Sums of Fractional Parts of Integer Multiples of an Irrational

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Abstract

Let α be a positive irrational real number, and let $C_\alpha(n) = \sum_{1\leq k\leq n}(\{k\alpha\} - \frac{1}{2})$, $n \geq 1$, where $\{x\}$ denotes the fractional part of x. We give an explicit formula for $C_\alpha(n)$ in terms of the simple continued fraction for α, and use this formula to give simple proofs of several results of A. Ostrowski, G. H. Hardy and J. E. Littlewood, and V. T. Sós. We also show that there exist positive constants d_A such that if $\alpha = [a_0, a_1, a_2, \ldots]$ and $(1/t)\sum_{1\leq j\leq t} a_j \leq A$ holds for infinitely many t, then $C_\alpha(x) > d_A \log x$ and $C_\alpha(x) < -d_A \log x$ each hold for infinitely many x.

1 Introduction

For real numbers β, let us agree to write $\{\beta\}$ for the fractional part of β, that is, $\{\beta\} = \beta - [\beta]$, where $[\beta]$ is the greatest integer less than or equal to β.

For an irrational number α, $0 < \alpha < 1$, and for integers $n \geq 1$, we write

$$C_\alpha(n) = \sum_{1\leq k\leq n}(\{k\alpha\} - \frac{1}{2}).$$

There are a number of papers dealing with estimates of $C_\alpha(n)$ as a function of n. Most of the known results are contained in two long 1922 papers by G. H. Hardy and J. E. Littlewood [2, 3], a long 1922 paper by A. Ostrowski [5], and a 1957 paper by Vera T. Sós [8]. Further references can be found in [6].

In the present note we give a simple and self-contained proof of a formula for $C_\alpha(n)$ which is more explicit than those which have appeared before, namely in [5, 8]. We then use this formula to derive several of the main results in [2, 3, 5] and the main result in [8]. It seems to us that our proofs are far simpler than those given previously. In some cases they lead to improvements.

In addition, we extend one of the main results in [3, 5]. Ostrowski and Hardy and Littlewood showed independently that there exist positive constants c and c_A with the following properties. If α is an

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arbitrarily positive irrational number, then the inequality $|C_a(x)| > c\log x$ holds for infinitely many x. If $\alpha = [a_0, a_1, a_2, \ldots]$ and $a_i \leq A$ for all i, then $C_a(x) > c_A \log x$ and $C_a(x) < -c_A \log x$ each hold for infinitely many x. (Hardy and Littlewood [3] call the proofs of these two results the most difficult in their paper. They give no values for the constants c and c_A. Ostrowski gives $c \geq \frac{1}{720}$ and $c_A \geq 1/8(A+1)^6$.) We show that there exist positive constants d_A such that if $\alpha[a_0, a_1, a_2, \ldots]$ and $(1/5)\sum_{1\leq j\leq t} a_j \leq A$ holds for infinitely many t, then $C_\alpha(x) > d_A \log x$ and $C_\alpha(x) < -d_A \log x$ each hold for infinitely many x. (We show that $d_A \geq 1/(7\cdot 64(A+1)^2 \log(A+1))$. Since $c_A \geq d_A$, this improves Ostrowski's bound. We also show that $c \geq \frac{1}{256}$.)

2 Notation

Let α be a positive irrational real number, and let

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

be the simple continued fraction for α, which we abbreviate as $\alpha = [a_0, a_1, a_2, \ldots]$. We write $p_n/q_n = [a_0, a_1, a_2, \ldots, a_n]$ and $d_{2k} = q_{2k}\alpha - p_{2k}$, $d_{2k+1} = p_{2k+1} - q_{2k+1}\alpha$, $k \geq 0$. Then $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$, $n \geq 2$, $p_{2k}/q_{2k} < \alpha < p_{2k+1}/q_{2k+1}$, $k \geq 0$, and $0 < d_n < 1/q_{n+1}$, $n \geq 0$.

