A Simple Proof of a Remarkable Continued Fraction Identity

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Abstract

We give a simple proof of a generalization of the equality

$$\sum_{n=1}^{\infty} \frac{1}{2^{[n/\tau]}} = [2, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots],$$

where $\tau = (1+\sqrt{5})/2$ and the exponents of the partial quotients are the Fibonacci numbers, and some closely related results.

P. E. Böhmer [2], L. V. Danilov [4], and W. W. Adams and J. L. Davison [1] showed independently that if $\alpha > 0$ is irrational, $b > 1$ is an integer, and $S_b(\alpha) = (b-1)\sum_{k=1}^{\infty} \frac{1}{b^{[k/\alpha]}}$, then the simple continued fraction for $S_b(\alpha)$ can be described explicitly in the following way. Let α have simple continued fraction

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [a_0, a_1, \ldots],$$

with $\frac{p_n}{q_n} = [a_0, \ldots, a_n]$, $n \geq 0$. Let $t_0 = a_0 b$, $t_n = \frac{b^{q_n} - b^{q_{n-2}}}{b^{q_{n-1}} - 1}$, $n \geq 1$. Then $S_b(\alpha) = [t_0, t_1, \ldots]$. Thus in the case $\alpha = \tau = (1+\sqrt{5})/2$, the golden ratio, and $b = 2$, one gets the remarkable equality $\sum_{n=1}^{\infty} \frac{1}{2^{[n/\tau]}} = [2, 2^0, 2^1, 2^2, 2^3, 2^5, \ldots]$, where the exponents of the partial quotients are the Fibonacci numbers.

More recently, R. L. Graham, D. E. Knuth, and O. Patashnik [7] indicated how to give a very different proof of the power series version of this result, where the number b is replaced by an indeterminate (they carried out the proof for the case $\alpha = (1+\sqrt{5})/2$), using the continuant polynomials of Euler [5].

In this note we give a proof, which we feel is simpler than the others, which makes use of a property of the “characteristic sequence” of α discovered by H. J. S. Smith [12]. The crucial idea of our approach appears in Lemma 2 below, where we regard certain initial segments of the characteristic sequence of α as base b representations of integers.

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(Böhmer, Danilov, and Adams and Davison also show that $S_b(\alpha)$ is transcendental for every irrational α. We omit the proof of this fact, which is an easy application of a theorem of Roth [10], using Lemma 3 and Theorem B below.)

Preliminaries. Let α be an irrational number with $0 < \alpha < 1$. (At the end, we will remove the restriction $\alpha < 1$.) Let $\alpha = [0, a_1, a_2, \ldots]$ and $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$, $n \geq 0$, where p_n, q_n are relatively prime non-negative integers. (As usual, we put $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$, so that $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for all $n \geq 0$.) For $n \geq 1$, define $f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$, and consider the infinite binary sequence $f_\alpha = (f_\alpha(n))_{n \geq 1}$, which is sometimes called the *characteristic sequence* of α. Define binary words X_n, $n \geq 0$, by $X_0 = 0$, $X_1 = 0^{a_1 - 1}1$, $X_k = X_{k-1}^{a_k} X_{k-2}$, $k \geq 2$, where X^a denotes the word X repeated a times, and $X_1 = 1$ if $a_1 = 1$.

The following result was first proved by Smith [12]. Other proofs can be found in [3, 6, 11, 13], and further references to the characteristic sequence can be found in [3]. Nishioka, Shiokawa, and Tamura [8] treat the more general case $[(n+1)\alpha + \beta] - [n\alpha + \beta]$.

Lemma 1. *For each $n \geq 1$, X_n is a prefix of f_α. That is, $X_n = f_\alpha(1) f_\alpha(2) \cdots f_\alpha(s)$, where s is the length of X_n.*

The main proof. We are now ready to prove the result stated in the Introduction. (However, we will keep the restriction $\alpha < 1$ until the following section.) Let $b > 1$ be an integer, let $0 < \alpha < 1$ be irrational, $\alpha = [0, a_1, a_2, \ldots]$, let $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$, $n \geq 0$, and let the binary words X_n, $n \geq 0$, be defined as above.

According to Lemma 1, the binary word X_n (which has length q_n by a trivial induction using $q_n = a_n q_{n-1} + q_{n-2}$) is identical with the binary word $f_\alpha(1) f_\alpha(2) \cdots f_\alpha(q_n)$. If we let x_n denote the integer whose base b representation is X_n, i.e., $x_n = f_\alpha(1) b^{q_n - 1} + f_\alpha(2) b^{q_n - 2} + \cdots + f_\alpha(q_n) b^0$, then we can write

$$x_n = b^{q_n} \cdot \sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k}.$$

Now we come to the crucial step.

Lemma 2. *For $n \geq 0$, let $t_{n+1} = \frac{b^{q_{n+1}} - b^{q_{n-1}}}{b^n - 1}$. Then for $n \geq 1$,*

$$x_{n+1} = t_{n+1} x_n + x_{n-1}.$$

Proof. Using the facts that X_n has length q_n, X_{n-1} has length q_{n-1}, x_{n+1} is the integer whose base b representation is X_{n+1}, and $X_{n+1} = X_n^{a_{n+1}} X_{n-1}$, it follows that

$$\begin{aligned} x_{n+1} &= b^{q_{n-1}}(1 + b^{q_n} + b^{2q_n} + \cdots + b^{(a_{n+1}-1)q_n}) x_n + x_{n-1} \\ &= \frac{b^{q_{n-1}}(b^{a_{n+1}q_n} - 1)}{(b^{q_n} - 1)} x_n + x_{n-1} = t_{n+1} x_n + x_{n-1} \end{aligned}$$

