Squares of Second-Order Linear Recurrence Sequences

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Abstract

We discuss integer sequences $\{T_n\}$ such that $\{T_n\}$ satisfies a second-order homogeneous linear recurrence relation with constant integer coefficients, and $\{T_n^2\}$ satisfies a second-order linear recurrence relation with constant integer coefficients. We also prove some related results.

1 Introduction

Let us call a sequence $\{T_n\}$ an "mth-order sequence" if $\{T_n\}_{n\geq 0}$ satisfies an mth order linear recurrence relation with constant integer coefficients. It is well known [2, 3] that if $\{T_n\}_{n\geq 0}$ is a second-order sequence then the sequence of squares $\{T_n^2\}_{n\geq 0}$ is a third-order sequence. (It is also easy to show this directly.) It would be of interest to be able to describe all second-order sequences $\{T_n\}_{n\geq 0}$ such that $\{T_n^2\}_{n\geq 0}$ is a *second*-order sequence.

In this note we do this for certain *homogeneous* sequences $\{T_n\}_{n\geq 0}$. That is, we assume that $\{T_n\}_{n\geq 0}$ satisfies a recurrence of the form $T_0 = a$, $T_1 = b$, $T_{n+1} = cT_n - dT_{n-1}$, $n \geq 1$, where a, b, $c \neq 0$, $d \neq 0$ are integers, $ab \neq 0$, and $x^2 - cx + d = 0$ has distinct roots. It then turns out that $\{T_n^2\}_{n\geq 0}$ satisfies a second-order linear recurrence (which we describe in Theorem 6) if and only if $d = 1$.

As an illustration of this, consider the sequence $1, 2, 7, 26, 97, 362, \ldots$, which satisfies the second-order recurrence $B_0 = 1$, $B_1 = 2$, $B_{n+1} = 4B_n - B_{n-1}$, $n \geq 1$. The sequence of squares $1^2, 2^2, 7^2, 26^2, 97^2, 362^2, \ldots$ satisfies the second-order recurrence $S_0 = 1$, $S_1 = 4$, $S_{n+1} = 14S_n - S_{n-1} - 6$, $n \geq 1$.

We also consider second-order sequences $\{T_n\}_{n\geq 0}$ such that a slight perturbation of the sequence of squares $\{T_n^2\}_{n\geq 0}$ is a second-order sequence. For example, the sequence $1, 1, 3, 7, 17, 41, 99, \ldots$ satisfies the second-order recurrence $B_0 = B_1 = 1$, $B_{n+1} = 2B_n + B_{n-1}$, $n \geq 1$, and the "perturbed" sequence of squares 1^2, $1^2 + 1$, 3^2, $7^2 + 1$, 17^2, $41^2 + 1$, $99^2, \ldots$, satisfies the second-order recurrence $S_0 = 1$, $S_1 = 2$, $S_{n+1} = 6S_n - S_{n-1} - 2$, $n \geq 1$.

We begin with some special cases using elementary techniques. Then, in the last section, we handle the general case using an old result of E. S. Selmer [2], which states that if $T_{n+1} = AT_n + BT_{n-1}$, $n \geq 1$, and $x^2 - Ax - B = (x-\alpha)(x-\beta)$, $\alpha \neq \beta$, then $T_{n+1}^2 = CT_n^2 + DT_{n-1}^2 + ET_{n-2}^2$, $n \geq 2$, where $x^2 - Cx^2 - Dx - E = (x-\alpha^2)(x-\beta^2)(x-\alpha\beta)$.

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2 Some special cases

We begin with some special cases, for which we will use the following Lemma.

Lemma. *Let $p \geq 4$ be any integer, let $\delta = \sqrt{\frac{p}{4}} + \sqrt{\frac{p}{4} - 1}$, and let $S_n = \left(\delta^n + \frac{1}{\delta^n}\right)^2$, $n \geq 0$. Then these numbers S_n satisfy the following identities.*

(a) *For all $0 \leq m \leq n$,*
$$(S_n - 2)(S_m - 2) = S_{n+m} + S_{n-m} - 4.$$
(In particular, $(S_n - 2)^2 = S_{2n}$, so S_{2n} is always a perfect square.)