3 A formula for $C_\alpha(n)$

Throughout this section, α is a fixed irrational number, $0 < \alpha < 1$, $\alpha = [a_0, a_1, a_2, \ldots]$. Let us use the notation

$$S_\alpha(n) = \sum_{1\leq k\leq n} [k\alpha], \qquad n \geq 1.$$

Lemma 1. *If $1 \leq k \leq q_n$, then $[k\alpha] = [k(p_n/q_n)]$. Also, $[q_n\alpha] = p_n$, n even, $[q_n\alpha] = p_n - 1$, n odd.*

Proof. This follows from $d_n < 1/q_{n+1}$. □

Lemma 2. *For $n \geq 1$,*

$$S_\alpha(q_n) = \frac{1}{2}(p_n q_n - q_n + p_n + (-1)^n),$$
$$C_\alpha(q_n) = \frac{1}{2}(-1)^n(d_n(q_n+1) - 1).$$

Proof. The second part follows from the first using $[\beta] + \{\beta\} = \beta$ and the definition of d_n. To prove the first part, observe that since $(p_n, q_n) = 1$, kp_n runs through all the non-zero residue classes modulo q_n, therefore

$$\sum_{1\leq k\leq q_n-1} \left\{k\frac{p_n}{q_n}\right\} = \sum_{1\leq t\leq q_n-1} \frac{t}{q_n} = \frac{q_n-1}{2},$$

hence

$$\sum_{1\leq k\leq q_n-1} \left[k\frac{p_n}{q_n}\right] = \sum_{1\leq k\leq q_n-1} \left(k\frac{p_n}{q_n} - \left\{k\frac{p_n}{q_n}\right\}\right) = \frac{(p_n-1)(q_n-1)}{2}.$$

Now apply Lemma 1. ∎

Lemma 3. *If $n \geq 1$, $q_n \leq N < q_{n+1}$, $N = bq_n + k$, $1 \leq k < q_n$, then $[N\alpha] = bp_n + [k\alpha]$. If $N = bq_n$, then $[N\alpha] = bp_n - 1$ if n is odd, $[N\alpha] = bp_n$ if n is even.*

Proof. Assume $N = bq_n + k$, $1 \leq k < q_n$. Let $L = [k(p_n/q_n)]$. Then $L + 1/q_n \leq k(p_n/q_n) \leq L + (q_n - 1)/q_n$. If n is even, then $0 < (bq_n + k)(\alpha - p_n/q_n) < q_{n+1}/q_{n+1}q_n = q_n$, or $0 < N\alpha - bp_n - k(p_n/q_n) < 1/q_n$. Adding to the second preceding inequality gives $L < N\alpha - bp_n < L + 1$, hence $[N\alpha] = bp_n + L = bp_n + [k(p_n/q_n)] = bp_n + [k\alpha]$. (For the last equality we used Lemma 1.)

If n is odd the calculation is similar. The second statement of the lemma is easy. (Actually, even more is true, namely if $0 < q < q_{n+1}$, then $[(q + q_n)\alpha] = p_n + [q\alpha]$; see [1].) ∎

Lemma 4. *(a) Let $n \geq 1$, $q_n < bq_n + m < q_{n+1}$, $1 \leq m < q_n$. Then*

$$S_\alpha(bq_n + m) = S_\alpha(bq_n) + S_\alpha(m) + mbp_n.$$

(b) Let $n \geq 1$, $q_n < bq_n < q_{n+1}$. Then

$$S_\alpha(bq_n) = \frac{1}{2}b(bp_nq_n - q_n + p_n + (-1)^n).$$

Proof. (a)

$$\begin{aligned} S_\alpha(bq_n + m) &= \sum_{1 \leq k \leq bq_n + m} [k\alpha] = S_\alpha(bq_n) + \sum_{1 \leq k \leq m} [(bq_n + k)\alpha] \\ &= S_\alpha(bq_n) + \sum_{1 \leq k \leq m} (bp_n + [k\alpha]). \end{aligned}$$

(b) For $b = 1$, Lemma 2 applies. Now induction using part (a) does the rest.

∎

Lemma 5. *(a) Let $n \geq 1$, $q_n < bq_n + m < q_{n+1}$, $1 \leq m < q_n$. Then*

$$C_\alpha(bq_n + m) = C_\alpha(bq_n) + C_\alpha(m) + mb(q_n\alpha - p_n).$$

(b) Let $n \geq 1$, $q_n \leq bq_n < q_{n+1}$. Then

$$C_\alpha(bq_n) = \frac{1}{2}(-1)^n b(d_n(bq_n + 1) - 1).$$

Proof. These follow from Lemma 4. ∎

Theorem 1. *For any $m \geq 1$, let $m = z_tq_{t-1} + \cdots + z_2q_1 + z_1q_0$, where*

1. *$0 \leq z_1 \leq a_1 - 1$,*
2. *$0 \leq z_i \leq a_i$, $2 \leq i \leq t$,*
3. *If $z_i = a_i$ then $z_{i-1} = 0$, $2 \leq i \leq t$.*

(This is the so-called "Zeckendorff representation of m." To find it, subtract the largest possible q_j from m and repeat.)

