Powers of Digital Sums

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1 Introduction

Let $s(n)$ denote the sum of the base 10 digits of the nonnegative integer n, and let $\log x$ denote the base 10 logarithm of x. R. E. Kennedy and C. Cooper have shown [1] that for any positive integer k,

$$\frac{1}{x}\sum_{n\leq x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-\frac{1}{3}} x),$$

and they conjectured that for any positive integer k,

$$\frac{1}{x}\sum_{n\leq x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x).$$

Recently [2] the same authors have shown, providing some evidence for the truth of this conjecture, that for each fixed positive integer k,

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In this note, we extend the result just mentioned. When k is a fixed positive integer, we show that for each m it is true that

$$\frac{1}{x}\sum_{n\leq x} s(n)^k = \left(\frac{9}{2}\right)^k \log^k x + O(\log^{k-1} x),$$

provided that x is restricted to the set of those positive integers having at most m nonzero digits in their base 10 representations. (Thus, the Kennedy & Cooper result is exactly the case $m = 1$.) We use the Kennedy & Cooper result in the course of our proof.

We state our result in the following form.

Proposition. *Let $m \geq 1$ and $k \geq 1$ be fixed integers. Then there is a constant $A = A(k,m)$ such that if x*

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is a positive integer with at most m nonzero digits in its base 10 representation,

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (\log x)^k\right| < A(\log x)^{k-1}.$$

2 Remarks and Lemmas

Remark 1. *It is easy to check that if m, k are fixed positive integers and*

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k [\log x]^k\right| < c[\log x]^{k-1},$$

then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (\log x)^k\right| < d(\log x)^{k-1},$$

where $[\cdot]$ denotes the greatest integer function and d is a constant that depends only on c and k.

To see this, suppose $10^n \le x \le 10^{n+1}$, so that $n = [\log x]$. Let $\log x = n + \alpha$, where $0 \le \alpha < 1$. Then

$$\begin{aligned}
&\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k (n+\alpha)^k\right| \\
&\quad < \left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| + \left|\left(\frac{9}{2}\right)^k n^k - \left(\frac{9}{2}\right)^k (n+\alpha)^k\right| \\
&\quad = \left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| + \left(\frac{9}{2}\right)^k \left\{k\alpha n^{k-1} + \binom{k}{2}\alpha^2 n^{k-2} + \cdots + \binom{k}{k}\alpha^k\right\} \\
&\quad < cn^{k-1} + c'n^{k-1} = dn^{k-1} \le d(n+\alpha)^{k-1}.
\end{aligned}$$

Remark 2. *In view of Remark 1, to prove the Proposition above, it is sufficient (and convenient) to prove the following statement, which will be done by induction on m.*

For fixed positive integers m, k, there is an $A = A(k,m)$ such that if $10^n \le x < 10^{n+1}$ and x has at most m nonzero digits in its base 10 representation, then

$$\left|\frac{1}{x}\sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| < An^{k-1}.$$

In the following three lemmas, k, n, p, y and t all denote integers.

Lemma 1. *For each $k \ge 1$, there is an $A(k)$ such that*

$$\frac{n \cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1 - \left(\frac{p}{n}\right)^k\right) < A(k) \text{ for all } n, p \text{ with } 1 \le p \le n.$$

Proof. Let $s = n - p - 1$. Then $-1 \le s \le n-2$ and

$$\begin{aligned}
\frac{n\cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1-\left(\frac{p}{n}\right)^k\right) &= \frac{n}{10^s+1}\left(1-\left(1-\frac{s+1}{n}\right)^k\right)\\
&= \frac{n}{10^s+1}\left(k\cdot\frac{s+1}{n} - \binom{k}{2}\frac{(s+1)^2}{n^2} + \binom{k}{2}\frac{(s+1)^3}{n^3} - \cdots - (-1)^k\binom{k}{k}\frac{(s+1)^k}{n^k}\right)\\
&= \frac{k\cdot(s+1)}{10^s+1} + o(1) < A(k).
\end{aligned}$$

□

Note that for $i \ge 2$,

$$\frac{n}{10^s+1}\frac{(s+1)^i}{n^i} \le \left(\max_{-1\le s<\infty}\frac{(s+1)^i}{10^s+1}\right)\cdot\frac{1}{n^{i-1}} \to 0 \text{ as } n\to\infty.$$

Lemma 2. *Fix $k \ge 1$. Let $A(k)$ be as in Lemma 1. Then for any y in the interval $10^p \le y \le 10^{p+1} \le 10^n$ and any $t \ge 1$,*

$$\left|\frac{t\cdot 10^n}{t\cdot 10^n + y}\left(\frac{9}{2}\right)^k n^k + \frac{y}{t\cdot 10^n + y}\left(\frac{9}{2}\right)^k p^k - \left(\frac{9}{2}\right)^k n^k\right| < \left(\frac{9}{2}\right)^k A(k) n^{k-1}.$$

Proof.