(b) *For all $0 \leq m \leq n$, $m \equiv n \pmod 2$,*
$$S_n S_m = (S_{(n+m)/2} + S_{(n-m)/2} - 4)^2.$$
(In particular, $S_{n+k}S_{n-k} = (S_n + S_k - 4)^2$ and $pS_{2n+1} = S_1 S_{2n+1} = (S_n + S_{n+1} - 4)^2$, so that S_{2n+1} is always a perfect square provided p is a perfect square.)

(c) *For all $0 \leq m \leq n$, $m \equiv n \pmod 2$,*
$$(S_n - 4)(S_m - 4) = S_{(n+m)/2} + S_{(n-m)/2})^2.$$
(In particular, $(p-4)(S_{2n+2} - 4) = (S_1 - 4)(S_{2n+1} - 4) = (S_{n+1} - S_n)^2$, so that $S_{2n+1} - 4$ is always a perfect square provided $p - 4$ is a perfect square.)

(d) $S_{n+1} = (p-2)S_n - S_{n-1} - 2(p-4)$, $n \geq 1$.

Proof. We prove part (d) in detail. The proofs of parts (a), (b), and (c) are very similar, and are omitted.

Note that $\frac{1}{\delta} = \sqrt{\frac{p}{4}} - \sqrt{\frac{p}{4} - 1}$, so that $\left(\delta + \frac{1}{\delta}\right)^2 = p$. Then

$$\begin{aligned} pS_{n+1} &= \left(\delta + \frac{1}{\delta}\right)^2 S_{n+1} = \left[\left(\delta + \frac{1}{\delta}\right)\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)\right]^2 \\ &= \left[\left(\delta^{n+2} + \frac{1}{\delta^{n+2}}\right) + \left(\delta^n + \frac{1}{\delta^n}\right)\right]^2 = S_{n+2} + S_n + 2\left[\delta^{2n+2} + \frac{1}{\delta^{2n+2}} + \delta^2 + \frac{1}{\delta^2}\right] \\ &= S_{n+2} + S_n + 2\left[\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)^2 - 2 + \left(\delta + \frac{1}{\delta}\right)^2 - 2\right] = S_{n+2} + S_n + 2S_{n+1} + 2(p-4), \end{aligned}$$

that is, $S_{n+2} = (p-2)S_{n+1} - S_n - 2(p-4)$, $n \geq 0$. $\square$

3 Second order sequences $\{T_n\}_{n\geq 0}$ whose squares $\{T_n^2\}_{n\geq 0}$ are also second order sequences. Special Cases.

Theorem 1. *Let $d \geq 3$ be an integer. Define the sequence $\{B_n\}_{n\geq 0}$ by $B_0 = 2$, $B_1 = d$, $B_{n+2} = dB_{n+1} - B_n$, $n \geq 0$. Then the sequence of squares $\{B_n^2\}_{n\geq 0}$ satisfies the second-order recurrence $B_{n+2}^2 = (d^2 - 2)B_{n+1}^2 - B_n^2 - 2(d^2 - 4)$, $n \geq 0$.*

Proof. Solving the recurrence $B_0 = 2$, $B_1 = d$, $B_{n+2} = dB_{n+1} - B_n$, $n \geq 0$ in the usual way gives $B_n = \delta^n + \frac{1}{\delta^n}$, $n \geq 0$, where $\delta = \sqrt{\frac{d^2}{4}} + \sqrt{\frac{d^2}{4} - 1}$, $\frac{1}{\delta} = \sqrt{\frac{d^2}{4}} - \sqrt{\frac{d^2}{4} - 1}$. Let us now simplify the notation by setting $S_n = B_n^2$, $n \geq 0$. Then $S_n = \left(\delta + \frac{1}{\delta}\right)^2$, $n \geq 0$, and by part (d) of the Lemma (with $p = d^2$), $S_{n+2} = (d^2 - 2)S_{n+1} - S_n - 2(d^2 - 4)$, $n \geq 0$. □

4 Perturbed sequences

Here we give a second-order sequence whose squares, when slightly perturbed, form a second-order sequence.