(a)

$$S_\alpha(m) = \frac{1}{2}\sum_{1\le i\le t} z_i(z_i p_{i-1}q_{i-1} - q_{i-1} + p_{i-1} + (-1)^{i-1}) + \sum_{1\le i<j\le t} z_i z_j q_{i-1} p_{j-1}.$$

(b)

$$C_\alpha(m) = \frac{1}{2}\sum_{1\le i\le t} (z_i(q_{i-1}\alpha - p_{i-1})(z_i q_{i-1}+1) + (-1)^i z_i) + \sum_{1\le i<j\le t} z_i z_j q_{i-1}(q_{j-1}\alpha - p_{j-1}).$$

(c)

$$C_\alpha(m) = \sum_{1\le j\le t} (-1)^j z_j \left(\frac{1}{2} - d_{j-1}\left(m_{j-1} + \frac{1}{2} z_j q_{j-1} + \frac{1}{2}\right)\right),$$

where $m_j = \sum_{1\le i\le j} z_i q_{i-1}$, $m_0 = 0$.

Proof. Part (a) follows from Lemma 4 by induction on t. Part (b) then follows from part (a) (or from Lemma 5 and induction). Part (c) is a rearrangement of part (b). □

4 A Bound for $\max_{0<m<q_t} |C_\alpha(m)|$

Theorem 2. *For* $t \ge 1$,

$$\frac{1}{32}\sum_{1\le j\le t}(a_j - 1) < \max_{0<m<q_t} |C_\alpha(m)| < \frac{1}{2}\sum_{1\le j\le t} a_j.$$

Proof. Let $m = \sum_{1\le j\le t} z_j q_{j-1}$. Since $0 < m_{j-1} + \frac{1}{2}z_j q_{j-1} + \frac{1}{2} \le q_j$ and $0 < d_{j-1} < 1/q_j$, Theorem 1(c) gives $|C_\alpha(m)| < \frac{1}{2}\sum_{1\le j\le t} z_j \le \frac{1}{2}\sum_{1\le j\le t} a_j$.

For the other side, define $M_1 = \sum_{1\le j\le t} z_j q_{j-1}$ by $z_j = [a_j/2]$ if j is odd, $z_j = 0$ if j is even. Then $m_j \le \sum_{1\le i\le t}(a_i/2)q_{i-1} \le (q_j - 1)/2$, so if j is odd, $m_{j-1} + \frac{1}{2}z_j q_{j-1} + \frac{1}{2} = m_{j-2} + \frac{1}{2}[a_j/2]q_{j-1} + \frac{1}{2} \le \frac{1}{2}(q_{j-1} - 1 + \frac{1}{4}a_j q_{j-1} + 1) = \frac{1}{4}(2q_{j-2} + a_j q_{j-1}) = \frac{1}{4}(q_j + q_{j-2}) < \frac{3}{8}q_j$, therefore $C_\alpha(M_1) < -\sum_{j \text{ odd}} z_j(\frac{1}{2} - \frac{3}{8}) = -\frac{1}{8}\sum_{j \text{ odd}} z_j \le -\frac{1}{16}\sum_{j \text{ odd}}(a_j - 1)$. Define $M_2 = \sum_{1\le j\le t} z_j q_{j-1}$ by $z_j = [a_j/2]$ if j is even, $z_j = 0$ if j is odd; then a similar calculation gives $C_\alpha(M_2) > \frac{1}{16}\sum_{j \text{ even}}(a_j - 1)$. Thus $C_\alpha(M_2) - C_\alpha(M_1) > \frac{1}{16}\sum_{1\le j\le t}(a_j - 1)$, therefore $\frac{1}{32}\sum_{1\le j\le t}(a_j - 1) < \max_{0<m<q_t} |C_\alpha(m)|$. □

5 Known Asymptotic Bounds and Inequalities for $C_\alpha(n)$

The following facts are known, but the proofs we give are simpler (and shorter) than those given before. In some cases our results are slight improvements. We use the notation $\alpha = [0, a_1, a_2, \ldots]$ for each α, $0 < \alpha < 1$.

Fact 1. *(Sierpenski [7]). For every* α, $C_\alpha(n) = o(n)$.

