a), b\,(\mathrm{mod}\; 2bc), 2c\,(\mathrm{mod}\; 2abc)\}$ where a, b and c are odd, $a > 1$, $b > 1$ and $\gcd(a, bc) = 1$.

Proposition 2 gives a very special class of sets S for which the converse of Theorem 4 fails. The next theorem gives sufficient conditions on S for $f_S(\vec{k})$ to be infinite for some $\vec{k}$, which are satisfied by a large class of sets S. Note, in particular, that Theorem 5 implies that the converse of Theorem 4 does hold if $\gcd(a_i, m_i) = 1$ for all i.

Theorem 5. *Let $S = \{a_i\,(\mathrm{mod}\; m_i) : 1 \le i \le n\}$. Let $e_i = \gcd(a_i, m_i)$ for $1 \le i \le n$, and $d_{ij} = \gcd\left(\frac{m_i}{e_i}, \frac{m_j}{e_j}\right)$ for $i \ne j$. Assume that for every pair $i \ne j$, d_{ij} divides neither a_i nor a_j. Then $f_S(\vec{k}^*) = \infty$ where $\vec{k}^* = (m_1/e_1, \ldots, m_n/e_n)$.*

Proof. We first prove the result for the case in which $e_1 = 1$ for all i. Let $m = \mathrm{lcm}\{m_1, \ldots, m_n\}$. By Lemma 2, it suffices to find a 2-coloring of $[1,m]$ that avoids m_i-term monochromatic semi $a_i\,(\mathrm{mod}\; m_i)$-sequences for all i.

Define the coloring $\chi : [1,m] \to \{0,1\}$ as follows:

$$\chi(x) = \begin{cases} 1 & \text{if } x \equiv 0\,(\mathrm{mod}\; m_i) \text{ for some } i \\ 0 & \text{if } x \equiv a_i\,(\mathrm{mod}\; m_i) \text{ for some } i \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

Note that χ is well-defined on $[1,m]$ because if $i \ne j$ and $x \equiv 0\,(\mathrm{mod}\; m_i)$, then (since d_{ij} does not divide a_j) $x \not\equiv a_j\,(\mathrm{mod}\; m_j)$. Since $e_1 = 1$ for all i, any m_i-term semi $a_i\,(\mathrm{mod}\; m_i)$-sequence must contain an element from each congruence class modulo m_i. Hence, no such sequence can be monochromatic under χ. This proves the theorem in case all $e_i = 1$.

Now let the e_i be arbitrary. Consider

$$S' = \{a_i\,(\mathrm{mod}\; m_i/e_i) : 1 \le i \le n\}.$$

Since $\gcd\left(a_i, \frac{m_i}{e_i}\right) = 1$ and since $\gcd\left(\frac{a_i}{m_i}, \frac{a_j}{m_j}\right)$ divides neither a_i nor a_j, we can apply the case proven above (the case of all $e_i = 1$). Thus, there is a coloring of the positive integers that avoids, for all i, (m_i/e_i)-term monochromatic $a_i\,(\mathrm{mod}\; m_i/e_i)$-sequences. Observing that every $a_i\,(\mathrm{mod}\; m_i)$-sequence is also an $a_i\,(\mathrm{mod}\; m_i/e_i)$-sequence, the proof is complete. □

4 A Special Case

Theorem 3 gives an upper bound, using 2 colors, for $f_{a(m_1),b(m_2)}(k)$, when $d = \gcd(m_1, m_2)$ divides a or b. If we use the stronger hypothesis that d divides both a and b, we are able to give a much better upper bound. In fact, we shall prove a result which deals with the more general case in which there are r colors and $|S| = r$. We first prove the following theorem.