Theorem 2. *Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}_{n\geq 0}$ by $C_0 = 2$, $C_1 = d$, $C_{n+2} = dC_{n+1} + C_n$, $n \geq 0$. Let $S_{2n} = C_{2n}^2$, $S_{2n+1} = C_{2n+1}^2 + 4$, $n \geq 0$. Then $S_{n+2} = (d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$.*

Proof. Solving the recurrence $C_0 = 2$, $C_1 = d$, $C_{n+1} = dC_{n+1} + C_n$, $n \geq 0$ in the usual way gives $C_n = \delta^n + \left(\frac{-1}{\delta}\right)^n$, where $\delta = \sqrt{\frac{d^2}{4} + 1} + \sqrt{\frac{d^2}{4}}$, $\frac{1}{\delta} = \sqrt{\frac{d^2}{4} + 1} - \sqrt{\frac{d^2}{4}}$. Then $S_{2n} = C_{2n}^2 = \left(\delta^{2n} + \frac{1}{\delta^{2n}}\right)^2$, $S_{2n+1} = C_{2n+1}^2 + 4 = \left(\delta^{2n+1} + \frac{1}{\delta^{2n+1}}\right)^2$, $n \geq 0$.

Since $\left(\delta + \frac{1}{\delta}\right)^2 = d^2 + 4$, we obtain $(d^2 + 4)S_{n+1} = \left[\left(\delta + \frac{1}{\delta}\right)\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)\right]^2$, and the calculations used in the proof of part (d) of the Lemma now give $S_{n+2} = (d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$. □

Corollary 1. *Let $S_{2n} = L_{2n}^2$, $S_{2n+1} = L_{2n+1}^2 + 4$, $n \geq 0$, where $\{L_n\}$ is the Lucas sequence. Then $S_{n+2} = 3S_{n+1} - S_n - 2$, $n \geq 0$.*

Proof. This is the case $d = 1$ of Theorem 2. □

Corollary 2. *Let $T_{2n} = F_{2n}^2 + \frac{4}{5}$, $T_{2n+1} = F_{2n+1}^2$, $n \geq 0$, where $\{F_n\}$ is the Fibonacci sequence. Then $T_{n+2} = 3T_{n+1} - T_n - 2$, $n \geq 0$.*

Proof. This follows from Corollary 1 and the identity [1, pp. 56] $5F_n^2 = L_n^2 - 4(-1)^n$. □

5 Additional special cases

If we now write $\delta = \sqrt{s} - \sqrt{s-1}$, $S_n = \frac{1}{4}\left(\delta^n + \frac{1}{\delta^n}\right)^2$, $n \geq 0$, we obtain, just as in the Lemma, $S_0 = 1$, $S_1 = s$, $S_{n+2} = 4(s-2)S_{n+1} - S_n - 2(s-1)$, $n \geq 0$.

The following two results can now be proved in essentially the same way as Theorems 1 and 2.

Theorem 3. *Let $d \geq 2$ be an integer. Define the sequence $\{B_n\}_{n\geq 0}$ by $B_0 = 1$, $B_1 = d$, $B_{n+2} = 2dB_{n+1} - B_n$, $n \geq 0$. Then the sequence of squares $\{B_n^2\}_{n\geq 0}$ satisfies the second-order recurrence*

$$B_{n+2}^2 = (4d^2 - 2)B_{n+1} - B_n - 2d^2, \quad n \geq 0.$$

Theorem 4. *Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}_{n\geq 0}$ by $C_0 = 1$, $C_1 = d$, $C_{n+2} = 2dC_{n+1} + C_n$, $n \geq 0$. Let $S_{2n} = C_{2n}^2$, $S_{2n+1} = C_{2n+1}^2$, $n \geq 0$. Then $S_{n+2} = (4d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$.*