Proof. Given $\alpha = [0, a_1, a_2, \ldots]$ and $\varepsilon > 0$, choose n so that $(\sqrt{2})^{-(n-1)} < \varepsilon$, and let $P = a_1 + \cdots + a_n$. Choose $s > n$ so that $P/q_n < \varepsilon$. For $m \geq q_s$, let $m = \sum_{1 \leq j \leq t+1} z_j q_{j-1}$; then $t \geq s$ and $z_{t+1} \geq 1$. Then

$$\begin{aligned}
\frac{1}{m}|C_\alpha(m)| &< \frac{1}{2m}\sum_{1 \leq j \leq t} z_j \leq \frac{1}{2}\frac{P}{q_t} + \frac{1}{2}\frac{\sum_{n+1 \leq j \leq t+1} z_j}{\sum_{n+1 \leq j \leq t+1} z_j q_{j-1}} \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\frac{\sum_{n+1 \leq j \leq t+1} z_j}{(\sqrt{2})^{n-1}\sum_{n+1 \leq j \leq t+1} z_j} < \varepsilon.
\end{aligned}$$

(Here we used $q_n > (\sqrt{2})^{n-1}$.) □

Fact 2. *(Lerch (without proof) [4], Hardy and Littlewood [2], and Ostrowski [5] have this result under the stronger hypothesis that $a_i \leq A$ for all i.) If $\alpha = [0, a_1, a_2, \ldots]$ and $(1/t)\sum_{1 \leq j \leq t} a_j \leq A$ for all i, then $C_\alpha(n) = O(\log n)$. In fact,*

$$|C_\alpha(n)| < \frac{A}{2\log\tau}\log n + \left(\frac{\log\sqrt{5}}{2\log\tau} - \frac{1}{2}\right)A, \qquad n \geq 2, \text{ where } \tau = \frac{1+\sqrt{5}}{2}.$$

Consequently, $|C_\alpha(n)| < \frac{3}{2}A\log n$, $n \geq 1$, and for every $\varepsilon > 0$ there is N_0 such that $|C_\alpha(n)| < (1/(2\log\tau) + \varepsilon)A\log n$, $n \geq N_0$. In particular, $|C_\alpha(n)| < (1.04)A\log n$, $n \geq N_0$.

Proof. Let

$$\frac{P_n}{Q_n} = \overbrace{[1, 1, \ldots, 1]}^{n+1}.$$

Then $Q_0 = Q_1 = 1$, $Q_{n+1} = Q_n + Q_{n-1}$, and $Q_t = (1/\sqrt{5})(\tau^{t+1} - (-1/\tau)^{t+1}) > (1/\sqrt{5})(\tau^{t+1} - 1)$. Since $|C_\alpha(q_t)| < \frac{1}{2}$, we can assume without loss of generality that $q_t < n < q_{t+1}$. Then $\sqrt{5}n > \sqrt{5}q_t + 1 \geq \sqrt{5}Q_t + 1 > \tau^{t+1}$, so $t + 1 < (\log\sqrt{5} + \log n)/\log\tau$, $\frac{1}{2}t < \frac{1}{2}(\log\sqrt{5}/\log\tau - 1) + \log n/(2\log\tau)$. Since $|C_\alpha(n)| < \frac{1}{2}\sum_{1 \leq j \leq t} a_j \leq \frac{1}{2}tA$, the result follows. □

Fact 3. *If $f(n) = o(n)$, then there exists α such that*

$$\limsup_{n\to\infty}\left|\frac{C_\alpha(n)}{f(n)}\right| = \infty.$$

Proof. Assume that $f(n) > 0$ for all n. Suppose $a_1, \ldots, a_t$ have already been chosen. Choose M so that $f(x)/x < 1/64(t+1)q_t$, $x > M$. Let $P = \max\{f(x) : 1 \leq x \leq M\}$. Choose $a_{t+1} \geq 3$ so that $\frac{1}{32}(a_{t+1} - 1) > (t+1)P$. Finally, choose x, $0 < x < q_{t+1}$, so that $\frac{1}{32}(a_{t+1} - 1) \leq \frac{1}{32}\sum_{1 \leq j \leq t+1}(a_j - 1) < |C_\alpha(x)|$. If $1 \leq x \leq M$, then

$$\frac{|C_\alpha(x)|}{f(x)} > \frac{(a_{t+1} - 1)}{32P} > (t+1).$$

If $M < x < q_{t+1}$, then

$$\frac{|C_\alpha(x)|}{f(x)} = \frac{|C_\alpha(x)|}{x}\frac{x}{f(x)} > \frac{(a_{t+1} - 1)}{32q_{t+1}} \cdot 64(t+1)q_t > t+1,$$

