Chaotic orderings of the rationals and reals

Hayri Ardal, Tom Brown, and Veselin Jungić*

Citation data: Hayri Ardal, Tom Brown, and Veselin Jungić, *Chaotic orderings of the rationals and reals*, Amer. Math. Monthly **118** (2011), 921–925.

Abstract

In this note we prove that there is a linear ordering of the set of real numbers for which there is no monotonic 3-term arithmetic progression. This answers the question (asked by Erdős and Graham) of whether or not every linear ordering of the reals must have a monotonic k-term arithmetic progression for every k.

1 Introduction.

Over the last few years there has been an increasing interest in combinatorial properties of linear orderings, also called infinite permutations, of various sets of real numbers. See for example [1, 5, 7, 8].

In this note, answering a question of Erdős and Graham [4, pp. 21–22], we show that there is a linear ordering $\ll$ of the set $\mathbb{R}$ of real numbers which has no monotonic 3-term arithmetic progression (AP). This means that there do not exist distinct real numbers x and y such that $x \ll \frac{1}{2}(x+y) \ll y$. We call such an ordering *chaotic*. In general, for any set $X \subseteq \mathbb{R}$, a linear ordering $\ll$ of X is *chaotic* if there do not exist distinct $x,y,z \in X$ such that $y = \frac{1}{2}(x+z)$ and $x \ll y \ll z$.

Symbols such as "$<$" or "$\geq$" always refer to the usual order relation on $\mathbb{R}$.

Our methods show that for every $k \geq 2$ there is a linear ordering of $\mathbb{R}$ for which there are monotonic k-term AP's, but no monotonic $(k+1)$-term AP. A *monotonic k-term AP* in $\mathbb{R}$ (monotonic with respect to an ordering $\ll$) is a set $\{a_i : 0 \leq i \leq k-1\}$ with $a_i = a_0 + id$, $0 \leq i \leq k-1$, $d \neq 0$, such that $a_0 \ll a_1 \ll \cdots \ll a_{k-1}$.

It has long been known that there are chaotic linear orderings of $\{1,2,\ldots,n\}$ for every positive integer n [3, 6, 9–11], but that for any arrangement of the positive integers into a sequence $a_1, a_2, \ldots$, or even a doubly-infinite sequence $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$, there *does* exist a monotonic 3-term AP, that is, there are $i < j < k$ such that $a_j = \frac{1}{2}(a_i + a_k)$ [2]. It is a well-known open question whether every sequence $a_1, a_2, \ldots$ which contains each positive integer exactly once must contain a monotonic 4-term AP. It is known that such a sequence need not contain a monotonic 5-term AP [2].

We start by constructing an explicit chaotic linear ordering of $\mathbb{Z}$, the set of integers. (Our method is similar to that of T. Odda [9].) This ordering of $\mathbb{Z}$ has 0 as its smallest element and -1 as its largest

*Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada, V5A 1S6. hardal@sfu.ca, tbrown@sfu.ca, vjungic@sfu.ca.

element. We then use König's infinity lemma to obtain a chaotic linear ordering of $\mathbb{Q}$, the set of rational numbers. Finally, we use the axiom of choice (in the form "every vector space has a basis") to obtain a chaotic linear ordering of $\mathbb{R}$. It would be of interest to know whether such an ordering can be constructed without using the axiom of choice.

For distinct real numbers $a_1, a_2, \ldots, a_n$ we write $\langle a_1, a_2, \ldots, a_n \rangle$ to denote the ordering $\ll$ on the set $\{a_1, a_2, \ldots, a_n\}$ in which $a_1 \ll a_2 \ll \cdots \ll a_n$. If $A = \langle a_1, a_2, \ldots, a_n \rangle$ and k is a real number, then $A + k$ denotes the ordering $\langle a_1 + k, a_2 + k, \ldots, a_n + k \rangle$, and for a nonzero k, kA denotes the ordering $\langle ka_1, ka_2, \ldots, ka_n \rangle$. For disjoint sets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$, if $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, then AB denotes the ordering $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \rangle$.

2 A chaotic ordering of $\mathbb{Z}$

Definition 2.1. Define the ordering A_n on $[-2^{n-1}, 2^{n-1} - 1]$, $n \geq 1$, inductively by setting $A_1 = \langle 0, -1 \rangle$ and $A_{n+1} = (2A_n)(2A_n + 1)$.

Thus $A_2 = \langle 0, -2, 1, -1 \rangle$ and $A_3 = \langle 0, -4, 2, -2, 1, -3, 3, -1 \rangle$.