Theorem 6. *Let $r \geq 2$ and, for $1 \leq i \leq r$, let $m_i > 1$ and $1 \leq a_i < m_i$. Let $a_0 = \min\{a_i\}$, $d = \gcd(m_1, \ldots, m_r)$, and $m = \operatorname{lcm}\{m_1, \ldots, m_r\}$. Assume that d divides a_i for all i and that the m_i/d are pairwise relatively prime. Then, for each $t \geq 2$, every r-coloring χ of $[1, m(t-2) + a_0 + 1]$ contains, for $1 \leq i \leq r$, a monochromatic k_i-term $a_i \pmod{m_i}$-sequence X_i, with*

$$\chi(X_i) \neq \chi(X_j) \text{ for } i \neq j \text{ and}$$

$$\sum_{i=1} rk_i = t \text{ with } k_i \geq 0.$$

Proof. We first prove the theorem for the case in which $d = 1$, using induction on t.

If $t = 2$ then since $\{1, a_0 + 1\}$ is a 2-term $a_i \pmod{m_i}$-sequence for some i, $[1, a_0 + 1]$ will either contain a monochromatic 2-term $a_i \pmod{m_i}$-sequence or else will contain a 1-term monochromatic $a_i \pmod{m_i}$-sequence and a 1-term monochromatic $a_j \pmod{m_j}$-sequence, with $j \neq i$, not of the same color. Hence the result holds for $t = 2$.

Now assume that for a fixed t the result holds (still assuming $d = 1$). Let $F(t) = m(t-2) + a_0 + 1$ and let χ be an r-coloring of $[1, F(t+1)]$. For $i = r$, let

$$X_i = \{x_{i_1}, \ldots, x_{i_{k_i}}\} \subseteq [1, F(t)]$$

be the monochromatic $a_i \pmod{m_i}$-sequences given by the induction hypothesis. By the Chinese remainder theorem there is a z with $F(t) < z \leq F(t) + m$ satisfying

$$z \equiv (a_i + x_{i_{k_i}}) \pmod{m_i}$$

for $i = 1, \ldots, r$. Then, for some i, $X_i \cup \{z\}$ is a $(k_i + 1)$-term monochromatic $a_i \pmod{m_i}$-sequence such that $\chi(X_i \cup \{z\}) \neq \chi(X_j)$ for all $j \neq i$. Hence in $[1, F(t) + m] = [1, F(t+1)]$ there is, for each $i = 1, \ldots, r$ a monochromatic $a_i \pmod{m_i}$-sequence, with no two sequences of the same color, such that the sum of the length of the sequences is $t + 1$. This completes the proof for the case in which $d = 1$.

We now assume $d \neq 1$. Let α be any r-coloring of $[1, F(t)]$. Let

$$N = 1 + \frac{a_0}{d} + \frac{m_1 \cdots m_r}{d^r}(t-2) = \frac{F(t) - 1}{d} + 1.$$

Define α' on $[1, N]$ by $\alpha'(i) = \alpha(d(i-1) + 1)$. Since $\gcd(m_i/d, m_j/d) = 1$ for $i \neq j$ we know that by the previous case, under α', $[1, N]$ contains, for each $i = 1, \ldots, r$, a monochromatic k_i-term $\frac{a_i}{d} \left(\bmod \frac{m_i}{d}\right)$-sequence $X_i = \{x_{i_1}, \ldots, x_{i_{k_i}}\}$ such that

$$\sum_{i=1}^{r} k_i = t \text{ and } \alpha'(X_{i_1}) = \alpha'(X_{i_2}) \text{ for } i_1 \neq i_2.$$

Then under α, for each i,

$$Y_i = \{d(x_{i_s} - 1) + 1 : s = 1, \ldots, k_i\}$$

is a monochromatic $a_i \pmod{m_i}$-sequence contained in $[1, F(t)]$ with $\alpha(Y_i) \neq \alpha(Y_j)$ for $i \neq j$. □

Corollary 2. *Let $S = \{a_i \pmod{m_i} : 1 \leq i \leq r\}$ and assume r, d, m, all a_i, and all m_i satisfy the hypotheses of Theorem 6. Then for all $k \geq 2$*

$$f_S^{(r)}(k) \leq m[r(k-1)-1] + a_0 + 1.$$

Proof. By Theorem 6, each r-coloring of $[1, m(r(k-1)-1)+a_0+1]$ contains, for each i, a monochromatic k_i-term $a_i \pmod{m_i}$-sequence, each of a different color, such that $\sum_{i=1}^{r} k_i = r(k-1)+1$. Then at least one of these sequences has length at least k. □

We state separately the upper bound for $f_{a(m_1),b(m_2)}(k)$ provided by Corollary 2.