since $2(a_{t+1} - 1)q_t \geq (a_{t+1} + 1)q_t > q_{t+1}$. □

Fact 4. *If $A \geq 2$ and $f(n) = o(\log n)$, then there exists $\alpha = [0, a_1, a_2, \ldots]$ with $a_i \leq A$ for all i and*

$$\limsup_{n\to\infty} \left| \frac{C_\alpha(n)}{f(n)} \right| = \infty.$$

In fact, this is true for every α such that $a_i < A$ for all i and $\sum_{1\leq i\leq t} a_i \geq ct$ infinitely often, for some fixed $c > 1$.

Proof. Assume that $f(n) > 0$ for all n. Fix $L \geq 1$. Choose M so that $f(x)/\log x < (c-1)/32L\log(A+1)$, $x > M$. Let $P = \max\{f(x) : 1 \leq x \leq M\}$. Choose t so that $\frac{1}{32}\sum_{1\leq i\leq t}(a_j - 1) \geq \frac{1}{32}(c-1)t > LP$. Choose x, $0 < x < q_t$, so $|C_\alpha(x)| > \frac{1}{32}\sum_{1\leq i\leq t}(a_j - 1) > LP$. If $1 \leq x \leq M$ then

$$\frac{|C_\alpha(x)|}{f(x)} \geq \frac{|C_\alpha(x)|}{P} > L.$$

If $M < x < q_t$ then

$$\frac{|C_\alpha(x)|}{f(x)} = \frac{|C_\alpha(x)|}{\log x}\frac{\log x}{f(x)} > \frac{(c-1)t}{32\log q_t} \cdot \frac{32L\log(A+1)}{(c-1)} > L,$$

where the last inequality holds since $q_{s+1} \leq Aq_s + q_{s-1} < (A+1)q_s$, so $\log q_t < t\log(A+1)$. □

6 New Results

Theorem 3. *Let $\alpha = [a_0, a_1, a_2, \ldots]$.*

(a) If $\sum_{1\leq i\leq t} a_j \geq (1+\frac{1}{7})t$ infinitely often, then the inequality $|C_\alpha(x)| > \frac{1}{256}\log x$ holds for infinitely many x.

(b) If $\sum_{1\leq i\leq t} a_j \leq (1+\frac{1}{7})t$ infinitely often, then $C_\alpha(x) > \frac{1}{56}\log x$ and $C_\alpha(x) < -\frac{1}{56}\log x$ each hold for infinitely many x.

Lemma 6. *([5]). For each $t \geq 1$, $\sum_{1\leq i\leq t} a_i > \log q_t$.*

Proof of Lemma. $q_t \leq (a_t+1)q_{t-1}\cdots \leq (a_t+1)\cdots(a_2+1)(a_1+1) \leq e^{a_t}\cdots e^{a_2}e^{a_1}$. □

Proof of Theorem. (a) For t such that $\sum_{1\leq i\leq t} a_j \leq (1+\frac{1}{7})t$, apply Theorem 2 to get x, $0 < x < q_t$, with $|C_\alpha(x)| > \frac{1}{32}\sum_{1\leq j\leq t}(a_j - 1) \geq \frac{1}{32}(t/7)$. By the Lemma, either $\sum_{1\leq j\leq t}(a_j-1) < \frac{1}{8}\log q_t$ or $t > \frac{7}{8}\log q_t$; in either case we get $|C_\alpha(x)| > (1/(8\cdot 32))\log x$.

(b) For t such that $\sum_{1\leq j\leq t} a_j \leq (1+\frac{1}{7})t$, let $G = \{j : 6j+1 \leq t, a_{6j} = a_{6j+1} = 1\}$. Since the number of $a_j > 1$ among $a_1, \ldots, a_t$ is at most $[t/7]$, it follows that $|G| \geq 6t/7$. Let $x = \sum_{j\in G} q_{6j}$. Note that $0 < x < q_t$, so that $\frac{8}{7}t \geq \sum_{1\leq i\leq t} a_j > \log q_t > \log x$. According to Theorem 1(c), $C_\alpha(x) = -\sum_{j\in G}(\frac{1}{2} - d_{6j}(m_{6j} + \frac{1}{2}q_{6j} + \frac{1}{2}))$. To estimate $d_{6j}m_{6j}$, we use $d_{6j} < 1/q_{6j+1}$, $m_{6j} \leq q_6 + q_{12} + \cdots + q_{6j+1}$ (since $q_s < 2q_{s+2}$), $q_{12} < (1/8^{j-2})q_{6j+1}$, etc., to get

$$d_{6j}m_{6j} < \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^{j-1}} = \frac{1}{7} - \frac{1}{7}\cdot\frac{1}{8^{j-1}}.$$