□

Lemma 3. *For $n \geq 1$,*

$$[0, t_1, \ldots, t_n] = \frac{b-1}{b^{q_n} - 1} \cdot x_n.$$

Proof. Let $y_n = \frac{b^{q_n}-1}{b-1}$, $n \geq 0$. We show by induction on n that $[0, t_1, \ldots, t_n] = \frac{x_n}{y_n}$. We start the induction at $n = 0$ by setting $t_0 = 0$. Note that $x_0 = 0$, $x_1 = 1$, $y_0 = 1$, $y_1 = \frac{b^{q_1}-1}{b-1} = t_1$. For the induction step, we simply note that $x_{n+1} = t_{n+1}x_n + x_{n-1}$ and $y_{n+1} = t_{n+1}y_n + y_{n-1}$. □

Theorem A. Let $b > 1$ be an integer, and let $0 < \alpha < 1$ be irrational, with $f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$, $n \geq 1$. Let $\alpha = [0, a_1, a_2, \ldots]$, let $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$, $n \geq 0$ (where p_n, q_n are relatively prime non-negative integers), and let $t_n = \frac{b^{q_n} - b^{q_{n-2}}}{b^{q_{n-1}} - 1}$, $n \geq 1$. Then

$$(b-1)\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = [0, t_1, t_2, \ldots].$$

Proof. We have seen that $x_n = b^{q_n} \sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k}$. Hence by Lemma 3,

$$(b-1)\left(\frac{b^{q_n}}{b^{q_n}-1}\right)\sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k} = [0, t_1, \ldots, t_n],$$

and we can take the limit as $n \to \infty$. □

Theorem B. With the same hypotheses as in Theorem A, we have

$$(b-1)\sum_{n=1}^{\infty} \frac{1}{b^{[n/\alpha]}} = [0, t_1, t_2, \ldots].$$

Proof. This is a restatement of Theorem A, using the easily verified fact (when $0 < \alpha < 1$) that $f_\alpha(k) = 1$ if and only if $k = [n/\alpha]$ for some n. □

Theorem C. With the same hypotheses as in Theorem A, we have

$$(b-1)^2 \sum_{k=1}^{\infty} \frac{[k\alpha]}{b^k} = [0, t_1, t_2, \ldots].$$

Proof. Using $f_\alpha(k) = [(k+1)\alpha] - [k\alpha]$ and $[\alpha] = 0$, the series in Theorem C is obtained from the series in Theorem A by a slight rearrangement. □

Theorem D. With the same hypotheses as in Theorem A, we have

$$\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = (b-1)\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(b^{q_k}-1)(b^{q_{k-1}}-1)}.$$

Proof. We say in the proof of Lemma 3 that $[0, t_1, \ldots, t_n] = \frac{x_n}{y_n}$, $n \geq 1$, where $y_n = \frac{b^{q_n}-1}{b-1}$, $n \geq 0$. By a well-known theorem (J. B. Roberts [9, pp. 101]), $\frac{x_n}{y_n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{y_k y_{k-1}}$, $n \geq 1$, and Theorem D now follows from Theorem A. □

Removing the restriction $\alpha < 1$. Now let $\alpha' = a_0 + \alpha$, where $a_0 \geq 0$ is an integer, α is irrational, and $0 < \alpha < 1$.

By Theorem A we get

$$\begin{aligned}(b-1)\sum_{k=1}^{\infty}\frac{f_{\alpha'}(k)}{b^k} &= (b-1)\sum_{k=1}^{\infty}\frac{a_0+f_{\alpha}(k)}{b^k}\\ &= (b-1)a_0\sum_{k=1}^{\infty}\frac{1}{b^k}+(b-1)\sum_{k=1}^{\infty}\frac{f_{\alpha}(k)}{b^k}\\ &= a_0+[0,t_1,t_2,\ldots]\\ &= [a_0,t_1,t_2,\ldots].\end{aligned}$$

To handle Theorem B we need to use the fact, whose simple proof we omit, that if $\alpha' = a_0+\alpha$, where $0<\alpha<1$, then for each $k=0,1,2,\ldots$, the value k is assumed by the expression $[n/\alpha']$ exactly a_0+1 times if $[n/\alpha]=k$ for some $n\geq 1$, and exactly a_0 times if $[n/\alpha]$ never equals k. It then follows from Theorem B that $(b-1)\sum_{n=1}^{\infty}\frac{1}{b^{[n/\alpha']}}=[a_0 b,t_1,t_2,\ldots]$.

By Theorem C and some careful rearrangement we get $(b-1)^2\sum_{k=1}^{\infty}\frac{[k\alpha']}{b^k}=[a_0 b,t_1,t_2,\ldots]$.

Finally, the modified Theorem D (using the modified Theorem A) is

$$(b-1)\sum_{k=1}^{\infty}\frac{f_{\alpha'}(k)}{b^k}=a_0+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(b-1)^2}{(b^{q_k}-1)(b^{q_{k-1}}-1)}.$$

Remark. This paper grew out of the first author's consideration of the number $\sum_{k=1}^{\infty}\frac{f_\alpha(k)}{2^k}$, where $\alpha=\frac{1+\sqrt{5}}{2}$, as the fixed point of the sequence $\{g_n(0)\}$, $n\geq 1$, where $g_1(x)=x/2$, $g_2(x)=(x+1)/2$, $g_n(x)=g_{n-1}(g_{n-2}(x))$, $n\geq 3$. This quickly leads (upon setting $g_n(x)=(x+a_n)/b_n$ and solving for a_n and b_n) to

$$\sum_{k=1}^{\infty}\frac{f_\alpha(k)}{2^k}=[2,2^0,2^1,2^1,2^2,2^3,2^5,\ldots].$$

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