Corollary 3. *Let $1 \leq a \leq m_1$, $1 \leq b \leq m_2$, and $a_0 = \min\{a, b\}$. Let $d = \gcd(m_1, m_2)$, $m = \mathrm{lcm}\{m_1, m_2\}$, and assume d divides both a and b. Then for all $k \geq 2$,*

$$f_{a(m_1),b(m_2)}(k) \leq (2k-3)m + a_0 + 1.$$

5 Remarks

The upper bounds given by Theorem 1 and Corollary 1 are very sharp. In fact, lower bounds for $f_{q(m),m}(k,n)$ were given in [5] which are of the same order of magnitude as the upper bounds of Theorem 1. For the case in which m/e is odd, the bounds of Theorem 1 are best possible; for example, $f_{5(7),7}(3,5) = 50$ and $f_{8(13),13}(3,4) = 76$ (these agree precisely with the bounds provide by Theorem 1). For m/e even, we have found instances in which the upper bound of Theorem 1 differs from the actual value of f by only 1; for example, $f_{9(10),10}(3,3) = 49$.

Since Theorem 1 is just a special case of Corollary 1 (the case in which $T = \{m\}$), the above examples also show that the bounds of Corollary 1 are sharp. We have found many additional examples, however, in which the bounds of Corollary 1 are sharp (and sometimes precise). For example, it is easy to see, from the proof of Corollary 1, that $f_{5(7),14}(3,3) \leq f_{5(7),7}(3,5) = 50$, and this is the exact value of $f_{5(7),14}(3,3)$. Perhaps more surprisingly, when $T = \{9, 14\}$, the value of $f_{5(7),T}(3,3)$ is still 50, again coinciding with the upper bound of Corollary 1.

There are many cases in which the bound given by Theorem 3, and hence Theorem 4, is more than twice the actual value of f. However, we have also found many examples where the bound is very sharp. In particular, computer data suggests the following conjecture is true:

Conjecture 1. *Let $c \geq 2$, let m_1 be even and let $b = (c-1)m_1$. Then $f_{a(m_1),b(m_2)}(3) = 4(c-1)m_1 + 1$.*

We see that, under the same hypotheses of Conjecture 1, the upper bound for f given by Theorem 3 is $4cm_1$.

The bound given by Corollary 3 has order of magnitude $2mk$, where $m = \mathrm{lcm}\{m_1, m_2\}$. We have computed many values of f, where the hypotheses of Corollary 3 are satisfied, and have not found any where f grows quite that fast, but have found many examples in which it grows faster than mk. One interesting set of examples is where $a = 1$, $m_1 = 2$, b is odd, and m_2 is odd. In all such cases,

$f_{a(m_1),b(m_2)}(3) = m+a+2b$ and, whenever $k \geq 4$ and $b < m_2/2$, we found that $f_{a(m_1),b(m_2)}(k) = m+a+2b+(k-3)(m+b)$. In addition, when $b = (m_2+1)/2$ (so $m_2 \equiv 1 \pmod 4$), $f_{a(m_1),b(m_2)}(k) = m+a+2b+(k-3)(m+b-1)$. Hence, in the case of $b = (m_2 \pm 1)/2$, $f(k)$ has value $m+a+\frac{m}{2}\pm 1+(k-3)\left(m+\frac{1}{2}\left(\frac{m}{2}-1\right)\right)$. This would suggest a growth rate of f of $\frac{5}{4}mk$.

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