Next, for $j \in G$ we have $q_{6j}/q_{6j+1} = (q_{6j-1}+q_{6j-2})/(2q_{6j-1}+q_{6j-2}) < \frac{2}{3}$. Finally, $d_{6j} < 1/q_{6j+1} < 1/8^j$. Therefore

$$C_\alpha(x) < -\sum_{j\in G}\left(\frac{1}{2}-\left(\frac{1}{7}-\frac{1}{7}\cdot\frac{1}{8^{j-1}}+\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{8^j}\right)\right)$$
$$< -\sum_{j\in G}\left(\frac{1}{2}-\left(\frac{1}{7}+\frac{1}{3}\right)\right) = -|G|\frac{1}{42}.$$

Since $|G| \geq 6t/7$ and $8t/7 > \log x$, this gives $C_\alpha(x) < -\frac{1}{56}\log x$.

The proof that $C_\alpha(x) > \frac{1}{56}\log x$ for infinity many x is essentially the same. □

Theorem 4. *Let A be a positive constant. Then there exists a positive constant d_A such that if $\alpha = [0, a_1, a_2, \ldots]$ is any irrational such that $\sum_{1\leq i\leq t} a_j \leq At$ for infinitely many t, then each of $C_\alpha(x) > d_A \log x$ and $C_\alpha(x) < -d_A \log x$ holds for infinitely many x.*

Proof. We show that $C_\alpha(x) < -d_A \log x$ holds infinitely often. The proof of the other inequality is essentially the same. Our method is similar to the proof of Fact 5.

First, choose L so that $1/(2^L-1) < \frac{1}{4}\cdot(1/(4A+2))$. Let $t > 4L$ and $t > 12A$, with $\sum_{1\leq i\leq t} a_j \leq At$.

From among the even terms $a_2, a_4, \ldots, a_{2k}, \ldots$ with $2k < t$, choose $u = [t/4]$ terms each less than or equal to $4A$. From among these u terms choose the Lth, $(2L)$th, ..., (wL)th successive terms, where $w = [t/4L]$. Let us call these w terms $a_{j_1}, \ldots, a_{j_w}$.

Then $a_{j_1}, \ldots, a_{j_w}$ have the following properties.

1. j_k is even, $1 \leq k \leq w$.

2. $a_{j_k} \leq 4A$, $1 \leq j \leq w$.

3. $j_1 < \cdots < j_w$ and $j_{k+1} - j_k \geq 2L$, $1 \leq k \leq w-1$.

Now let $x = q_{j_1} + \cdots + q_{j_w}$. (Then $x < q_t$.) We will show using Theorem 1(c) (as in the proof of Fact 5) that $C_\alpha(x) < -dt$ for some constant d depending only on A. Then since $\log x < \log q_t < \sum_{1\leq i\leq t} a_j \leq At$, we will have $C_\alpha(x) < -(d/A)\log x$, and we can take $d_A = d/A$.

To apply Theorem 1(c), since the j_k's are even we have $C_\alpha(x) = -\sum_{1\leq k\leq w}(\frac{1}{2} - d_{j_k}(m_{j_k} + \frac{1}{2}q_{j_k} + \frac{1}{2}))$.

Since $d_{j_k} < 1/q_{j_k+1}$, $m_{j_k} = q_{j_1} + \cdots + q_{j_{k-1}}$, and $q_{j_2} > 2^L a_{j_1}$, etc. (since $j_2 - j_1 \geq 2L$), we get

$$d_{j_k} m_{j_k} < \frac{1}{2^L} + \frac{1}{2^{2L}} + \cdots + \frac{1}{2^{(k-1)L}} = \frac{1}{2^L-1}\left(1 - \frac{1}{2^{(k-1)L}}\right).$$

Next, we use the (easily verified) fact that if $a_{s+1} \geq 2$ then $q_s/q_{s+1} < \frac{1}{2}$ and if $a_{s+1} = 1$ then $q_s/q_{s+1} < 1 - 1/(a_s+2)$. Since $a_{j_k} \leq 4A$ (and $\frac{1}{2} < 1 - 1/(4A+2)$), $d_{j_k} q_{j_k} < q_{j_k}/q_{j_k+1} < 1 - 1/(4A+2)$.