Lemma 2.1. *(i) Each A_n is chaotic, $n \geq 1$. (This means that if $a, b, c \in [-2^{n-1}, 2^{n-1} - 1]$ and $a + c = 2b$, then b cannot lie between a and c, in the ordering A_n.)*

(ii) Each A_{n+1} extends A_n, $n \geq 1$. (This means that the numbers -2^{n-1}, $-2^{n-1} + 1, \ldots, 2^{n-1} - 1$ appear in A_{n+1} in the same order that they appear in A_n.)

Proof. (i) Clearly A_1 is chaotic. Assume that $n \geq 1$ and that A_n is chaotic, that is, contains no monotonic 3-term AP. Then so are $2A_n$ and $2A_n + 1$. Suppose a, $b = a + d$, $c = a + 2d$ is a 3-term AP contained in $A_{n+1} = (2A_n)(2A_n + 1)$. Since a and c have the same parity, a and c must both appear in $2A_n$ or in $2A_n + 1$. But $2A_n$ and $2A_n + 1$ are chaotic, and hence b cannot appear between a and c. Thus A_{n+1} is chaotic.

(ii) We see that A_2 extends A_1. Assume now that $n \geq 2$ and that A_n extends A_{n-1}. Then $2A_n$ extends $2A_{n-1}$ and $2A_n + 1$ extends $2A_{n-1} + 1$. Hence $A_{n+1} = (2A_n)(2A_n + 1)$ extends $A_n = (2A_{n-1})(2A_{n-1} + 1)$. □

Definition 2.2. The linear ordering $<_{\mathbb{Z}}$ on $\mathbb{Z}$ is defined as follows. For all $a, b \in \mathbb{Z}$,

$$a <_{\mathbb{Z}} b \Leftrightarrow \text{ whenever } a, b \in [-2^{n-1}, 2^{n-1} - 1],\ a \text{ precedes } b \text{ in } A_n.$$

This definition makes sense in view of part (ii) of Lemma 2.1. Part (i) clearly implies the following theorem.

Theorem 2.2. *The linear ordering $<_{\mathbb{Z}}$ of $\mathbb{Z}$ is chaotic.*

3 A chaotic ordering of $\mathbb{Q}$

Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ be a fixed enumeration of $\mathbb{Q}$, and for each $n \geq 1$ let $X_n = \{r_1, r_2, \ldots, r_n\}$. We can produce a chaotic ordering of X_n by choosing $k \in \mathbb{N}$ so that $kX_n \subseteq \mathbb{Z}$, restricting the ordering $<_{\mathbb{Z}}$ of $\mathbb{Z}$ to

kX_n to give a chaotic ordering of kX_n, and then dividing by k to give a chaotic ordering of X_n. Different values of k may give different chaotic orderings of X_n. The essential point is that there exists at least one chaotic ordering of X_n for each $n \geq 1$.

Construct a rooted tree T by letting the vertices at level n be the chaotic orderings of X_n, for each $n \geq 1$. A vertex p at level n is adjacent to a vertex q at level $n+1$ if and only if q extends p.

Some branches of T may terminate. For example, 6 cannot be inserted into $\langle 2,4,3,0 \rangle$.

By König's infinity lemma, T has an infinite branch $p_1 \subset p_2 \subset p_3 \subset \cdots$, where $p_n \subset p_{n+1}$ means that the ordering p_{n+1} of $X_{n+1} = \{r_1, r_2, \ldots, r_n, r_{n+1}\}$ extends the ordering p_n of $X_n = \{r_1, r_2, \ldots, r_n\}$.

The union of the chain of orderings $p_1 \subset p_2 \subset p_3 \subset \cdots$ is a linear ordering of $\mathbb{Q}$. Call this ordering $<_{\mathbb{Q}}$. Thus by definition, for $a, b \in \mathbb{Q}$

$$a <_{\mathbb{Q}} b \Leftrightarrow \text{ whenever } a, b \in X_n,\ a \text{ precedes } b \text{ in } p_n.$$

Since each p_n is a chaotic ordering of X_n, we have proved the following theorem.

