Squares of Second-Order Linear Recurrence Sequences

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Abstract

We discuss integer sequences \( \{T_n\} \) such that \( \{T_n\} \) satisfies a second-order homogeneous linear recurrence relation with constant integer coefficients, and \( \{T_n^2\} \) satisfies a second-order linear recurrence relation with constant integer coefficients. We also prove some related results.

1 Introduction

Let us call a sequence \( \{T_n\} \) an “\( n \)th-order sequence” if \( \{T_n\}_{n \geq 0} \) satisfies an \( n \)th order linear recurrence relation with constant integer coefficients. It is well known [2, 3] that if \( \{T_n\}_{n \geq 0} \) is a second-order sequence then the sequence of squares \( \{T_n^2\}_{n \geq 0} \) is a third-order sequence. (It is also easy to show this directly.) It would be of interest to be able to describe all second-order sequences \( \{T_n\}_{n \geq 0} \) such that \( \{T_n^2\}_{n \geq 0} \) is a second-order sequence.

In this note we do this for certain homogeneous sequences \( \{T_n\}_{n \geq 0} \). That is, we assume that \( \{T_n\}_{n \geq 0} \) satisfies a recurrence of the form \( T_0 = a, T_1 = b, T_{n+1} = cT_n - dT_{n-1}, n \geq 1 \), where \( a, b, c \neq 0, d \neq 0 \) are integers, \( ab \neq 0 \), and \( x^2 - cx + d = 0 \) has distinct roots. It then turns out that \( \{T_n^2\}_{n \geq 0} \) satisfies a second-order linear recurrence (which we describe in Theorem 6) if and only if \( d = 1 \).

As an illustration of this, consider the sequence 1, 2, 7, 26, 97, 362, ..., which satisfies the second-order recurrence \( B_0 = 1, B_1 = 2, B_{n+1} = 4B_n - B_{n-1}, n \geq 1 \). The sequence of squares \( 1^2, 2^2, 7^2, 26^2, 97^2, 362^2, \ldots \) satisfies the second-order recurrence \( S_0 = 1, S_1 = 4, S_{n+1} = 14S_n - S_{n-1} - 6, n \geq 1 \).

We also consider second-order sequences \( \{T_n\}_{n \geq 0} \) such that a slight perturbation of the sequence of squares \( \{T_n^2\}_{n \geq 0} \) is a second-order sequence. For example, the sequence 1, 3, 7, 17, 41, 99, ... satisfies the second-order recurrence \( B_0 = B_1 = 1, B_{n+1} = 2B_n + B_{n-1}, n \geq 1 \), and the “perturbed” sequence of squares \( 1^2, 1^2 + 1, 3^2, 7^2 + 1, 17^2, 41^2 + 1, 99^2, \ldots \), satisfies the second-order recurrence \( S_0 = 1, S_1 = 2, S_{n+1} = 6S_n - S_{n-1} - 2, n \geq 1 \).

We begin with some special cases using elementary techniques. Then, in the last section, we handle the general case using an old result of E. S. Selmer [2], which states that if \( f_{n+1} = AT_n + BT_{n-1}, n \geq 1 \), and \( x^2 - Ax - B = (x - \alpha)(x - \beta), \alpha \neq \beta \), then \( T_{n+1} = C T_n^2 + D T_{n-1}^2 + E T_{n-2}^2, n \geq 2 \), where \( x^2 - Cx^2 - Dx - E = (x - \alpha^2)(x - \beta^2)(x - \alpha\beta) \).

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2 Some special cases

We begin with some special cases, for which we will use the following Lemma.

**Lemma.** Let \( p \geq 4 \) be any integer, let \( \delta = \sqrt{\frac{p}{4} + \frac{1}{p^2 - 1}} \), and let \( S_n = \left( \delta^n + \frac{1}{\delta^n} \right)^2, \ n \geq 0. \) Then these numbers \( S_n \) satisfy the following identities.

(a) For all \( 0 \leq m \leq n, \)
\[
(S_n - 2)(S_m - 2) = S_{n+m} + S_{n-m} - 4.
\]
(In particular, \( (S_n - 2)^2 = S_{2n}, \) so \( S_{2n} \) is always a perfect square.)