Finally, $d_{j_k} < 1/q_{j_k+1} < 1/2^{kL}$. Putting all these together gives

$$C_\alpha(x) < -\sum_{1\leq k\leq w}\left(\frac{1}{2} - \frac{1}{2^L-1}\left(1-\frac{1}{2^{(k-1)L}}\right) - \frac{1}{2}\left(1-\frac{1}{4A+2}\right) - \frac{1}{2}\frac{1}{2^{kL}}\right)$$
$$< -\sum_{1\leq k\leq w}\left(\frac{1}{2}\left(\frac{1}{4A+2}\right) - \frac{1}{2^L-1}\right) < -w\cdot\frac{1}{4}\cdot\frac{1}{4A+2} < -dt,$$

since $w = [t/4L]$ and L depends only on A. This completes the proof. □

By tracing back through the choice of L, one can see that $d_A \geq 1/(7 \cdot 64(A+1)^2 \log(A+1))$.

Theorem 5. *Let* $\alpha = [0, a_1, a_2, \ldots]$.

(a) If $\liminf_{t\to\infty}(1/t)\sum_{1\leq j\leq t} a_j = 1$, *then* $C_\alpha(x) > \frac{1}{56}\log x$, $C_\alpha(x) < -\frac{1}{56}\log x$ *hold infinitely often.*

(b) If $\liminf_{t\to\infty}(1/t)\sum_{1\leq j\leq t} a_j < \infty$, *then* $C_\alpha(x) > d\log x$, $C_\alpha(x) < -d\log x$ *hold infinitely often, for some* $d > 0$.

(c) If $\liminf_{t\to\infty}(1/t)\sum_{1\leq j\leq t} a_j = \infty$, *then such a* d *may fail to exist. (See the theorem of Vera T. Sós below.) However, for every* $\varepsilon > 0$, $|C_\alpha(x)| > (\frac{1}{32} - \varepsilon)\log x$ *holds infinitely often.*

Proof. Part (a) follows from Theorem 3(b). Part (b) follows from Theorem 4. Part (c) is proved in the same way as Theorem 3(a). □

7 The Vera T. Sós Theorem

Our final application of Theorem 1(c) will be a simplified proof of the following result of Vera T. Sós [8], which answered a question of Ostrowski [5].

Fact 5. *Let* $\alpha = [0, a_1, a_2, \ldots]$, *where* $a_{2n+1} = 1$, $a_{2n} = n^2$, $n \geq 0$. *Then there exists a constant* C *such that* $C_\alpha(n) > C$ *for all* $n \geq 1$.

(In Sós's paper, the a_i's are indexed differently, and n^3 appears rather than n^2.)

Lemma 1. *For* $k \geq 1$, $\frac{1}{2}(q_{2k-2}/q_{2k-1} + d_{2k}/d_{2k-1} - 2) < k^2(\frac{1}{2} - d_{2k-1}q_{2k-1}(1 + \frac{1}{2}k^2)) < 0$.

Proof. Using $d_{s+1} < d_s$, $q_s < q_{s+1}$, $q_{s+1}d_s + q_s d_{s+1} = 1$, and $q_{2k} = k^2 q_{2k-1} + q_{2k-2}$, we get

$$\frac{1}{d_{2k-1}q_{2k-1}} = \frac{q_{2k}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}} = k^2 + \frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}} < k^2 + 2,$$

so $d_{2k-1}q_{2k-1}(1 + \frac{1}{2}k^2) > (1 + \frac{1}{2}k^2)/(k^2 + 2) = \frac{1}{2}$, which gives the right-hand inequality. Next,

$$\begin{aligned}
0 &> \frac{1}{2} - d_{2k-1}q_{2k-1}\left(1 + \frac{1}{2}k^2\right) \\
&= \frac{1}{2} - \frac{1 + \frac{1}{2}k^2}{k^2 + \frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}}} \\
&= \frac{\frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}} - 2}{2\left(k^2 + \frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}}\right)} \\
&> \frac{\frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}} - 2}{2k^2},
\end{aligned}$$

which gives the left-hand inequality. □

Lemma 2. *For* $k \geq 2$, $q_{2k-2}/q_{2k-1} > 1 - 1/(k-1)^2$. *For* $k \geq 1$, $d_{2k}/d_{2k-1} > d_{2k}q_{2k} > 1 - 2/k^2$.