Theorem 3.1. *The linear ordering $<_{\mathbb{Q}}$ of $\mathbb{Q}$ (defined above) is chaotic.*

4 A chaotic ordering of $\mathbb{R}$

Let B be a basis for the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Let $<_{\mathbb{Q}}$ be a chaotic ordering of $\mathbb{Q}$. Extend $<_{\mathbb{Q}}$ to a linear ordering of $\mathbb{R}$ as follows. For distinct $a, b \in \mathbb{R}$ write

$$a = \sum_{i=1}^{k} a_i \gamma_i \text{ and } b = \sum_{i=1}^{k} b_i \gamma_i,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k \in B$, $\gamma_1 < \gamma_2 < \cdots < \gamma_k$, and $a_1, \ldots a_k, b_1, \ldots, b_k \in \mathbb{Q}$.

Let $j = \min\{i : 1 \leq i \leq k \text{ and } a_i \neq b_i\}$. Then we define the ordering $<_{\mathbb{R}}$ on $\mathbb{R}$ by

$$a <_{\mathbb{R}} b \Leftrightarrow a_j <_{\mathbb{Q}} b_j.$$

We claim that $<_{\mathbb{R}}$ is a chaotic ordering of $\mathbb{R}$. (It is clearly a linear order.)

For suppose that $a, b, c \in \mathbb{R}$, a, b, c distinct, with $a + c = 2b$. Write

$$a = \sum_{i=1}^{k} a_i \gamma_i,\ b = \sum_{i=1}^{k} b_i \gamma_i, \text{ and } c = \sum_{i=1}^{k} c_i \gamma_i,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k \in B$, $\gamma_1 < \gamma_2 < \cdots < \gamma_k$, $a_i, b_i, c_i \in \mathbb{Q}$, $1 \leq i \leq k$.

Note that since $a + c = 2b$,

$$a_i + c_i = 2b_i \text{ for each } i,\ 1 \leq i \leq k.$$

Let $j = \min\{i : 1 \leq i \leq k, \text{ and not all of } a_i, b_i, c_i \text{ are equal}\}$. Then no two of a_j, b_j, c_j are equal since $a_j + c_j = 2b_j$ would imply that all three are equal.

Thus for $1 \leq i \leq j-1$, all three of a_i, b_i, c_i are equal. Then $a <_{\mathbb{R}} b <_{\mathbb{R}} < c$ implies that $a_j <_{\mathbb{Q}} b_j <_{\mathbb{Q}} c_j$ in $\mathbb{Q}$, which is not possible since $<_{\mathbb{Q}}$ is a chaotic ordering of $\mathbb{Q}$.

This establishes our claim, and completes the proof of the following theorem.

Theorem 4.1. *The linear ordering $<_{\mathbb{R}}$ of $\mathbb{R}$ (defined above) is chaotic.*

Note that in Theorem 4.1, $\mathbb{R}$ can be replaced by any field of characteristic 0.

5 Remarks

1. Definition 2.1 can be generalized by replacing 2 by k, for any $k \geq 2$. Fix $k \geq 2$. Define the permutation C_n of $[-(k-1)k^{n-1}, k^{n-1}-1]$, $n \geq 1$, inductively by setting $C_1 = \langle 0, -1, -2, \cdots, -(k-1) \rangle$ and $C_{n+1} = (kC_n)(kC_n+1)\cdots(kC_n+k-1)$, $n \geq 1$.

   All of the above arguments in Sections 2, 3, and 4 can now be repeated with little change, resulting in a linear ordering of $\mathbb{R}$ with respect to which there is no monotonic $(k+1)$-term AP, but there are monotonic k-term AP's.

2. Let $\mathbb{N}$ denote the set of nonnegative integers, and for $a, b \in \mathbb{N}$, let $a = \sum_{i=0}^{k} a_i 2^i$ and $b = \sum_{i=0}^{k} b_i 2^i$, $a_i, b_i \in \{0, 1\}$. From Definition 2.1, restricted to $\mathbb{N}$, it is not hard to see that $a <_{\mathbb{Z}} b$ if and only if $(a_0, a_1, \ldots, a_k)$ precedes $(b_0, b_1, \ldots, b_k)$ in the standard lexicographic ordering of finite binary sequences. It easily follows that

   $$a <_{\mathbb{Z}} b \Leftrightarrow \sum_{i=0}^{k} \frac{a_i}{2^i} < \sum_{i=0}^{k} \frac{b_i}{2^i}.$$

   Thus $<_{\mathbb{Z}}$, restricted to $\mathbb{N}$, can be defined in terms of $<$.