(b) For all \( 0 \leq m \leq n, m \equiv n (\mod 2), \)
\[
S_n S_m = (S_{(n+m)/2} + S_{(n-m)/2} - 4)^2.
\]
(In particular, \( S_{n+k}S_{n-k} = (S_n + S_k - 4)^2 \) and \( pS_{2n+1} = S_1S_{2n+1} = (S_n + S_{n+1} - 4)^2, \) so that \( S_{2n+1} \) is always a perfect square provided \( p \) is a perfect square.)

(c) For all \( 0 \leq m \leq n, m \equiv n (\mod 2), \)
\[
(S_n - 4)(S_m - 4) = (S_{(n+m)/2} + S_{(n-m)/2} - 4)^2.
\]
(In particular, \( (p-4)(S_{2n+1} - 4) = (S_1 - 4)(S_{2n+1} - 4) = (S_{n+1} - S_n)^2, \) so that \( S_{2n+1} - 4 \) is always a perfect square provided \( p-4 \) is a perfect square.)

(d) \( S_{n+1} = (p-2)S_n - S_{n-1} - 2(p-4), \ n \geq 1. \)

**Proof.** We prove part (d) in detail. The proofs of parts (a), (b), and (c) are very similar, and are omitted.

Note that \( \frac{1}{\delta} = \sqrt{\frac{p}{4} - \frac{1}{p^2 - 1}}, \) so that \( \left( \delta + \frac{1}{\delta} \right)^2 = p. \) Then
\[
pS_{n+1} = \left( \delta + \frac{1}{\delta} \right)^2 S_{n+1} = \left[ \left( \delta + \frac{1}{\delta} \right) \left( \delta^{n+1} + \frac{1}{\delta^{n+1}} \right) \right]^2
= \left[ \left( \delta^{n+2} + \frac{1}{\delta^{n+2}} \right) + \left( \delta^n + \frac{1}{\delta^n} \right) \right]^2
= S_{n+2} + S_n + 2 \left[ \delta^{2n+2} + \frac{1}{\delta^{2n+2}} + \delta^2 + \frac{1}{\delta^2} \right]
= S_{n+2} + S_n + 2 \left[ \delta^{n+1} + \frac{1}{\delta^{n+1}} \right]^2 - 2 + \left( \delta + \frac{1}{\delta} \right)^2 - 2
= S_{n+2} + S_n + 2S_{n+1} + 2(p-4),
\]
that is, \( S_{n+2} = (p-2)S_{n+1} - S_n - 2(p-4), \ n \geq 0. \)

3 Second order sequences \( \{T_n\}_{n \geq 0} \) whose squares \( \{T_n^2\}_{n \geq 0} \) are also second order sequences. Special Cases.

**Theorem 1.** Let \( d \geq 3 \) be an integer. Define the sequence \( \{B_n\}_{n \geq 0} \) by \( B_0 = 2, B_1 = d, B_{n+2} = dB_{n+1} - B_n, \ n \geq 0. \) Then the sequence of squares \( \{B_n^2\}_{n \geq 0} \) satisfies the second-order recurrence \( B_{n+2}^2 = (d^2 - 2)B_{n+1}^2 - B_n^2 - 2(d^2 - 4), \ n \geq 0. \)
Proof. Solving the recurrence $B_0 = 2, B_1 = d, B_{n+2} = dB_{n+1} - B_n$, $n \geq 0$ in the usual way gives $B_n = \delta^n + \frac{1}{\delta^n}$, $n \geq 0$, where $\delta = \sqrt{\frac{d^2}{4} + \frac{d^2}{4} - 1} = \sqrt{\frac{d^2}{4} - \frac{d^2}{4} - 1}$. Let us now simplify the notation by setting $S_n = B_n^2$, $n \geq 0$. Then $S_n = (\delta + \frac{1}{\delta})^2$, $n \geq 0$, and by part (d) of the Lemma (with $p = d^2$), $S_{n+2} = (d^2 - 2)S_{n+1} - S_n - 2(d^2 - 4)$, $n \geq 0$. □

4 Perturbed sequences

Here we give a second-order sequence whose squares, when slightly perturbed, form a second-order sequence.