Proof. For $k \geq 2$,

$$\frac{q_{2k-2}}{q_{2k-1}} = \frac{q_{2k-2}}{q_{2k-2}+q_{2k-3}} = 1 - \frac{q_{2k-3}}{q_{2k-2}+q_{2k-3}} > 1 - \frac{q_{2k-3}}{q_{2k-2}} > 1 - \frac{1}{(k-1)^2}.$$

For $k \geq 1$,

$$\begin{aligned}\frac{d_{2k}}{d_{2k-1}} &> d_{2k}q_{2k} \\ &= \frac{1}{\frac{q_{2k+1}}{q_{2k}} + \frac{d_{2k+1}}{d_{2k}}} \\ &= \frac{1}{1+\frac{q_{2k-1}}{q_{2k}} + \frac{d_{2k+1}}{d_{2k}}} \\ &> \frac{1}{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} > 1 - \frac{2}{k^2}\end{aligned}$$

(Here we used $q_{2k} = k^2 q_{2k-1} + q_{2k-2}$ and $d_{2k} = (k+1)^2 d_{2k+1} + d_{2k+2}$.) $\square$

Proof of Fact 5. Let $m = \sum_{1\leq j\leq t} z_j q_{j-1}$ be the Zeckendorff representation of m. Then

$$C_\alpha(m) = \sum_{1\leq j\leq t} (-1)^j z_j (\frac{1}{2} - d_{j-1}(m_{j-1} + \frac{1}{2}z_j q_{j-1} + \frac{1}{2})) = D_0(m) - D_1(m),$$

where

$$D_0(m) = \sum_{1\leq 2k\leq t} z_{2k}(\frac{1}{2} - d_{2k-1}(m_{2k-1} + \frac{1}{2}z_{2k}q_{2k-1} + \frac{1}{2})),$$

$$D_1(m) = \sum_{1\leq 2k+1\leq t} z_{2k+1}(\frac{1}{2} - d_{2k}(m_{2k} + \frac{1}{2}z_{2k+1}q_{2k} + \frac{1}{2})).$$

We wish to find constants A and B such that $D_0(m) \geq A$, $D_1(m) \geq B$, $m \geq 1$, so that $C_\alpha(m) \geq A - B$, $m \geq 1$.

Since $m_{2k-1} \leq q_{2k-1}$, we have

$$\begin{aligned}D_0(m) &> \sum_{1\leq 2k\leq t} z_{2k}(\frac{1}{2} - d_{2k-1}(q_{2k-1} + \frac{1}{2}z_{2k}q_{2k-1})) \\ &= \sum_{1\leq 2k\leq t} z_{2k}(\frac{1}{2} - d_{2k-1}q_{2k-1}(1+\frac{1}{2}z_{2k})).\end{aligned}$$

Using $z_{2k} \leq a_{2k} = k^2$ and part of Lemma 1, this gives $D_0(m) \geq \sum_{1\leq 2k\leq t} k^2(\frac{1}{2} - d_{2k-1}q_{2k-1}(1+\frac{1}{2}k^2))$. The other part of Lemma 1, followed by Lemma 2, now gives

$$\begin{aligned}D_0(m) &> \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{q_{2k-2}}{q_{2k-1}} + \frac{d_{2k}}{d_{2k-1}} - 2\right) \qquad > \frac{1}{2}\left(\frac{q_0}{q_1} + \frac{d_2}{d_1} - 2\right) + \frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{(k-1)^2} - \frac{2}{k^2}\right) \\ &= A.\end{aligned}$$

Next, first omitting some negative terms from $D_1(m)$ gives

$$D_1(m) \leq \sum_{1\leq 2k+1\leq t} z_{2k+1}(\frac{1}{2} - \frac{1}{2}d_{2k}z_{2k+1}q_{2k}).$$

Using Lemma 2 gives

$$\begin{aligned} D_1(m) &\leq \frac{1}{2}z_1(1-d_0z_1q_0) + \frac{1}{2}\sum_{3\leq 2k+1\leq t} z_{2k+1}\left(1 - z_{2k+1}\left(1-\frac{2}{k^2}\right)\right) \\ &\leq \frac{1}{2}z_1(1-d_0z_1q_0) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{2}{k^2} = B. \end{aligned}$$

(For the last inequality, we used $0 \leq z_{2k+1} \leq a_{2k+1} = 1$.) □

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