3. The last equivalence suggests defining an explicit chaotic linear ordering $<_{\mathbb{D}}$ of the set $\mathbb{D}$ of nonnegative dyadic rationals,

   $$\mathbb{D} = \left\{ \frac{p}{2^n} : p, n \in \mathbb{N} \right\}.$$

   For $a, b \in \mathbb{D}$, let $a = \sum_{i=-\infty}^{\infty} a_i 2^i$, $b = \sum_{i=-\infty}^{\infty} b_i 2^i$, $a_i, b_i \in \{0, 1\}$. Let

   $$a^* = \sum_{i=-\infty}^{\infty} a_i 2^{-i} \text{ and } b^* = \sum_{i=-\infty}^{\infty} b_i 2^{-i}.$$

   As remarked above, in case $a, b \in \mathbb{N}$ then

   $$a <_{\mathbb{Z}} b \Leftrightarrow a^* < b^*.$$

   Now we define, for all $a, b \in \mathbb{D}$,

   $$a <_{\mathbb{D}} b \Leftrightarrow a^* < b^*.$$

   To see that $<_{\mathbb{D}}$ is chaotic, for $a, b \in \mathbb{D}$ choose q so that $2^q a, 2^q b \in \mathbb{N}$. Then $(2^q a)^* = \frac{1}{2^q} \cdot a^*$ and

   $$a <_{\mathbb{D}} b \Leftrightarrow a^* < b^* \Leftrightarrow \frac{1}{2^q} \cdot a^* < \frac{1}{2^q} \cdot b^* \Leftrightarrow (2^q a)^* < (2^q b)^* \Leftrightarrow 2^q a <_{\mathbb{Z}} 2^q b.$$

Since $<_{\mathbb{Z}}$ is chaotic, so is $<_{\mathbb{D}}$.

4. Note that the mapping ϕ from $\mathbb{D}$ to $\mathbb{D}$ defined by $\phi(a) = a^*, a \in \mathbb{D}$, is a bijection and, by definition, $a <_{\mathbb{D}} b$ if and only if $\phi(a) < \phi(b)$ for all $a, b \in \mathbb{D}$. Thus, $(\mathbb{D}, <_{\mathbb{D}})$ and $(\mathbb{D}, <)$ are *order-isomorphic*.

   Since $\phi(\mathbb{N}) = \mathbb{D} \cap [0,2)$ and $\phi(\mathbb{D} \cap [0,2)) = \mathbb{N}$, we have that $(\mathbb{N}, <_{\mathbb{D}})$ and $(\mathbb{D} \cap [0,2), <)$, as well as $(\mathbb{D} \cap [0,2), <_{\mathbb{D}})$ and $(\mathbb{N}, <)$, are order-isomorphic.

   Since $<_{\mathbb{D}}$ and $<_{\mathbb{Z}}$ agree on $\mathbb{N}$, we conclude, as in Remark 2, that $(\mathbb{N}, <_{\mathbb{Z}})$ and $(\mathbb{D} \cap [0,2), <)$ are order-isomorphic.

Acknowledgments. The authors would like to thank Radoš Radoičić for initiating their interest in the combinatorial properties of linear orderings. The authors would also like to acknowledge the IRMACS Centre at Simon Fraser University for its support while they were conducting the research presented in this work.

References

[1] S.V. Avgustinovich, A. Frid, T. Kamae, and P.V. Samilov, *Infinite permutations of lowest maximal pattern complexity*, Theoretical Computer Science **412** (2011), 2911–2921.

[2] J.A. Davis, R.C. Entringer, R.L. Graham, and G.J. Simmons, *On permutations containing no long arithmetic progressions*, Acta Arith. **34** (1977/1978), 81–90.

[3] R.C. Entringer and D.E. Jackson, *Problem e 2440*, Amer. Math. Monthly **82** (1975), 74.

[4] P. Erdős and R.L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[5] D.G. Fon-Der-Flaass and A.E. Frid, *On periodicity and low complexity of infinite permutations*, European J. Combin. **28** (2007), 2106–2114.

[6] R.C. Lyndon, *Solution to problem e 2440*, Amer. Math. Monthly **82** (1975), 74–75.

[7] M.A. Makarov, *On an infinite permutation similar to the Thue-Morse word*, Discrete Math. **309** (2009), 6641–6643.

[8] M.A. Makarow, *On the infinite permutation generated by the period doubling word*, European J. Combin. **31** (2010), 368–378.

[9] T. Odda, *Solution to problem e 2440*, Amer. Math. Monthly **82** (1975), 74.

[10] G.J. Simmons, *Solution to problem e 2440*, Amer. Math. Monthly **82** (1975), 76–77.

[11] Jr. Thomas, H.E., *Solution to problem e 2440*, Amer. Math. Monthly **82** (1975), 75–76.