Theorem 2. Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}_{n \geq 0}$ by $C_0 = 2, C_1 = d, C_{n+2} = dC_{n+1} + C_n$, $n \geq 0$. Let $S_{2n} = C_{2n}^2$, $S_{2n+1} = C_{2n+1}^2 + 4$, $n \geq 0$. Then $S_{n+2} = (d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$.

Proof. Solving the recurrence $C_0 = 2, C_1 = d, C_{n+2} = dC_{n+1} + C_n$, $n \geq 0$ in the usual way gives $C_n = \delta^n + \left(\frac{1}{\delta}\right)^n$, where $\delta = \sqrt{\frac{d^2}{4} + 1 + \frac{d^2}{4}} = \sqrt{\frac{d^2}{4} + 1 - \frac{d^2}{4}}$. Then $S_{2n} = C_{2n}^2 = \left(\delta^{2n} + \frac{1}{\delta^{2n}}\right)^2$, $S_{2n+1} = C_{2n+1}^2 + 4 = \left(\delta^{2n+1} + \frac{1}{\delta^{2n+1}}\right)^2$, $n \geq 0$.

Since $\left(\delta + \frac{1}{\delta}\right)^2 = d^2 + 4$, we obtain $(d^2 + 4)S_{n+1} = \left[\left(\delta + \frac{1}{\delta}\right)\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)\right]^2$, and the calculations used in the proof of part (d) of the Lemma now give $S_{n+2} = (d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$. □

Corollary 1. Let $S_{2n} = L_{2n}^2$, $S_{2n+1} = L_{2n+1}^2 + 4$, $n \geq 0$, where $\{L_n\}$ is the Lucas sequence. Then $S_{n+2} = 3S_{n+1} - S_n - 2$, $n \geq 0$.

Proof. This is the case $d = 1$ of Theorem 2. □

Corollary 2. Let $T_{2n} = F_{2n}^2 + \frac{4}{5}$, $T_{2n+1} = F_{2n+1}^2$, $n \geq 0$, where $\{F_n\}$ is the Fibonacci sequence. Then $T_{n+2} = 3T_{n+1} - T_n - 2$, $n \geq 0$.

Proof. This follows from Corollary 1 and the identity [1, pp. 56] $5F_n^2 = L_n^2 - 4(-1)^n$. □

5 Additional special cases

If we now write $\delta = \sqrt{3} - \sqrt{3 - 1}$, $S_n = \frac{1}{4} \left(\delta^n + \frac{1}{\delta^n}\right)^2$, $n \geq 0$, we obtain, just as in the Lemma, $S_0 = 1$, $S_1 = s$, $S_{n+2} = 4(s - 2)S_{n+1} - S_n - 2(s - 1)$, $n \geq 0$.

The following two results can now be proved in essentially the same way as Theorems 1 and 2.

Theorem 3. Let $d \geq 2$ be an integer. Define the sequence $\{B_n\}_{n \geq 0}$ by $B_0 = 1, B_1 = d, B_{n+2} = 2dB_{n+1} - B_n$, $n \geq 0$. Then the sequence of squares $\{B_n^2\}_{n \geq 0}$ satisfies the second-order recurrence

$$B_{n+2}^2 = (4d^2 - 2)B_{n+1} - B_n - 2d^2, \quad n \geq 0.$$  

Theorem 4. Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}_{n \geq 0}$ by $C_0 = 1, C_1 = d, C_{n+2} = 2dC_{n+1} + C_n$, $n \geq 0$. Let $S_{2n} = C_{2n}^2$, $S_{2n+1} = C_{2n+1}^2$, $n \geq 0$. Then $S_{n+2} = (4d^2 + 2)S_{n+1} - S_n - 2d^2$, $n \geq 0$.
6 The more general homogeneous case

**Theorem 5.** Let \( a, b, c \neq 0, d \neq 0 \) be integers, with \( ab \neq 0 \) and \( c^2 \neq 4d \). Let \( B_0 = a, B_1 = b, B_{n+1} = cB_n - dB_{n-1}, n \geq 1 \). Then \( B_{n+1}^2 = (c^2 - 2d)B_n^2 - d^2B_{n-1}^2 + 2(b^2 - a^2d - abc)d^n, n \geq 1 \).