6 The more general homogeneous case

Theorem 5. *Let $a, b, c \neq 0$, $d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$. Let $B_0 = a$, $B_1 = b$, $B_{n+1} = cB_n - dB_{n-1}$, $n \geq 1$. Then $B_{n+1}^2 = (c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + 2(b^2 - a^2 d - abc)d^n$, $n \geq 1$.*

Proof. Let α, β be the roots of $x^2 - cx + d = 0$. Then $\alpha, \beta = \frac{1}{2}(c \pm \sqrt{c^2 - 4d})$, $\alpha \neq \pm\beta$, $\alpha^2, \beta^2 = \frac{1}{2}(c^2 - 2d \pm c\sqrt{c^2 - 4d})$, $\alpha\beta = d$. Also $\alpha^2 \neq \beta^2 \neq d$, since $c \neq 0$, $d \neq 0$, $c^2 \neq 4d$.

According to the result of Selmer stated in the Introduction, there are constants A, B, C such that $B_n^2 = A\alpha^{2n} + B\beta^{2n} + Cd^n$, $n \geq 0$.

Solving the system

$$\begin{aligned} a^2 &= B_0^2 &&= A + B + C \\ b^2 &= B_1^2 &&= A\alpha^2 + B\beta^2 + Cd \\ (bc - ad)^2 &= B_2^2 &&= A\alpha^4 + B\beta^4 + Cd^2 \end{aligned}$$

for C gives $C = \frac{2(b^2 + a^2 d - abc)}{4d - c^2}$.

Using $(c^2 - 2d)\alpha^{2n} - d^2\alpha^{2n-2} = \alpha^{2n+2}$ and $(c^2 - 2d)\beta^{2n} - d^2\beta^{2n-2} = \beta^{2n+2}$ gives $(c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + C[(c^2 - 2d)d^n - d^{n+1}] + ed^n$. Now choosing e so that $C[(c^2 - 2d)d^n - d^{n+1}] + ed^n = Cd^{n+1}$ (namely $e = C(4d - c^2) = 2(b^2 + a^2 d - abc)$), finally gives $(c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + Cd^{n+1} = B_{n+1}^2$, which completes the proof. □

Remark. *The result of Theorem 5 appears in [4].*

Applying Theorem 5 to the question raised in the Introduction, we immediately get the following result.

Theorem 6. *Let $a, b, c \neq 0$, $d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$. Let $B_0 = a$, $B_1 = b$, $B_{n+1} = cB_n - dB_{n-1}$, $n \geq 1$. Then the sequence of squares $\{B_n^2\}_{n \geq 0}$ satisfies a second-order linear recurrence (with constant coefficients) if and only if $d = 1$, in which case $B_{n+1}^2 = (c^2 - 2)B_n^2 - B_{n-1}^2 + 2(b^2 + a^2 - abc)$, $n \geq 1$.*

Our final result is the general version of Theorem 2, in which we consider a perturbation of the sequence of squares.

Theorem 7. *Let $a, b, c \neq 0$, $d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$, such that $e = \frac{4(a^2 + abc - b^2)}{c^2 + 4}$ is an integer. Define the sequence $\{B_n\}_{n \geq 0}$ by $B_0 = a$, $B_1 = b$, $B_{n+1} = cB_n + B_{n-1}$, $n \geq 1$. Let $S_{2n} = B_{2n}^2$, $S_{2n+1} = B_{2n+1}^2 + e$, $n \geq 0$. Then $\{S_n\}_{n \geq 0}$ satisfies the second-order recurrence $S_{n+1} = (c^2 + 2)S_n - S_{n-1} + 2e + 2(b^2 - a^2 - abc)$, $n \geq 1$.*

Proof. This is a direct application of Theorem 5 with $d = -1$, according to which $B_{n+1}^2 = (c^2 + 2)B_n^2 - B_{n-1}^2 + 2(b^2 - a - abc)(-1)^n$. □

References

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