$$\begin{aligned}
\left|\frac{t\cdot 10^n}{t\cdot 10^n + y}n^k + \frac{y}{t\cdot 10^n + y}p^k - n^k\right| &= n^k\frac{y}{t\cdot 10^n + y}\left(1-\left(\frac{p}{n}\right)^k\right)\\
&< n^k\frac{10^{p+1}}{t\cdot 10^n + 10^{p+1}}\cdot\left(1-\left(\frac{p}{n}\right)^k\right)\\
&\le n^{k-1}\frac{n\cdot 10^{p+1}}{10^n + 10^{p+1}}\left(1-\left(\frac{p}{n}\right)^k\right)\\
&< A(k)\cdot n^{k-1},
\end{aligned}$$

where the last inequality is given by Lemma 1. □

Lemma 3. *Let $t \ge 1$ and $k \ge 1$ be fixed. Then*

$$\frac{1}{t\cdot 10^n}\sum_{i=0}^{t\cdot 10^n - 1} s(i)^k = \left(\frac{9}{2}\right)^k n^k + O(n^{k-1}).$$

In other words, there is $B(t,k)$ with

$$\left|\frac{1}{t\cdot 10^n}\sum_{i=0}^{t\cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k n^k\right| < B(t,k) n^{k-1}.$$

To prove the Proposition, we will only need this lemma for $t = 1, 2, \ldots, 9$.

Proof. We use induction on t. For $t = 1$, this is the result of Kennedy & Cooper mentioned above.

Now fix $t \geq 1$ and assume the result for this t. Then, using the fact that $s(t \cdot 10^n + i) = s(t) + s(i)$, $0 \leq i \leq 10^n - 1$,

$$\begin{aligned}
\left| \frac{1}{(t+1)\cdot 10^n} \sum_{i=0}^{t\cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k n^k \right| &\leq \left| \frac{t}{t+1} \left(\frac{1}{t\cdot 10^n} \sum_{i=0}^{t\cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| \\
&+ \left| \frac{1}{t+1} \left(\frac{1}{\cdot 10^n} \sum_{i=0}^{t\cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| \\
&+ \frac{1}{t+1} \cdot \frac{1}{10^n} \sum_{i=0}^{t\cdot 10^n - 1} (c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k) \\
&< \frac{t}{t+1} B(t,k) n^{k-1} + \frac{1}{t+1} B(1,k) n^{k-1} + C n^{k-1} \\
&< B(t+1,k) n^{k-1}.
\end{aligned}$$

□

Note that we used the result of Kennedy & Cooper a second time. Here $c_1, \ldots, c_k$, C are constants that depend only on k and t.

3 Proof of the Proposition

According to Remark 2, we need to show that, for each $m \geq 1$ and $k \geq 1$, there is a constant $A(k,m)$ such that if $10^n \leq x < 10^{n+1}$ and x has at most m nonzero digits in its base 10 representation, then

$$\left| \frac{1}{x} \sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k \right| < A(k,m) n^{k-1}.$$

If $m = 1$, this follows from Lemma 3 (for all k), since then $x = t \cdot 10^n$, $1 \leq t \leq 9$.

Now assume the result for a given $m \geq 1$, and let x have $m+1$ nonzero digits, say,

$$10^n \leq x < 10^{n+1}, x = t \cdot 10^n + y, 1 \leq t \leq 9, 10^p \leq y < 10^{p+1} \leq 10^n,$$

where y has m nonzero digits. Then, using $s(t \cdot 10^n + i) = t + s(i)$,

$$\begin{aligned}
\left| \frac{1}{x} \sum_{i=0}^{x-1} s(i)^k - \left(\frac{9}{2}\right)^k \right| &\leq \left| \frac{t\cdot 10^n}{t\cdot 10^n + y} \left(\frac{1}{t\cdot 10^n} \sum_{i=0}^{t\cdot 10^n - 1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| \\
&+ \left| \frac{y}{t\cdot 10^n + y} \left(\frac{1}{y} \sum_{i=0}^{y-1} s(i)^k - \left(\frac{9}{2}\right)^k \right) \right| \\
&+ \frac{y}{t\cdot 10^n + y} \cdot \frac{1}{y} \sum_{i=0}^{y-1} (c_1 s(i)^{k-1} + c_2 s(i)^{k-2} + \cdots + c_k) \\
&< A(k,1) n^{k-1} + A(k,m) p^{k-1} + D p^{k-1} \\
&< A(k,m+1) n^{k-1}.
\end{aligned}$$

Here $c_1, \ldots, c_k, D$ are constants that depend on k and t, but since $1 \leq t \leq 9$, they in fact depend only on k. For the second equality, we used Lemma 3 as well as the induction hypothesis.

References

[1] C. Cooper and R.E. Kennedy, *Digit sum sums*, J. Inst. Math. Comp. Sci. **5** (1992), 45–49.

[2] ______, *Sums of powers of digital sums*, The Fibonacci Quarterly **31.4** (1993), 341–345.