**Proof.** Let \( \alpha, \beta \) be the roots of \( x^2 - cx + d = 0 \). Then \( \alpha, \beta = \frac{1}{2}(c \pm \sqrt{c^2 - 4d}), \alpha \neq \pm \beta, \alpha^2, \beta^2 = \frac{1}{2}(c^2 - 2d \pm c\sqrt{c^2 - 4d}), \alpha \beta = d \). Also \( \alpha^2 \neq \beta^2 \neq d \), since \( c \neq 0, d \neq 0, c^2 \neq 4d \).

According to the result of Selmer stated in the Introduction, there are constants \( A, B, C \) such that \( B_n^2 = A\alpha^{2n} + B\beta^{2n} + Cd^n, n \geq 0 \).

Solving the system

\[
\begin{align*}
    a^2 &= B_0^2 = A + B + C \\
    b^2 &= B_1^2 = B^2 + B\beta^2 + Cd \\
    (bc - ad)^2 &= B_2^2 = A\alpha^4 + B\beta^4 + Cd^2
\end{align*}
\]

for \( C \) gives \( C = \frac{2(ab^2c^2d - abc)}{4d^2 - c^2} \).

Using \( (c^2 - 2d)\alpha^{2n} - d^2\alpha^{2n-2} = \alpha^{2n+2} \) and \( (c^2 - 2d)\beta^{2n} - d^2\beta^{2n-2} = \beta^{2n+2} \) gives \( (c^2 - 2d)B_n^2 - d^2B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + C[(c^2 - 2d)d^n - d^{n+1}] + ed^n \). Now choosing \( e \) so that \( C[(c^2 - 2d)d^n - d^{n+1}] + ed^n = Cd^{n+1} \) (namely \( e = C(4d - c^2) = 2(b^2 + a^2d - abc) \)), finally gives \( (c^2 - 2d)B_n^2 - d^2B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + Cd^{n+1} = B_{n+1}^2 \), which completes the proof. \( \square \)

**Remark.** The result of Theorem 5 appears in [4].

Applying Theorem 5 to the question raised in the Introduction, we immediately get the following result.

**Theorem 6.** Let \( a, b, c \neq 0, d \neq 0 \) be integers, with \( ab \neq 0 \) and \( c^2 \neq 4d \). Let \( B_0 = a, B_1 = b, B_{n+1} = cB_n - dB_{n-1}, n \geq 1 \). Then the sequence of squares \( \{B_n^2\}_{n \geq 0} \) satisfies a second-order linear recurrence (with constant coefficients) if and only if \( d = 1 \), in which case \( B_{n+1}^2 = (c^2 - 2)B_n^2 - B_{n-1}^2 + 2(b^2 + a^2d - abc), n \geq 1 \).

Our final result is the general version of Theorem 2, in which we consider a perturbation of the sequence of squares.

**Theorem 7.** Let \( a, b, c \neq 0, d \neq 0 \) be integers, with \( ab \neq 0 \) and \( c^2 \neq 4d \), such that \( e = \frac{4(ab^2c^2d - abc)}{4d^2 - c^2} \) is an integer. Define the sequence \( \{B_n\}_{n \geq 0} \) by \( B_0 = a, B_1 = b, B_{n+1} = cB_n + dB_{n-1}, n \geq 1 \). Let \( S_{2n} = B_{2n}^2, S_{2n+1} = B_{2n+1}^2 + e, n \geq 0 \). Then \( \{S_n\}_{n \geq 0} \) satisfies the second-order recurrence \( S_{n+1} = (c^2 + 2)S_n - S_{n-1} + 2e + 2(b^2 - a^2d - abc), n \geq 1 \).

**Proof.** This is a direct application of Theorem 5 with \( d = -1 \), according to which \( B_{n+1}^2 = (c^2 + 2)B_n^2 - B_{n-1}^2 + 2(b^2 - a - abc)(-1)^n \).

\( \square \)

**References**


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